Space Flight Mechanics
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Module No # 17
Lecture No # 82
Trajectory Transfer (Contd.)

Now welcome to lecture 82 so we have been doing the generalized trajectory transfer in a planer orbit or coplanar orbit we will continue with that. So going back into the previous lecture and looking for that. Here tan ϕ_i and tan ϕ_f both of them involved θ_i and θ_f so that means the true anomaly for both these orbits need to be determined. If it is circular orbit so it can be from any reference line you can do but if it is an elliptic orbit then we have to measure it from the proper reference line which in this case constitutes your perigee apogee okay.

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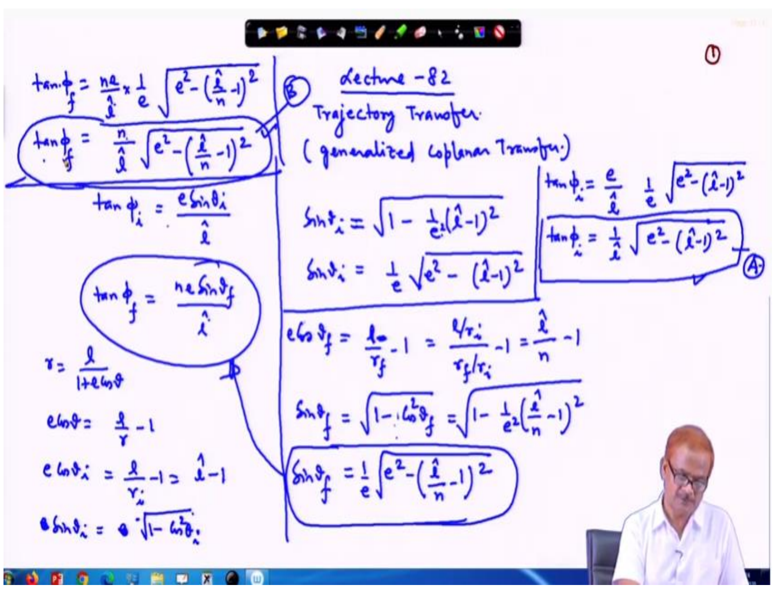

$$tan\phi_i = \frac{esin\theta_i}{\hat{l}}$$

$$tan\phi_f = \frac{esin\theta_f}{\hat{l}}$$

$$r = \frac{l}{1+ecos\,\theta}$$

$$ecos\theta_i = \frac{l}{r_i} - 1$$

So we have defined tanϕ_i equal to tan ϕ_i this we have got as e sinθ_i this divided by $\hat{l}$ and tan ϕ_f. Now we utilize this relation r = l / 1 + e Cosθ to get the value for θ. So e Cosθ this will be 1 / r – 1 and e Cosθ_i can write as l / r_i – 1 so this gets reduced to $\hat{l}$ – 1 and based on this we can it is write e sin θ_i equal to e sinθ is let us write it like this sinθ = 1 – Cosθ_i^2 under root equal to 1 – Cosθ is here in this case 1 / e times $\hat{l}$– 1.

$$\sin\theta_i = \frac{1}{e}\sqrt{e^2 - \left(\hat{l} - 1\right)^2}$$

$$ecos\theta_f = \frac{l}{r_f} - 1 = \frac{\frac{l}{r_i}}{\frac{r_f}{r_i}} - 1 = \frac{\hat{l}}{n} - 1$$

So this will be a^2 in this also get a^2 and this under root okay. So this is 1 part here this we can write as 1 / e times e^2 – $\hat{l}$– 1 whole^2 under. Similarly we will have sinθ_f = 1- first let us write Cosθ_f and from there then we write as sinθ here. Cosθ_f = e Cosθ_f into l / r_f l – 1 1 / r_i divided by $\hat{l}$ / n – 1. And therefore sinθ_f Cosθ_f^2 1 – Cosθ_f^2 under root.

