Space Flight Mechanics
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Module No # 16
Lecture No # 85
Trajectory Transfer (Contd.)

Welcome to the lecture 85 so we have discussed last time about the Hohmann transfer and the bi-elliptic transfer. Now we look into when the in the Hohmann transfer the maximum impulse will be required under what condition. So this we are going to derive but I will not do the complete derivation I will give you the idea and rest of the materials I will provide as the written material type material for this particular part ok.

And there after we will look when the bi-elliptical transfer and the Hohmann transfer there will be equivalent. That means in both of them the same energy will be required. And then I will show you graphs for graphs I will have to download the material and show you here okay. So for that it will take little time because I do not want to interfere by presenting it here. So let me see how to proceed. So we will start here ok so for our convenience I have just written from the previous lecture.

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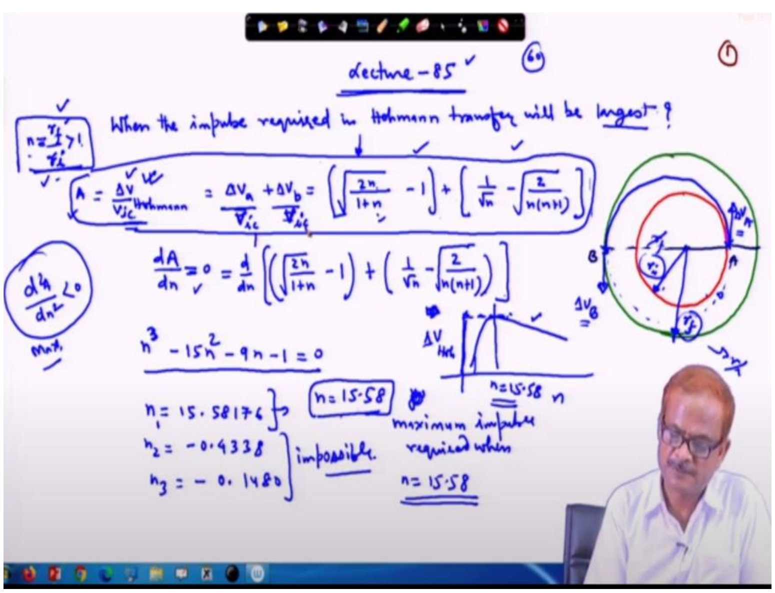

$$\Delta v_{Hohmann} = \Delta v_a + \Delta v_b = \left[\sqrt{\frac{2n}{1+n}} - 1\right] + \left[\frac{1}{\sqrt{n}} - \sqrt{\frac{2}{n(n+1)}}\right]$$

This is the total impulse required in the Hohmann transfer. So at A the delta v A is required which I have written here. And at this delta v B required which I have written here ok and what is n?

$$n = \frac{r_f}{r_i} > 1$$

n is a quantity r f / r i where i is the inner orbit radius and f is the outer orbit radius here in this case and this quantity is greater than 1 for this particular. It may happen that you are going from the inner to the outer to the inner orbit.

That means we are going at the opposite way from here that means here we are starting and going to this point. This also either going from inner to outer, outer to inner. And depending on the situation if we are going from the outer to the inner orbit at that time r f becomes your the initial orbit. So instead of writing this as rf then this will be replaced by r i ok and this will be replaced by rf. So this is the thing that we need to do ok. But n always indicates r f by r i in our derivation this we have to maintain ok.

And this ratio will change only once rf is smaller and ri is larger that means n will be less than 1 ok. So for irrespective of that this results whatever we have derived it remains valid it does not depend on that. So somewhere you have to reduce the velocity so in that case what we will do then if it is not positive and at that time you take the magnitude of this ok. So that will give you the proper result or either you work out from the scratch and then do your calculation.

