Space Flight Mechanics
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Lecture – 89
Interplanetary Transfer (Contd.,)

Welcome to lecture 89 we have done one problem related to interplanetary is transferred. So you might have got an idea of what is happening. Now we will put whatever the processing we have done in terms of mathematics.

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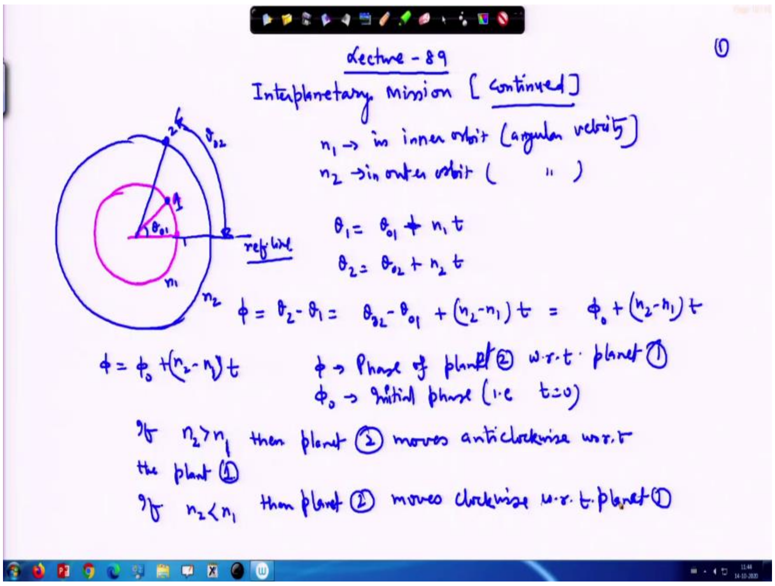

$$\theta_1 = \theta_{o1} + n_1 t$$

$$\theta_2 = \theta_{o2} + n_2\, t$$

So let us start with; there are 2 planets are shown here this is one and this is the reference line we take reference line. So, the planet 1 and 2 they are in the inner and outer orbit. And let us say n_1 is angular velocity and this and n_2 is the angular velocity in outer one. So, n_1 in inner orbit angular velocity and n_2 in the outer orbit in outer orbit; angular velocity. So, with the reference line the angle for these two planets it can be written as $n_1 t$ and θ_2 is equal to θ_{o2} + $n_2 t$.

$$\phi = \theta_2 - \theta_1 = \theta_{o2} - \theta_{o1} + (n_2 - n_1)t$$

$$\phi = \phi_o + (n_2 - n_1)t$$

So these are the position which is very simple. So θ_2 – θ_1 therefore can be written as θ_{o2} - θ_{o1} + $(n_2 - n_1)t$ which we can write as plus into minus and 13 and on the different side. We can write the search file. So ϕ_0 equal to $(n_2 - n_1)t$ on the left hand side you can write this as ϕ. So ϕ is equal to $\phi_o + (n_2 - n_1)t$, ϕ is called phase angle of planet 2 with respect to planet 1.

ϕ_0 is initial phase that is at t is equal to 0. If n_2 is greater than n_1 which will happen only if n_2 is lying in the inner orbit and n_1 is lying in the outer orbit. So n_2 is greater than n_1 then planet 2 moves anticlockwise with respect to the planet 1. If n_2 is less than n_1 then planet 2 appears to move clockwise with respect to planet 1.

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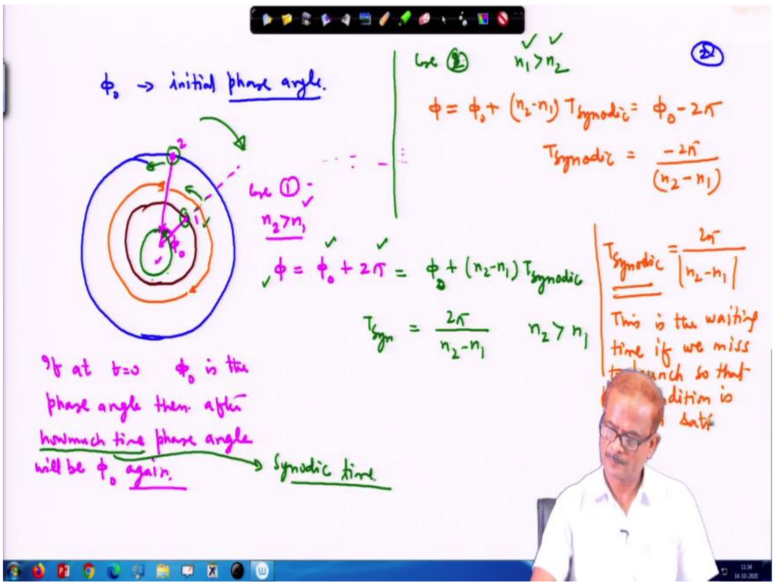

So ϕ_o this is the initial phase angle. So the questions can be asked now let us take this figure again here in this case. This was a position of 1 and this was position of 2 this angle is your ϕ_o the difference between them. The question arises when again this phase difference between these two because they are moving the phase difference will change when again the same phase difference will appear.