$$sin\theta_f = \frac{1}{e}\sqrt{e^2 - \left(\frac{\hat{l}}{n} - 1\right)^2}$$

Cosθ_f is 1 / e times ($\hat{l}$ / n)-1 this is^2 under root sinθ_f so by this statement you can work equally for the circular or the elliptical orbit and whatever it may have. Because I have done a generalized here treatment for this so with the circular orbit case it becomes easier to work out when the elliptic orbit we have to take into account all this angles so okay anyway. We have been able to work out all this things so finally we can summarize here.

$$tan\phi_f = \frac{e}{\hat{l}}\frac{1}{e}\sqrt{e^2 - \left(\hat{l} - 1\right)^2}$$

$$\tan\phi_i = \frac{1}{\hat{l}}\sqrt{e^2 - \left(\hat{l} - 1\right)^2}$$

Therefore tanϕ_i this will be equal to e / $\hat{l}$ times sinθ_i which is 1 / e times e^2 / $\hat{l}$– 1^2 or 1 / $\hat{l}$ e^2 minus let us name this as equation A here. In the same way the next thing I will write here in this place tanϕ_f this will be given by ne divided by $\hat{l}$ times 1 / e e^2 – here we have written sinθ_f. So sinθ_f we need to replace here in this equation we are going to insert this okay.

$$tan\phi_f = \frac{ne}{\hat{l}}\frac{1}{e}\sqrt{e^2 - \left(\frac{\hat{l}}{n} - 1\right)^2}$$

$$= \frac{n}{\hat{l}}\sqrt{e^2 - \left(\frac{\hat{l}}{n} - 1\right)^2}$$

So this is ne / $\hat{l}$ 1 / e e^2 – $\hat{l}$ / n – 1 whole square n / $\hat{l}$and this equation we term as B. Okay so we are got till this point but still our job is not one because we will look here in this place the we require Cosϕ in both the places Cosϕ_i and Cosϕ_f. So till now we have been able to resolve it in terms of tanϕ_i times ϕ_f and where that here on the right hand side the θ was appearing then we eliminate θ.

So dependence on the true anomaly as gone in this equation A and B okay but still it is in terms of tanϕ so we need to convert this into in terms of Cos this in terms of tan we have to change it to Cosine okay.

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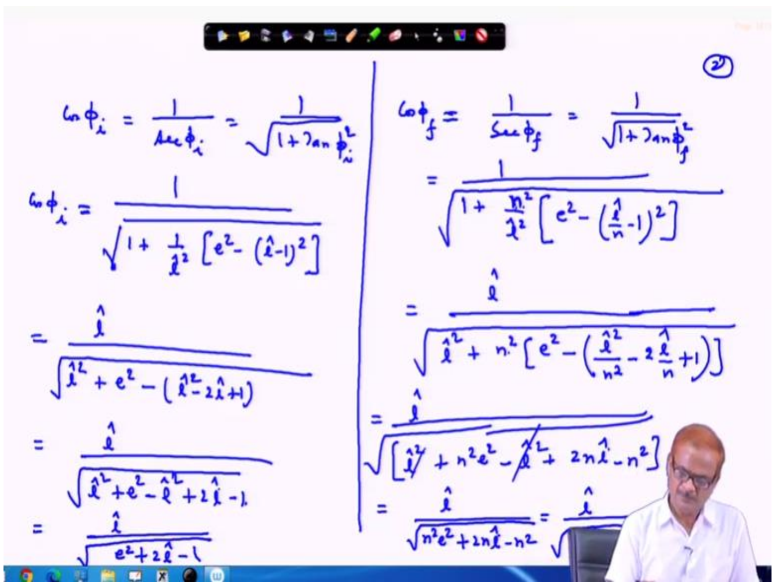

$$\cos\phi_i = \frac{1}{sec\phi_i} = \frac{1}{\sqrt{1+tan\phi^2}}$$

$$\cos\phi_i = \frac{1}{\sqrt{1+\frac{1}{\hat{l}}\left(e^2-(\hat{l}-1)^2\right)}}$$

$$= \frac{\hat{l}}{\sqrt{\hat{l}^2+e^2-(\hat{l}^2-2\hat{l}+1)}}$$

$$= \frac{\hat{l}}{\sqrt{e^2+2\hat{l}-1}}$$

So Cosϕ_i this will be equal to 1 /ϕ_i 1 / 1 + tanϕ_i^2 under root tanϕ_i^2 just now we have worked out 1 / $\hat{l}$ this is 1 / $\hat{l}^2$ and times e^2 – $\hat{l}$ – 1 whole^2. And this under root $\hat{l}^2$ + e^2 – $\hat{l}^2$ 2$\hat{l}$ + 1 $\hat{l}^2$ + e^2 under root $\hat{l}$ cancels out this is e^2 2$\hat{l}$– 1. This is called ϕ_i similarly it Cosϕ_f can be obtained cos ϕ_f = 1 / 1 +

