Space Flight Mechanics
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Module No # 16
Lecture No # 83
Trajectory Transfer (Contd.)

Ok welcome to lecture 83. So we have discussed last time about the trajectory transfer and in that context we looked at that the minimum eccentricity transfer is the Hohmann transfer. And also, not always the Hohmann transfer is the minimum energy transfer but most of the time we will see that it is the minimum energy transfer. I will write the context in which the Hohmann transfer happens to be the minimum energy transfer.

Later on also I am going to work on that when there is one more transfer we call it as bi elliptic transfer. So it is the combination of 2 Hohmann transfer. So using that also the transfer can be done but whether the bi elliptic will be the better or the Hohmann transfer will be better it depends on the intermediate orbit that is used for transferring the satellite on initial orbit to the final orbit. So that will come over course of time. So let us first start this first Hohmann transfer and finish it and then we will come to come over to that point.

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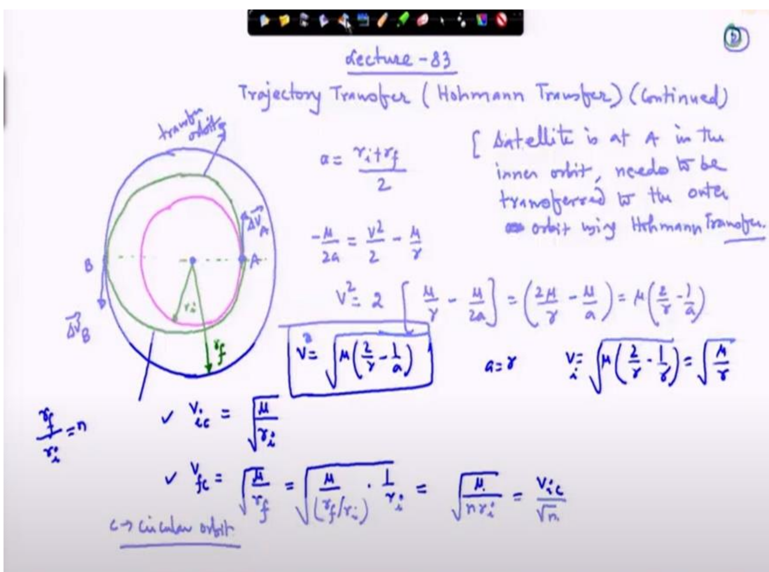

So we can see from this place that the semi major axis of the transfer orbit which is shown in green on the actually the satellite is here in this place. There is a center of attraction and from here let us say this is point A and this is point B. So it is been fade here in this direction the impulse Δv_A is given here and then impulse Δv_B is given here to achieve the. So what is required that the statement is satellite A in the inner orbit needs to be transferred to the outer orbit using Hohmann transfer.

So Hohmann transfer in general in the normal context if we look the outer boundary space so Hohmann transfer will be the minimum is the transfer in which minimum Δ which we required (Refer time: 04:15). The total Δ be the amount of the propellant unit to burn is the least one. So satellite is at A impulse Δv_A is given here and then again Δv_B impulse is given at B. So by giving impulse at A we are sending it into the transfer orbit. So here we can show this to be the transfer orbit this is the transfer orbit. Thus the working out is very easy as we have done earlier the specific energy is μ- μ/ 2a = v^2/ 2 - μ/ r.

$$-\frac{\mu}{2a} = \frac{V^2}{2} - \frac{\mu}{r}$$
$$V^2 = 2\left[\frac{\mu}{r} - \frac{\mu}{2a}\right] = \left(\frac{2\mu}{r} - \frac{\mu}{a}\right)$$

So from here we see v^2/ 2 times μ / r - μ/ 2a = 2 μ/ r - μ/ a. And μ either we can keep it outside μ2 / r - 1 / a. So v is available towards this way. So therefore in the initial orbit v will be given by μ/ r_i under root. Immediately you can see that for the circular orbit a = r and therefore v = v_i = this is the square so we will put it into the form of square root. So v_i = μ times 2 / r - 1 / r under root and this gives the result μ/ r under root.

$$V = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$
$$v_{ic} = \sqrt{\frac{\mu}{r_i}}$$
$$v_{fc} = \sqrt{\frac{\mu}{r_f}} = \sqrt{\frac{\mu}{nr_i}} = \frac{v_{ic}}{\sqrt{n}}$$

So therefore in the initial circular orbit we can put here c to indicate that this is in the circular orbit μ/ r_i. Similarly v_{fc} this in the final orbit the velocity will be given by μ/ r_f. And little bit of change we can do here to write it this way r_f / r_i 1 / r_i and r_f / r_i we write as n. This gives you μ/ n r_i under root. This is nothing but v_{ic} / under root n ok. So we have written the velocity in the initial and final circular orbit there c is stands for circular orbit. Ok thereafter what will be the velocity in the elliptical orbit at a.