So this is the question. That means ϕ + 2 π equal to ϕ_0 then this will occur. I will write it in the language if at t is equal to ϕ_o is the phase angle then after how much time phase angle will be ϕ_o again. So here ϕ_o is indicating the phase of two with respect to 1 initially. So let us consider this in two cases say the n_2 is greater than n_1. Case 1 this I will remove rather than writing like this I will write in terms of equations.

So, case 1 n_2 is greater than 1 so therefore ϕ will be equal to ϕ_o + 2π what does this mean? See here this is the phase of 2 n_2 it is suppose n_2 is faster than the n_1 greater than n_1. So then the same phase will come for this required that with respect to this with respect to 1 obviously n_1

is also moving. So while it moves in the orbits of with respect to 1 it has to go all the way from; 2 is here it has to go all the way from here and again go back to this place so gain 2π angle it has to cover.

So that means the phase angle of 2 with respect to 1 this becomes $\phi_o + 2\pi$ at that time. This is moving, this one is also moving 2 is also moving here in this direction. But 2 is moving faster of course, which is not the case, once 2 is inside only that will happen, but let us suppose that it is written like this. So, starting from this place and going back to this place. Total phase angle with respect to 1 then becomes equal to ϕ_o and $+ 2\pi$ and this must be equal to ϕ_o + into $-n_1$ times t.

And the period after which after how much time this period this is called the synodic time same as the moving reference frame rotating reference time, we have done in the restricted three body problem. So this phase is synodic and therefore from this place what we can see this case synodic, this will be equal to 2π $n_2 - n_1$ provided n_2 is greater than n_1. Of course, this can be case b note 2, n_1 greater than n_2. If n_1 is greater than n_2 so 2 will appear to move in the backward direction.

$$\phi = \phi_o + 2\pi = \phi_o + (n_2 - n_1)T_{synodic}$$

$$T_{syn} = \frac{2\pi}{n_2 - n_1}, \quad n_2 > n_1$$

If n_1 is greater than n_2 so with respect to 1, 2 will appear to move in the clockwise direction like this. So therefore, the angle to be covered in that cases it has to go from; 2 has to start from this place. Ok it will be going on in the back direction. So here in this case this will be ϕ equal to $\phi_0 + n_2 - n_1$ times T synodic. This will equal to ϕ_0 filter starts from this place, it goes all the way here in the negative direction and then comes back.

At that time the phase angle this becomes $\phi_0 - 2\pi$ this is in the opposite direction this 2π it comes to the same page again with respect to 1. Ok and once you solve this solve this so that is synodic this will be -2π divided by $n_2 - n_1$. So, both can be combined together and it is synodic can be written as 2π divided by $n_2 - n_1$ magnitude. And this is very important because this tells you as you have seen that before launching certain condition should be satisfied.

$$T_{syn} = \frac{-2\pi}{n_2 - n_1}, \quad n_1 > n_2$$

The phase angle between the planet A and planet B it should have a particular value so that if you are launching the satellite from planet A to planet B so your satellite goes through 180° of Hohmann transfer. This condition has to be satisfied. So, this particular s the phase angle differences between planet A and Planet B then it will occur again. So, once you missed that time then you have to wait for how much time you have to wait? It is given by the synodic time.

This is the waiting time if we miss to launch so that phase angle condition this condition is matched is satisfied.