$\tan\phi_f^2$ under root and the inserting the corresponding values from the previous equation $\hat{l}^2$times n / $\hat{l}$ this is n / $\hat{l}$.

$$Cos\ \phi_f = \frac{1}{\sqrt{1+tan\phi_f^2}}$$
$$= \frac{1}{\sqrt{1+\frac{n^2}{\hat{l}^2}\left[e^2-\left(\frac{\hat{l}}{n}-1\right)^2\right]}}$$
$$= \frac{\hat{l}}{\sqrt{\hat{l}^2+n^2e^2-\hat{l}^2+2n\hat{l}-n^2}}$$
$$= \frac{\hat{l}}{\sqrt{n^2e^2+2n\hat{l}-n^2}}$$

So here we have to out here n also n / $\hat{l}^2$ Cos tan ϕ_i n / $\hat{l}^2$ v^2 then e^2 – $\hat{l}$ divided by n – 1^2 this also goes as n^2 2n times $\hat{l}$+ and then minus this quantity can drops out and left with $\hat{l}$ divided by. So this way our so the angles are known if it try to put all the equations in the all this equation in the final thing it becomes very tough to manage okay equations becomes very long.

But if you have to do the computation then it is quite easy you keep computing the quantities at each step and then finally use it into the symbolic equation dot v r okay.

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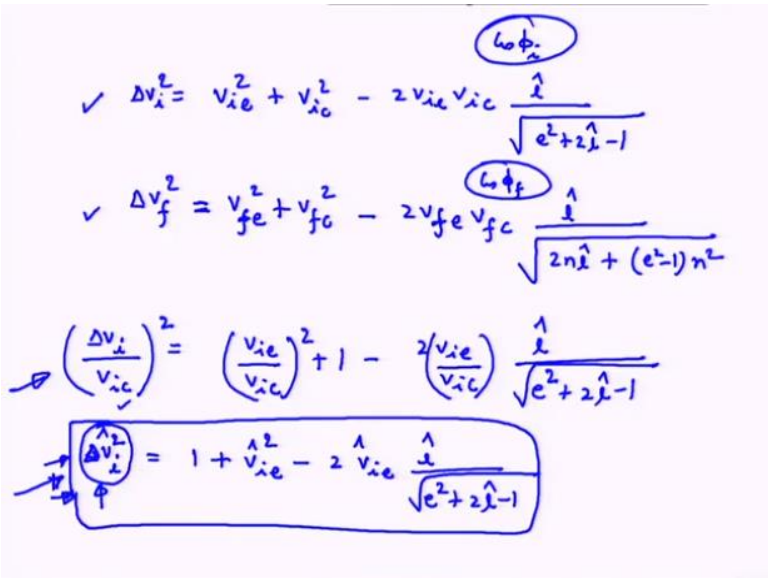

$$\Delta v_i^2 = v_{ie}^2 + v_{ic}^2 - 2v_{ie}v_{ic}\frac{\hat{l}}{\sqrt{e^2+2\hat{l}-1}}$$

$$\Delta v_f^2 = v_{fe}^2 + v_{fc}^2 - 2v_{fe}v_{fc}\frac{\hat{l}}{\sqrt{2n\hat{l}+(e^2-1)n^2}}$$