So that delta v and delta v B appears to be positive in all those places ok. Now starting with this I would like to write this quantity as A. And this quantity when this will be, maximum so for this its required that this quantity should be for extremum value this quantity should be 0 because this is only if A is a function of n here. And once we differentiate here this we get the result. So we need to differentiate this quantity on the right hand side.

$$\frac{dA}{dn} = 0 = \frac{d}{dn}\left[\left(\sqrt{\frac{2n}{1+n}} - 1\right) + \left(\frac{1}{\sqrt{n}} - \sqrt{\frac{2}{n(n+1)}}\right)\right]$$

So this exercise I am not going to do because I am going to give the material ok. It is a common differentiation you have to do here and after that what you get as the final result that I am going to write in this place because we do not have that much of time the lecture already we have the eighty fifth lecture is going. While we only 60 lectures we are schedule. So 25 lectures already we have exceeded ok.

So if you differentiate re arrange and all those things which I am going to give you the written material. So this gives you n cube -15 n square - 9 n - 1 this = 0. And if you solve so n you will get the value n = 15.58176 and then -0.4338 and -0.1480. So let us write this as n1, this is as n2, this as n3. So these are two are impossible why because n cannot be this is a ratio of 2 positive quantities so it cannot be negative.

$$n^3 - 15n^2 - 9n - 1 = 0$$

$$n_1 = 15.58176$$
$$n_2 = -0.4338$$
$$n_3 = -0.1480$$

So this is yours solution. So for n = 15.58 around for this value that means the ratio of the outer orbit and the inner orbit when this is 15.58 and then again this has to be checked. See this is only the extremum check whether this is the maximum that needs to be check by deriving d A square / d n square and for maximum this quantity should be turn out to be less than 0 for this is for maximum ok.

$$\frac{d^2A}{d^2n} < 0 \quad for\ maximum$$

So with this we get the extremum or the maximum impulse required when n = 15.58. Rest other will be less than corresponding to this value. So delta v if we plot it and then n we will plot here in this place. And we will see that this graph looks like this I will show you describe later on it goes like this. Somewhere you get here you the maxima which corresponds to n = 15.58. And how do we get this curve? We get this curve by plotting this delta v as we have written here.

This whole thing has to be coded and giving different values of n calculating this delta v here delta v Hohmann transfer and we need to plot for different values of n. So you change from n = 1 to say 100 or 60 whatever and plot it. So you will get this result. And this maximum then it is taking place here in this place which corresponds to 15 point this is 15.58 we have written here. And one thing here we have missed out which is v ic this is v ic is missing here that we need to write.

So on this is side we should write here rather v ic for which is the initial velocity delta v A / v i delta v b / v i. We derived it from the circular orbit and that is why I have given this notation this c as stated earlier also.

$$A = \frac{\Delta v_{Hohmann}}{v_{ic}} = \frac{\Delta v_a}{v_{ic}} + \frac{\Delta v_b}{v_{ic}} = \left[\sqrt{\frac{2n}{1+n}} - 1\right] + \left[\frac{1}{\sqrt{n}} - \sqrt{\frac{2}{n(n+1)}}\right]$$

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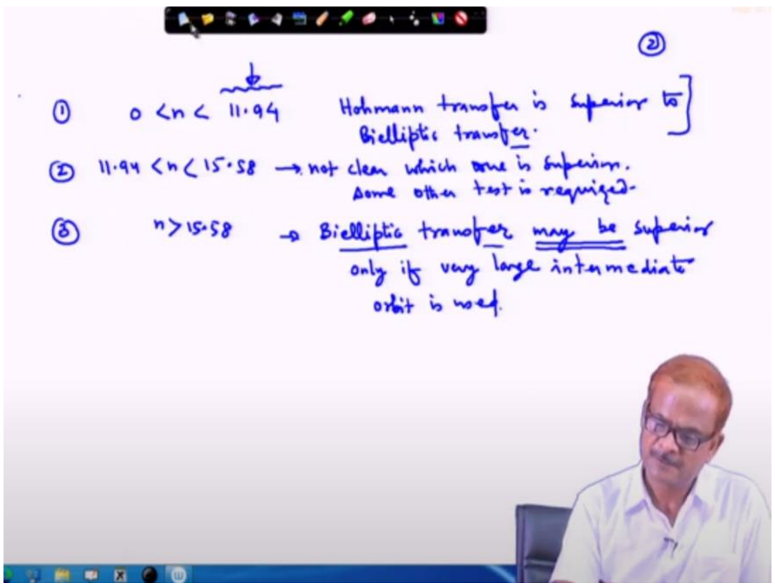