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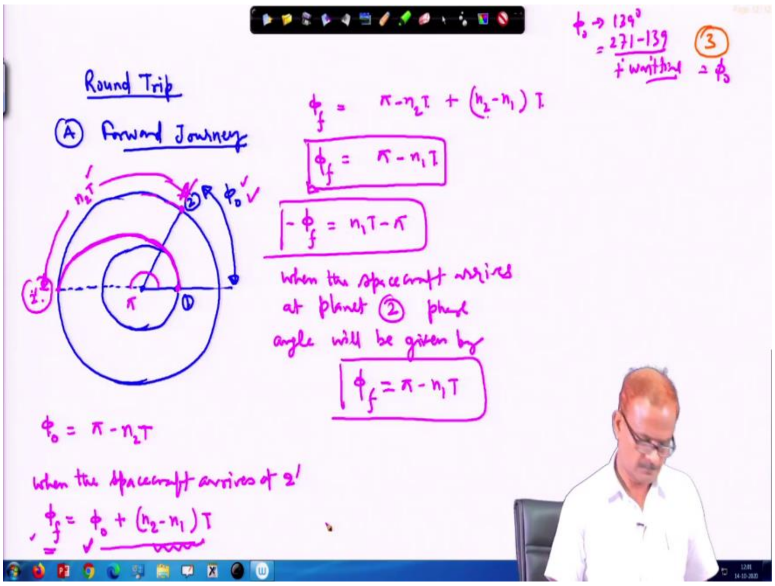

So we have looked into one aspect now we go for the round trip mission in terms of mathematics. Round trip, A forward journey one is here 2 is here. This angle is ϕ_0 the phase angle difference. And then from here the launching is done. So, if in this condition 1 has to go and catch up with 2. So if you launched a satellite at this time using the Hohmann transfer so it will catch up 2 here in this place.

So, this is 2′ it comes to 2′. So, the initial phase angle condition should be matched. So in our case ϕ_0 problem we have done this is corresponding to the initial angle was given which was 139° for 1 and to other was 271° so 271 - 139 and plus the wait time which we are used and plus the wait time that makes your ϕ_0. And then you will be able to launch so that in that time it the 2 goes from this place to this place.

So here this time will be two times the time of travel and 2T we can write the equation now. ϕ_0 this quantity and its related to this whole angle is π so ϕ minus and this angle is from here to here. This angle is n_2T. When the spacecraft arrives at 2′ so at that time is the phase angle and this quantity will be $\phi_0 + n_2 - n_1$ and this is our time t during this whole flight is taking place.

$$\phi_0 = \pi - n_2 T$$

So therefore, the phase angle difference it will be given by the initial phase and plus $n_2 - n_1$ times T dangle between these two vehicles. One will be at that time because ϕ_0 is the initial phase angle here. This is your ϕ_0 which is shown here. And then $n_2 - n_1$ so the angular velocity of 2 and angular velocity of 1 this we have to subtract and this flight time how much it will go ahead. So this will show you where your; the second one is lying with respect to 1.

$$\phi_f = \pi - n_2 T + (n_2 - n_1) T$$

$$\phi_f = \pi - n_1 T$$

This is the angular difference initially and later on this much of angular difference between 1 and 2 will be created. So ϕ_f then this becomes $\phi - n_2T + n_2 - n_1T$ or ϕ_f is equal to $\pi - n_2T$ n_2T cancels out n_1T because one is faster and therefore this ϕ f will turn out to be negative. So this something we can write clearly in this way also n_1T - π. So then the spacecraft arise at planet 2 phase angle will be given by ϕ_f is equal to $\pi - n_1T$.

$$-\phi_f = n_1 T - \pi$$

$$\phi_o = \pi - n_2 T$$

$$\phi_f = \phi_o + (n_2 - n_1) T$$

And this is very important and if you refer to the; our earlier problem initial phase difference we have calculated somewhere; see initial phase difference see this is the quantity we have calculated. So this is the initial phase difference between these two θ'_B and θ_A between the planet A asteroid A and asteroid B. So this is referring to the phase difference we are writing here. So this is written more in mathematical terms.

Sometimes it may happen that you do not remember the equation, but you can always work out using the problem the way I have solved it. Now during the return journey; so this is for the forward journey remember; that here your wait time is included in this wait time or initial any offset of one with respect 2 this is everything is included in this ϕ_0, which we are calling as the initial phase difference. Do not mistake this with the previous problem here. We have combined both of them.