So with this we have Δv_i^2 equal to $v_{ie}^2 + v_{ic}^2 - 2\, v_{ie}$ times v_{ic} times Cos if here ϕ_i so Cosϕ_i already we have worked out on the previous page. So Cos ϕ_i is here finally this is – so $\hat{l}$ divided by under root e^2 2l – 1 e^2 + 2l – 1 this is 1 equation and Δv_f^2 equal to $v_{fe}^2 + v_{fc}^2 - 2v_{fe}$ times v_{fc} and then here instead of Cosϕ for Cosϕ_f appears here. So for that we are writing here $\hat{l}$ divided by under root 2n $\hat{l}$ e^2 – 1 times n^2.

$$\frac{\Delta v_i^2}{\Delta v_{ic}^2} = \left(\frac{v_{ie}}{v_{ic}}\right)^2 + 1 - 2\left(\frac{v_{ie}}{v_{ic}}\right)\frac{\hat{l}}{\sqrt{e^2+2\hat{l}-1}}$$

$$\Delta\hat{v}_i^2 = 1 + \hat{v}_{ie}^2 - 2\hat{v}_{ie}^2\frac{\hat{l}}{\sqrt{e^2+2\hat{l}-1}}$$

So at the initial point what will be the impulse required it will be computed from this particular equation. Little bit simplification we can further do to represent it in the parametric terms so we can write this as Δv_i / v_{ic}^2 and then symbolically this is your Δv_i and this can be written as 1 + $\hat{v}_{ie}$ the notion we are using -2 $\hat{v}_{ie}$ $\hat{l}$ times e^2 + 2 $\hat{l}$– 1 under root.

So you may not be able to realize what is the benefit putting in this form okay this is at non dimensional quantity and this you are divided by v_{ic}. So whenever this quantity you are calculating and just multiplying by v_{ic} this is v_{ic}^2 was missing so we should put $\hat{v}_{ic}{}^2$ $\hat{v}_i{}^2$. So here you just need to multiply by any time by v_{ic}^2 so your Δv_i^2 will get recovered.

And this helps in parametric study this is the benefit but we are not going to do all the plots and other things at once you have in this format or either here in this format your equation is there that is very easy to work out the problem.

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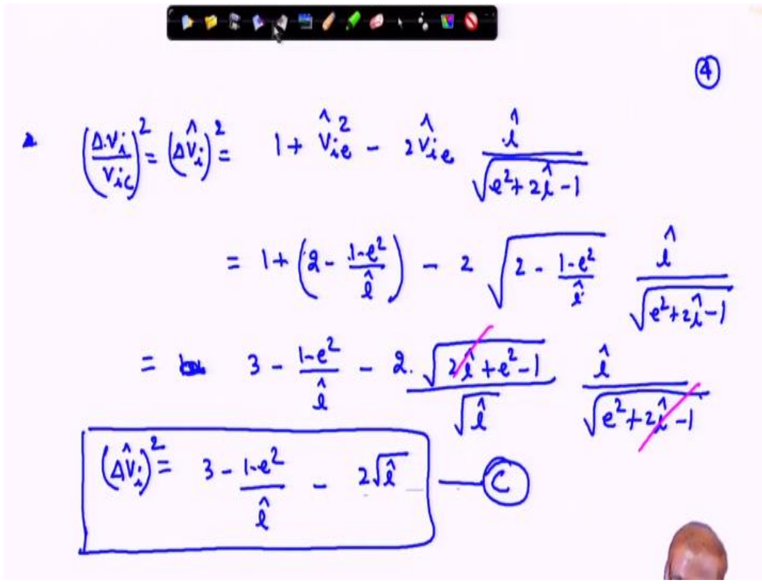

$$\frac{\Delta v_i^2}{\Delta v_{ic}^2} = (\Delta \hat{v}_i)^2 = 1 + \hat{v}_{ie}^2 - 2\hat{v}_{ie}\frac{\hat{l}}{\sqrt{e^2+2\hat{l}-1}}$$

$$= 1 + \left(2 - \frac{1-e^2}{\hat{l}}\right) - 2\sqrt{2 - \frac{1-e^2}{\hat{l}}}\ \frac{\hat{l}}{\sqrt{e^2+2\hat{l}-1}}$$

$$(\Delta \hat{v}_i)^2 = 3 - \frac{1-e^2}{\hat{l}} - 2\sqrt{\hat{l}}$$

Next we have so what we have got here Δv_i / Δv_{ic} we have written as $\Delta\ \hat{v}_i$ this we are squaring to get the square here. And this we have written as $1 + v_{ie}^2 - 2$ times $\hat{v}_{ie}$times $\hat{l}$ divided by $\hat{v}_i - 1$ under root this is what we are registering here. Now we replace vie here in this place and v_{ie} we have derived it earlier look here in this place v_{ie} and v_{fe}.