So to state for n lying between 0 and 11.94 Hohmann transfer is superior to bi-elliptic transfer. You may be wondering from where we are getting this quantity shortly we will see that. For n lying between 11.94 and between 15 point what we have derived here 15.58 let say not clear which one is superior some other test in required. For n greater than 15.58 bi-elliptic transfer may be superior this is may be ok this is what is important may be superior only if very large intermediate orbit is used.

So in that condition we get the bi-elliptic transfer to be betttter. And this is the reason I have earlier stated whether the Hohmann transfer is always superior to other transfer it is known ok. But in certain range Hohmann transfer is superior in terms of the impulse required which is definitely the representation of energy how much energy you are spending in terms of propellant okay. And next we look for when this Hohmann transfer and the bi-elliptic transfer they will match they will coincide for what value of n?

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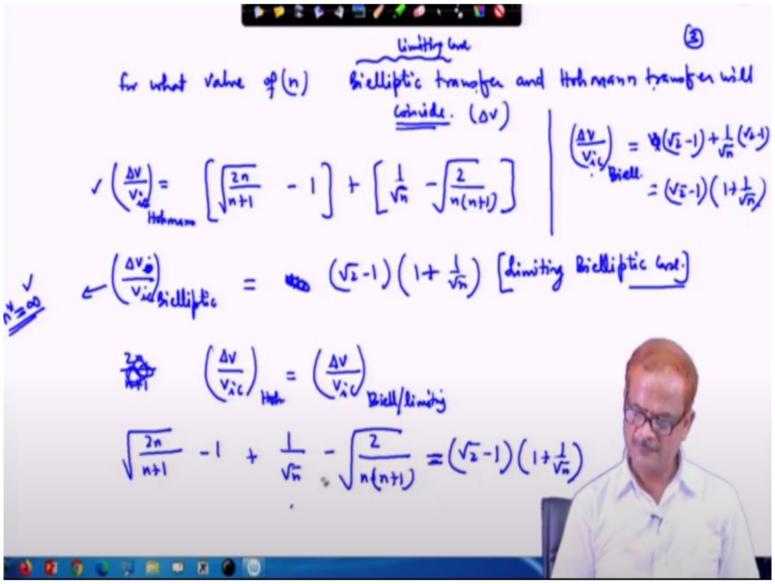

So the next question is what value of n in terms of delta v required in both the cases will be the same. This is what we are looking for. So already we have written delta v / v ic in the Hohmann transfer this is Hohmann and this part we have written as from the previous thing where 2n / 1 + n 2n / n+1 under root -1 and then + 1 / n under root -1 -2 / n times n + 1. n+1 under root. So this is in the Hohmann transfer. In the bi-elliptic transfer we have got very complex result ok.

$$\left(\frac{\Delta v_{ic}}{v_{ic}}\right)_{Hohmann} = \left[\sqrt{\frac{2n}{1+n}} - 1\right] + \left[\frac{1}{\sqrt{n}} - \sqrt{\frac{2}{n(n+1)}}\right]$$

$$n^* = \infty \leftarrow \left(\frac{\Delta v_{ic}}{v_{ic}}\right)_{Bielliptic} = \left(\sqrt{2}-1\right) + \frac{1}{\sqrt{n}}\left(\sqrt{2}-1\right)$$

$$= \left(\sqrt{2}-1\right)\left(1 + \frac{1}{\sqrt{n}}\right)$$

$$\left(\frac{\Delta v_{ic}}{v_{ic}}\right)_{Bielliptic} = \left(\sqrt{2}-1\right)\left(1 + \frac{1}{\sqrt{n}}\right) \quad \{Limiting\ Bielliptic\ case\}$$

It long equation we have got but if we choose here for this one n star to be infinity that means the intermediate radius intermediate transfer orbit is lying at infinity. So the case gets simplified. And that is what we have got delta these are all v ic so we have got in terms of I will write it here for the delta v / v ic in case of bi-elliptic we wrote it as vic. So v ic I am transferring here root 2 - 1. And then 1 / root n times root 2 - 1 this is what we have got.