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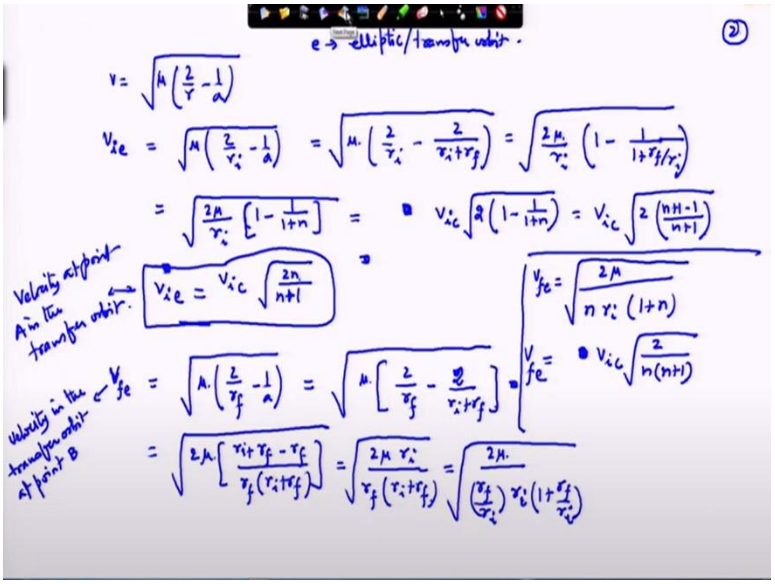

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$v_{ie} = \sqrt{\mu\left(\frac{2}{r_i} - \frac{1}{a}\right)} = \sqrt{\mu\left(\frac{2}{r_i} - \frac{2}{r_i + r_f}\right)} = v_{ic}\sqrt{2(1 - \frac{1}{1+n})}$$

We utilize this formula what we have derived here v = μ times 2 / r - 1 / a so v in the transfer orbit or in the elliptical orbit at e. So for simplicity let us write this as v_{ie} in the initial orbit point e we just write it as v_{ie} in that e indicates elliptical orbit e stands for elliptic / transfer orbit. And this can be simplified as 2 / r_i - e = r_i + r_f / 2 where 2 comes in the numerator. 2 μ/ r_i we can take it out rate this equals into 1 / 1 + r_f / r_i.

$$= v_{ic}\sqrt{2\left(\frac{n+1-1}{n+1}\right)}$$

$$v_{ie} = v_{ic}\sqrt{\frac{2n}{n+1}}$$

$$v_{fe} = \sqrt{\mu\left(\frac{2}{r_f} - \frac{1}{a}\right)}$$

So therefore, this gives you and this under root. 2 μ/ r_i 1 - 1 / 1 + n. 2 times μ/ r_i is μ/ r_i is v_{ic} so that we take it out and then we are left with 2 times 1 - 1 / 1 + n under root this can be further simplified as n + 1 - 1 / n + 1. This is the velocity at point A in the transfer orbit similarly v_{fc} which is the velocity in the transfer orbit at point B. This we can write as v fc and using the same equation you can write here A does not change for the transfer orbit and μ/ 2 / r_f - 2 divided by.

$$= \sqrt{\mu\left(\frac{2}{r_f} - \frac{2}{r_i+r_f}\right)}$$

$$= \sqrt{\frac{2\mu r_i}{r_f(r_i+r_f)}} = \sqrt{\frac{2\mu}{\left(\frac{r_f}{r_i}\right) r_i (1+\frac{r_f}{r_i})}}$$

We have got here in this format now we divide both the numerator denominator for r_i this gets this and this formats 2 μ_{rf} / r_i times and r_i it we take common from the bracket this gets reduced to r_f by r_i ok. Therefore, v_{fc} this gets reduced to 2 μ/ n times r_i times 1 + n under root and μ/ r_i is nothing but our v_{ic} we are left with 2 / n times n + 1 under root this is your v_{fc}. So now we know the velocity in the both the places ok.

$$v_{fe} = v_{ic}\sqrt{\frac{2}{n(n+1)}}$$

So we can utilize this result these 2 we have got here which we have written here in this way to find out how much the extra how much impulse is required at A and B.