So we can utilize this result to solve this problem so $1 + v_{ie}$ will be v_i. First we need to divide your of if you divide it v_{ie} / v_{ic} so this comes as v_{ie} / $v_{ie}^2 = \hat{v}_{ie}^2$and this is the quantity we are looking for that simply becomes $2 - 1 - e^2$ divided by $\hat{l}$. This quantity simply becomes $2 - 1 - e^2$ divided by $\hat{l}$ this becomes $3 - 1 - e^2$ divided by $\hat{l}$- 2 times this is reduced to $2\hat{l} + e^2 - 1$ divided by $\hat{l}$ under root times $\hat{l}$ divided by.

This term and this term they drop out $2\hat{l}$ under root $\Delta\hat{v}_i{}^2$ so this is the imposed required at the non-dimensional format impulse required at the initial point or in the initial orbit.

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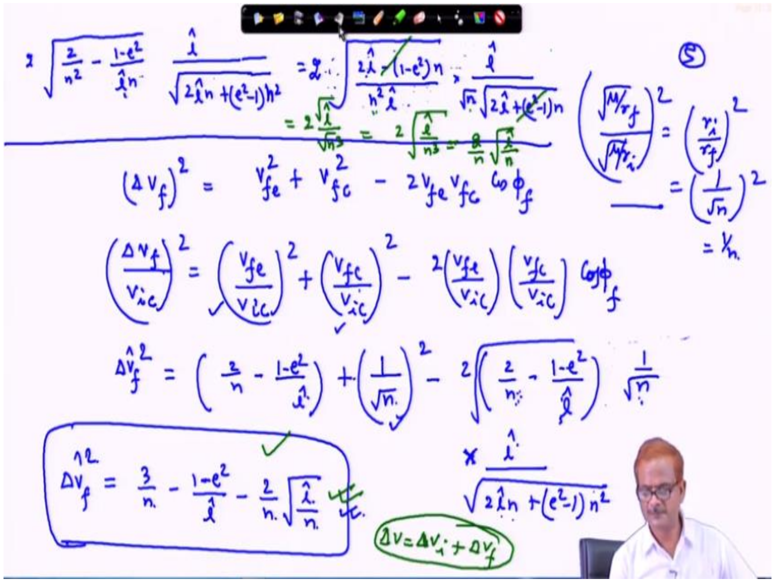

$$\Delta v_f^2 = \ v_{fe}^2 + v_{fc}^2 - 2 v_{fe} v_{fc} \cos \phi_f$$

$$\frac{\Delta v_f^2}{\Delta v_{ic}^2} = \left(\frac{v_{fe}}{v_{ic}}\right)^2 + \left(\frac{v_{fc}}{v_{ic}}\right)^2 - 2\left(\frac{v_{fc}}{v_{ic}}\right)\left(\frac{v_{fc}}{v_{ic}}\right) cos\phi_f$$

$$\Delta \hat{v}_f^2 = \frac{3}{n} - \frac{1-e^2}{\hat{l}} - \frac{2}{n}\sqrt{\frac{\hat{l}}{n}}$$

Similarly we can find out the impulse required in the final orbit Δv_f^2 we have written as $v_{fe}^2 + v_{fc}^2$ – 2 v_{fe} times v_{fc} Cosϕ_f into Cosϕ using the same symbol as we have used earlier. This quantity if we go back and look here v_{fe} we have written here this quantity is v_{ic}. So if we divide so this side here we have written on this equation v_{fe} / v this is r_i v_{fe} divided by v_{ic}^2 is equal to 2 / n – 1 – e^2 / $\hat{l}$.