So this thing we can bring here and work it out. That means this is root 2 - 1 times 1 + 1 / root n. So let us write this root 2 - 1 times 1 + root n this is here in bracket we write limiting bi-elliptic case. So limiting bi-elliptic case and the Hohmann transfer case we want to compare this. And when this two will be equal. So when bi-elliptic transfer and Hohmann transfer bi-elliptic transfer limiting case. When this two will cross each other, they will have a common solution.

$$\left(\frac{\Delta v_{ic}}{v_{ic}}\right)_{Hohmann} = \left(\frac{\Delta v_{ic}}{v_{ic}}\right)_{Bielliptic/limiting}$$

If we look for this then we will need to compute this quantity. At first let us write this as delta v / v ic Hohmann this equal to delta v / v ic bi elliptic with limiting case slash limiting. And doing this means we have to solve this equation 1 / root n - 2 divided by n times n + 1 under root = root 2 - 1 1 + 1 / root n this we need to solve. So again the solution of this I will be providing you the same time not to extend too many lectures.

$$Solve\left[\sqrt{\frac{2n}{n+1}}-1+\frac{1}{\sqrt{n}}-\sqrt{\frac{2}{n(n+1)}}=\left(\sqrt{2}-1\right)\left(1+\frac{1}{\sqrt{n}}\right)\right]$$

I am just escaping some of the mathematical details which you can do yourself because it is very simple calculus twelfth standard calculus involved and there is no other complexity only the equation you need to solve. There is no concept the physical concept is involved after this. And therefore I am avoiding doing anything at this stage. So this we need to solve.

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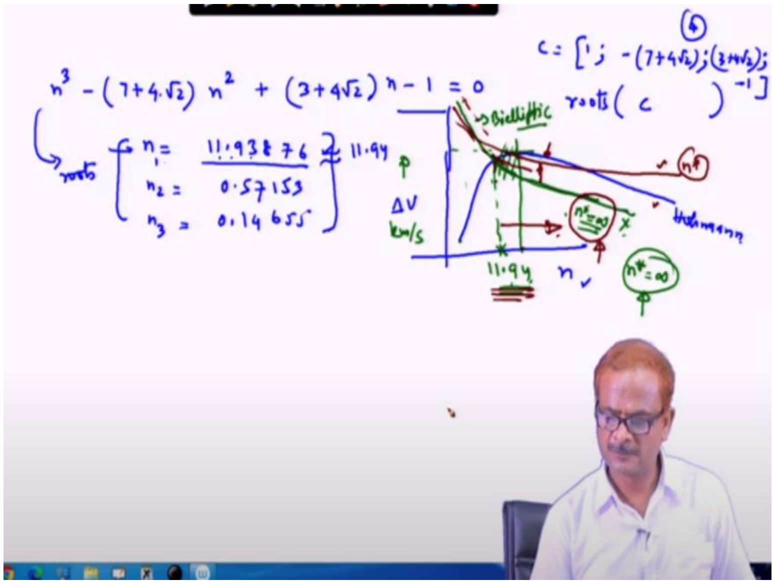
$$n^3-\left(7+4\sqrt{2}\right)n^2 + \left(3+4\sqrt{2}\right)n-1=0$$

$$n_1 = 11.93876$$
$$n_2 = 0.57153$$
$$n_3 = 0.14655$$



And once we solve this we get the equation in this format n cube this is a polynomial of degree 3 (18:01). And if we solve this we get n = 11.93. So you can see that 11.94 we have written here. So the same thing is appearing here 11.93876 and this we will write as n1. So these are the roots of this polynomial and n2 will be 0.57153. And you can do it on the matlab if you use this roots define c as 1 if you have in mat lab you can do it -7 + 4 root 2 and 3 + 4 root 2 these are semi colons and then -1 and then bracket close.