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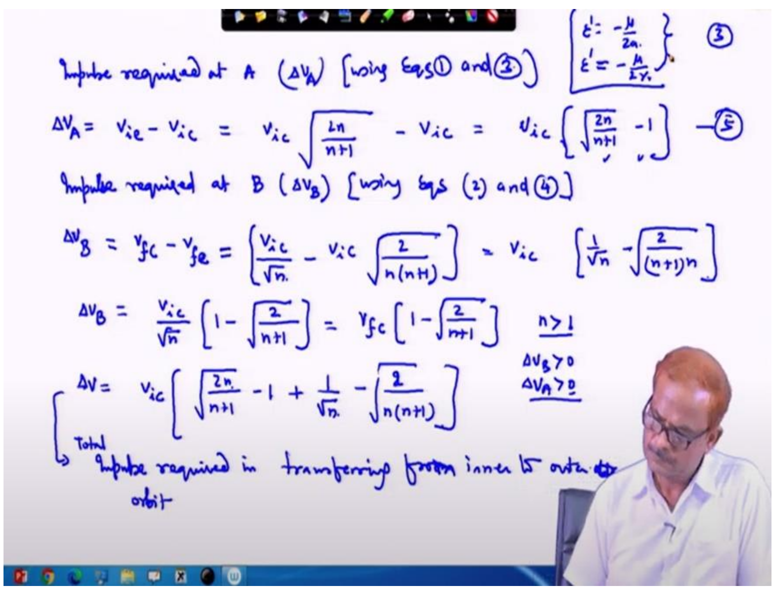

$$\Delta v_A = v_{ie} - v_{ic} = v_{ic}\sqrt{\frac{2n}{n+1}} - v_{ic} = v_{ic}\left\{\sqrt{\frac{2n}{n+1}} - 1\right\}$$

So impulse required at A we write as Δv_A. So Δv_A will be given by v_{ie} which we have derived here let us name this as this we name as 1, 2 then 3 this we name as 4 this one is the 4. So using 1 and 3 using equation 1 and 3 Δv_A = this is a v_{ic} v_{ie} will write this $v_{ie} - v_{ic}$ = v_{ic} under root 2n / n + 1 under root - v_{ic}. We can take it outside common this is 2n / n + 1 - 1. So this is the impulse required the initial point using equations 2 and 4.

$$E' = -\frac{\mu}{2a}$$

$$E' = -\frac{\mu}{2r}$$

$V_{fc} - v_{fe}$ and v_{fc} already we have seen this quantity is $v_{ic} / \sqrt{n}$ that means the inner orbit the velocity is higher because n is greater than 1. So in the inner orbit velocity is higher than the outer orbit the outer circular orbit velocity is lesser. But energy wise the outer orbit is having more energy. This should be remember if $E' = -\mu / 2a$. So here in this case if this is circular orbit you can just replace by 2r.

So greater the value of r the more positive this will proceed towards 0. As r tends to infinity $E' = 0$. Otherwise you can see that the larger the value of r the more energy the orbit will have as compare to a smaller orbit. But in the smaller orbit our vital velocity is high – v_{fc} v_{ic} times under root 2 / n times n + 1. And the same thing we can also write this way if we take v_{ic} under root n as common so this will appear like this and this is v_{fc} times 1 - 2 / n + 1 under root.

$$\Delta v_B = v_{fc} - v_{fe} = \left\{\frac{v_{ic}}{\sqrt{n}} - v_{ic}\sqrt{\frac{2}{n(n+1)}}\right\} = v_{ic}\left[\frac{1}{\sqrt{n}} - \sqrt{\frac{2}{(n+1)n}}\right]$$

Now this is the impulse required at point B. And you can notice that this quantity is positive because n is greater than 1 so Δv_B is greater than 0 and also Δv_A is greater than 0. Here also you can see this quantity is greater than 0. The way we have written okay sometimes the impulse is required opposite to the velocity vector ok. So, we but always we can calculate or even when it comes to negative you have to take this magnitude and work out.

$$\Delta v = v_{ic}\left[\sqrt{\frac{2n}{n+1}} - 1 + \frac{1}{\sqrt{n}} - \sqrt{\frac{2}{n(n+1)}}\right]$$

And therefore Δv the total impulse required is v_{ic} and it can be taken as common 2n / n + 1 under root - 1 + 1 / root n 2 / n times n + 1 ok. So this is the impulse required total impulse required in transferring from inner to outer orbit. The same way it can also be done for the inner to the outer to the inner orbit. There is no difference in the working out mechanism. We have to write it in the same way.

And this particular thing whatever we have worked out we could have gone by using this energy approach. And through that root we will find out then cent result will come but it takes later on you have to solve the quadratic equation and then you will get the result only. So the process is the same but that much longer so I will give you write up about the other way of doing the same problem. Here I am not going to discuss this because you do not get anything new in that particular way of working.