$$\left(\sqrt{\frac{\frac{\mu}{r_f}}{\frac{\mu}{r_i}}}\right)^2 = \left(\frac{r_i}{r_f}\right)^2 = \left(\frac{1}{\sqrt{n}}\right)^2 = \frac{1}{n}$$

So we utilize this lesson here so this quantity becomes 2 divided by n – 1 – e^2 / $\hat{l}$ and v_{fc} / vic. This is nothing but you have written in terms of v_{fc} equal to μ / r_f and μ / r_i and the square of that. Okay so this is under root so from here we get as 1 by and from there we wrote this as r_i / r_f^2 or 1 / under root in square. So / n under root square so this we have done earlier and same thing we have to insert here 2 divided by n + 1 – e^2 / $\hat{l}$ under root times 1 / $\hat{l}$ under root and multiplied by Cosϕ_f.

So Cosϕ_f we have to pick up from the previous derivation Cosϕ_f we have done here so this is the final thing $\hat{l}$ divided by 2nl + e^2 – 1^2 $\hat{l}$divided by 2 $\hat{l}$ + e^2 – 1 n^2. So this part can be simplified this becomes 1 / n so directly you can see this is 3 / n – 1 – v_e^2 divided by $\hat{l}$ and minus and then this quantity you will see from this place 2 times $\hat{l}$ this becomes and here.

So 2 times $\hat{l}$ and we have 1 / root $\hat{l}$ is here so suppose if we take it here inside here in this place so we will be able to simply it I will have to go and do in on the next phase. But the final result will turn out be in this way $\hat{l}$ divided by n 2 by this is let us check this value okay v_{fc} this equal to 1 / n this one is 1 / n but the square term already we have put here in this 1 / n. So here we have taken 2 times square root. So only one times square root should go okay and then we need to bring it inside.

So if we bring it inside so here this will be something like n^2 and times $\hat{l}$ n and then once you multiply it so the simplification will take space I do not have the space here let us say that I revoke this and work out here it will be easy in working 2 times under root 2 / n^2 – 1 - e^2 divided by $\hat{l}$n and then $\hat{l}$ divided by 2 $\hat{l}$ n + e^2 -1 times n^2 this will be 2 times 2 $\hat{l}$ n – l – e^2 times n^2 divided n^2 times $\hat{l}$ 2 / n^2.

$$2\sqrt{\frac{2}{n^2}-\frac{1-e^2}{\hat{l}n}}\frac{2}{\sqrt{2\hat{l}n+(e^2-1)n^2}}=\frac{2\sqrt{\hat{l}}}{\sqrt{n^3}}$$

We have done the mistake here n^2 $\hat{l}$ so n^2 we have 2 times and then we need to write in a proper way we are writing like that otherwise we can write here as. I will rewrite this whole thing 2 times n^2 $\hat{l}$ is the lcm so n^2 this is $2\hat{l}$ and then – 1 – e^2 n $\hat{l}$ this is n and then $\hat{l}$ divided by n we can take it out from this place of so that it is greater in representation to $\hat{l}$+ e^2 – 1 and now it is a okay.

So this term and this term this cancels out and here we are let with l to $\hat{l}$ under root and this is n cap and here this is n^2 under root so n give under root with 2 times $\hat{l}$ / n^3 under root. And the simply this is 2 divided by n times $\hat{l}$ / n this is what we have written here. So this way we have been able to find out all the corrections required or all the impulses required in different places and this quantities we have to take the magnitude of this mode of this and then add them to get the final what will be Δv.

So this will be $\Delta v_i + \Delta v_f$ whatever the technique we have applied the same thing as to be applied to the elliptic or bead circular the same technique will go only thing rather than working using this method I will not use for solving any problem this is only meant for parametric study. If I have to solve this problem at it as at each stage I will just keep calculating these values. What is the tan ϕ these are the things required okay and rather than putting in non-dimensional format I will just go with this equation okay.

And using those values I will calculate it and do this so I will never use this equation and this equation data I have written here. Because you need to remember this any time you have to use it and you have to do many reductions. So rather than doing this we can directly we need not remember all this equation we can directly work from the basics and any time you can solve the trajectory transfer problem if you start with the basic. Okay so we stop here today and we will continue in the next lecture thank you very much.