$$c = [1;\ -(7+4\sqrt{2})\ ;\ \ (3+4\sqrt{2});\ -1]$$

$$roots(c\ \ )$$

And then here if you write c so it will list you all this polynomial root of this polynomial. So this implies that this two methods the bi-elliptic and the Hohmann they cross each other at the point n1 = 11. 93876 means approximately this is 11.94 as I have written earlier. So if we write here delta v n on this axis. So this is for the Hohmann transfer which is only dependent on n. But here in in this case for the bi-elliptic transfer it is dependent on n star also.

So n star here in this bi-elliptic case we have already chosen. So this is limiting case we are writing here $n^* = \infty$ ok. So for this it is defined. So that case this curve will come like this ok. So you can see that where ever it cuts that point will be 11.94 so this is for bi-elliptic. So from this figure you can see that delta v which is in kilometer per second and n is a non-dimensional quantity.

So delta v here in this case of bi-elliptic it is a lesser and as we go it achieves a maximum here the maximum amount of impulse required for transfer and there after again reduces. But the bi elliptic transfer it goes like this. So if the intermediate orbit is very very large or it is a infinite which is not practical in that case ok. So here in this case this n star this equal to infinity. So in that case your Hohmann transfer will not be that superior as compared to the bi-elliptical transfer.

After this 11.94 and between this range ok. So in this range which will be greater this you need to explore. But here we can see that this is coming down so that means for n star = infinity, the bi-elliptic is superior after this. But what about the other n star, n star = infinity does not have any meaning. This is useless case because you cannot send your satellite to infinity and then from there this will come back. So this case is of no use.

So what ever is of use so those cases we need to explore so for the other cases it may appear something like this. It goes like this it may go like this and there after it crosses like this for other n star values ok. So then you can see that the Hohmann transfer here is better bi-elliptic is not better. But here bi-elliptic is better Hohmann transfer is lying up. So it depends on the value of n and n star which is going to be better.

But for n start equal to infinity which is the theoretically interesting your bi-elliptic transfer turns out to be better that the Hohmann transfer but not for all values. Only after 11.94 only after this value, this is going to be theoretically interesting but not of any practical interest ok. So, for that I have a figure here and for this I want to show you as the slide okay.

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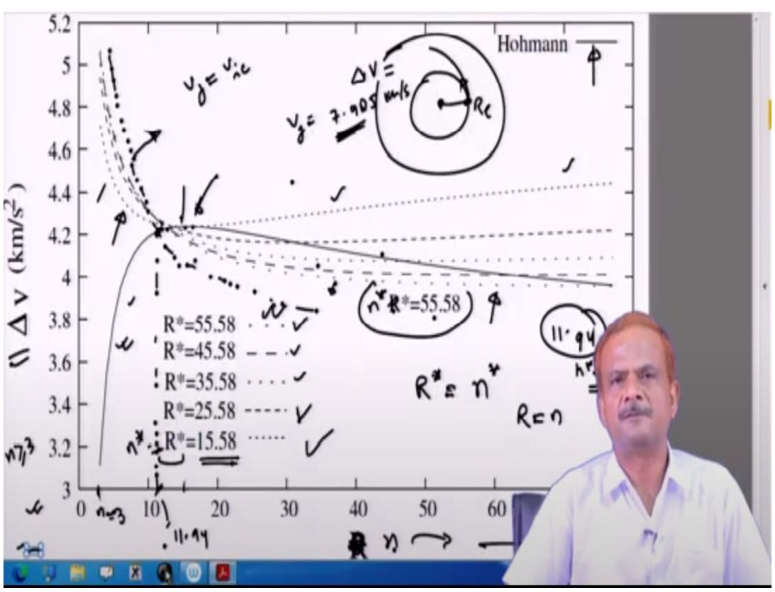

Now let us now let us look into the graph which is shown here I have done this on computer. So as visual from the graph this here it is written in terms of R star. So R star is here nothing but in our case we have written this as n star ($R^* = n^*$). R which was written here R is nothing but, in our case this equal to n ($R = n$). The different symbol is used delta v is on the left hand side plotted and n is plotted here on this side.