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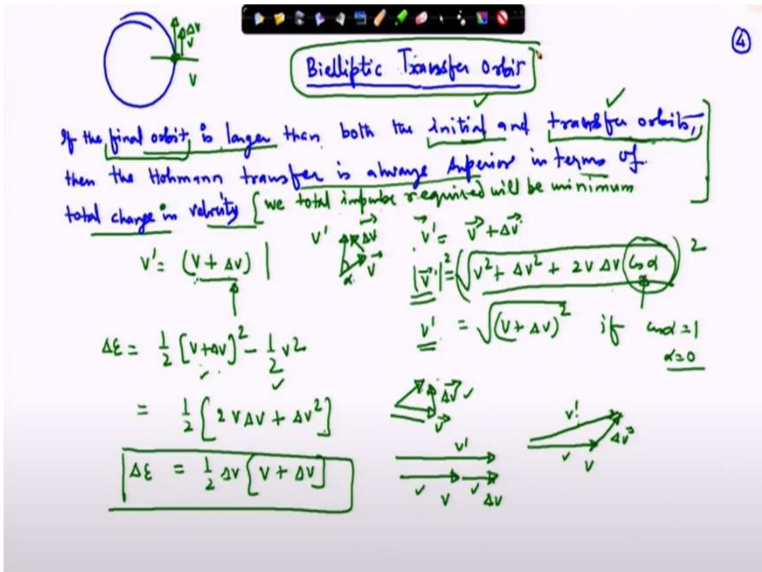

Next, we will look into the bi-elliptic orbit this is bi-elliptic transfer object. It is the combination of 2 Hohmann transfer so one result I will write it here. If the final orbit is larger than orbit is larger means it is the radius it is larger than both, the initial and transfer orbit, then the Hohmann transfer is always superior in terms of total change in velocity required. Ok in that case you require the least impulsion. What is the reason? That I have already explained you if you are starting from this place inner orbit and then applying Δv impulse here in this point.

$$v' = v + \Delta v$$

So the always transfer in the total change in velocity not the impulse referred. See what I want to state here that v + Δv. This is the if both had the velocity v and the Δv both had in the same direction ok so this is will be your change velocity. And there after if you have both the velocities not in the same direction this is your v and this is your Δv. And this angle let us say this angle is alpha this is your v′.

$$|\vec{v}'| = \sqrt{v^2 + \Delta v + 2v\Delta v cos\,\alpha}$$

So v' as we are looked into this will be given by v + Δv. And this will be equal to magnitude by v' magnitude will be v^2+ Δv^2+ 2v times Δv Cos α under root. Only when Cos α = 1 you will get this as v + Δv^2 under root if Cos α = 1 and α = 0. In that case only you will get change velocity to be the maximum okay. So other way I can state that if the tangential impulse is required so the change in velocity maximum.

$$v' = \sqrt{(v + \Delta v)^2}$$
$$\Delta E = \frac{1}{2}\ [\ v + \Delta v]^2 - \frac{1}{2}\ v^2$$

And energy change that depends on already I have dictated $(v + \Delta v)^2$ minus this is ΔE change in the energy because you are applying at the same point 1 / v^2 this is the final energy initial energy. So v^2 cancels out we get here 2v times $(v + \Delta v)^2$. So you get the energy there is a change in the energy this way. But if you have instead of v + Δv if we write this quantity because this is then the magnitude if we take the square on both side.

$$\Delta E = \frac{1}{2}\ \Delta v[v + \Delta v]$$

If we put the square here and this also v^2 immediately you can see the Cos α fact will come here okay. So the change in energy will not be the maximum. Here in this case you are getting the maximum possible change in energy. So therefore if the orbit final orbit is larger than, both the initial and transfer orbit Hohmann transfer is always superior in terms of total change in velocity. And also simultaneously I would like to state here that this part we have to remember that the transfer orbit is greater than the transfer orbit.

If the final orbit is large than both the initial and transfer orbit then only we will also have total impulse required will be minimum as we can see from this place because of this if the 2 vectors are incline v is here in this direction and Δv is here in this direction. So we are not going to get proper change in to achieve the same change in velocity. Here if this 2 vector we have planned v and Δv so to achieve the same velocity we require more energy or the more value of Δv.

What does this mean? That always this should apply if v is the velocity. So we should if we apply Δv in the same direction because we get the change velocity v' like this. But if it is v there is in direction Δv is in this direction so we are not going to in spite of both being of the same magnitude this v' will be lesser in length. So this is not efficient use of energy. So here in this case you if you are doing the Hohmann transfer here not only the minimum eccentricity transfer as we have proved earlier.

But this also requires lowest amount of energy provided the initial and the transfer of which are both are smaller than the final orbit. If this condition is satisfied this will be done otherwise later on we will look in this bi-elliptic transfer orbit how the whole thing how the bi-elliptic transfer is some cases it will be greater than the Hohmann transfer for that the certain conditions should be satisfied. And this we are going to explore in the next lecture thank you.