This is you're the solid line as it is shown here the solid line is stands for Hohmann transfer. So this is for Hohmann transfer ok. And dotted lines are for the bi-elliptic transfer. Now out of that, first I have done for 15. 58. So 15.58 what is that value that is corresponding to the value where the Hohmann transfer is the getting the maximum okay. So sorry n = 15. 58 somewhere it will lie here ok 15.58 on the right hand side little bit.

So it goes from this place somewhere here you get the maximum value for a Hohmann transfer if you look here on this side. How delta v we have calculated ? This delta v is actually this is delta v was calculated for a grazing orbit. Grazing orbit what does it means? The grazing orbit mean this is the radius of earth Re ok and for that it is a hypothetical orbit so this is the radius of the earth and the satellite is moving just in this. This is called the grazing orbit.

That means it is moving on the surface of the earth. Which is obviously assuming that earth is total mass is concentrated just at the center and the surface is not present. So this is we call as grazing orbit. And for grazing orbit somewhere I have noted this value for v. So v turns out to be the v in the grazing orbit which we write as vg this is around 7.91 or 905 kilometer per second ($v_g = 7.905\ km/s$). This is the velocity of the satellite ok. So in context of this everything is defined. So from this orbit you want to go to the higher orbit ok.

All your orbits are outside this ok. And from here then we are starting. So with respect to this, this relationship is derived and the plots are made here. And here the plots has been done for n greater than 3 n3 onwards greater than equal to 3. Here whatever you see here so it has start from n=3. So the plot is done like this so corresponding delta v how much is required? So total delta v is calculated in terms of this grazing orbit requirement where vg in this case this will turn out to be v ic ($v_g = v_{ic}$). as we have written earlier.

So this solid line then stands for the Hohmann transfer and the dotted lines are for bi-elliptic transfer. In the bi-elliptic transfer R star is nothing but your n star ok. So, n star as you can see 15.58 if we take for the highest value. So this is the curve which is passing from this point itself it is near to this ok. If I take higher value for the n star we have remember the one where we got the intersection point that we have obtained as 11.94.

This we did for n star equal to infinity. But I cannot here workout for n star equal to infinity or I could have done this but I have not done here in this particular graph. The relationship already we have written so this could have been done using that. So anyway I have not done it does not matter. But you can see that there is a movement this is for n star = 15.58 this is a first curve. And thereafter as you increase the n star value by 10 so here this is increased by 10, next again by 10, next again by 10 and next again by 10.

So as I increase the value of n star this curve shift here from this direction to this direction it goes here okay there is a shifting. And the curve also it comes here in this direction. You can see that this the n for higher n star value here n star = 55.58 is the lower dotted one which is this one. And somewhere for 11.94 where the intersection takes place which the curve I have not shown so that curve let us say that it comes something like this 11.94 it will come down and it will intersect here in this point ok.

So that only for that 11.9 n star = infinity, we have solved. And the corresponding intersection point will come here and which you drop it on the x axis. So this is your correspondingly 11.94. So the intersection then comes here in this place. So what is our conclusion that then for n star = infinity this will go down like this that means as we are increasing, the value of n star the energy required is lesser than the Hohmann transfer which is shown by the solid line ok after certain stage. So here in this case after 11.94 n = 11.94 Hohmann transfer is better if n star = 55.58.

But if n star = infinity. But if n star is having other values then this point will shift. Thus the intersection point will be different ok. As it is appearing from this graph here. And that we need to work out. So that cannot be turn in a single equation. Equation becomes very clumsy doing it numerically it is very easy. The intersection point you just plot 2 curves and then look into for what the corresponding intersection point is it is a easy to workout that. We stop here and next time we will continue thank you very much.