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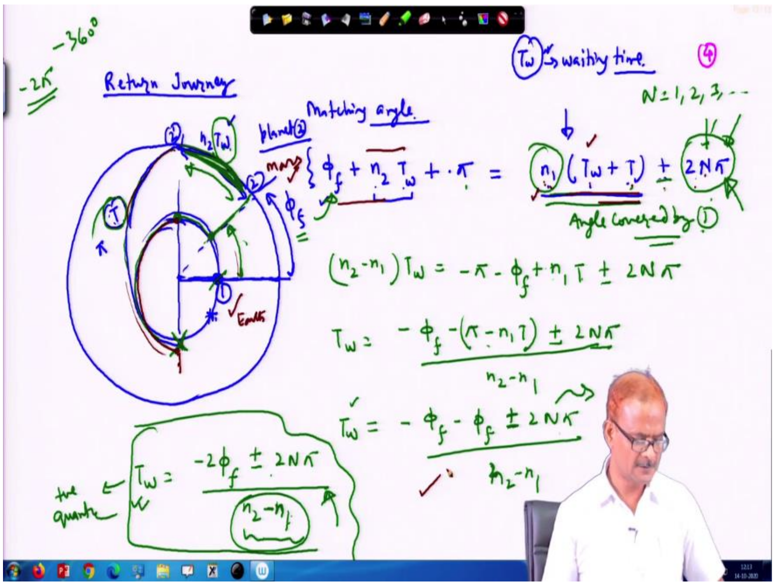

So return journey: Return journey, we can just write by matching the angles. So, ϕ_f is the initial phase difference and they $n_2$times T_w. Now the same thing happens in the return journey, you have to wait for certain time. So this we are writing as wait time; the waiting time you cannot immediately get back from planet 2 to planet 1. So n_2 times T_w this is your angle covered during the waiting time by 2 this is for on the left-hand side this is planet 2.

So, initial phase difference say here either 1 back direction on the front direction whatever it may be here in this case let us show this has ϕ_f and then the waiting time goes. Waiting time here is in this case is separately on n_2 times T_w and then from here the satellite is being launched. So the angle covered. This π corresponding to this; total angle covered from this place measuring from this place.

This is a position of one this is the position of 2 here initially is goes to 2'during this time your one is here and one during this wait time it will travel to say some where it travels maybe because it is a faster also so it may travel to this place. It may go all the way from here and it can come to this place. And then from again for in this time which is corresponding to your T_w. Sorry this is the flight time, which is corresponding; this is the flight time T.

And which is corresponding to angle ϕ that means; where the earth has to cover a certain distance such that your angle is matched. So here we write this as ϕ we will return back to this.

So n_1 times T_w + T $\pm 2N\pi$. Let us look here in this quantity first what this is n_1 times T_w + T what is this? n_1 times T_w corresponding this particular part this segment where the time T_w which is the waiting time this and plus T time.

$$\phi_f + n_2 T_w + \pi = n_1(T_w + T) + 2\pi N$$

So during this time T_w + T how much one is will be covering the angle. It will go from all the way that there is a speed of n_1 multiplied. So this is the angle this is going to cover. This must be lined opposite here in this case it if it has to catch the specific in the return journey. So this is angle covered by angle covered by 1. So angle covered by 1 it is getting back maybe 1 rotation or as I have written here this is plus minus this is plus and minus.

It may go a number of times but it has to come back. If it is going number of times for this plus minus sign has been added to Nπ, N is the number of times it goes there. So N equal to 1, 2, 3 and so on so this is angle covered by 1 and plus minus 2Nπ and this part is angle covered by 1. And this angle must be equal to ϕ_f which is written here plus n_2 times ω this angle.

So you can see here it starts from this place. If you just look from the angle wise so it goes to this place and from here. The launch is being done. So this is the total angle covered and this we have to add to ϕ_f. So, ϕ_f + n_2 T_w + π this should be equal to this much and thereafter it rearranges n_1 T_w this will be equal to π –ϕ_f then n_1 T $\pm 2N\pi$, π – n_1 T already we have this value here ϕ_f is equal to from the previous derivation so this is – ϕ_f – ϕ_f $\pm 2N\pi$ divided n_2 – n_1.

$$(n_2 - n_1)T_w = -\pi - \phi_f + n_1 T \pm 2\pi N$$

$$(n_2 - n_1)\ T_w = \frac{-\pi - \phi_f + n_1 T \pm 2N\pi}{n_2 - n_1}$$

T_w is the wait time is then -2 ϕ_f $\pm 2N\pi$ divided by $n_2 - n_1$. So other way also we can derive this equation expression where whatever the problem we have done mathematical 1 if you follow that notation add all the angles, we will get the same result only thing that you are to add here $\pm 2N\pi$. And this is to make this quantity positive and n is present in such a way so this is 1 2 3 4 like that for a minimum of that n has to be chosen.

$$T_w = \frac{-\phi_f - \phi_f \pm 2N\pi}{n_2 - n_1}$$

So that the state of positive here, here this T_w has to be this is a positive quantity because it depends on $n_2 - n_1$. If $n_2 - n_1$ is greater than zero so this denominator becomes positive accordingly we need to add that. If this is negative then you can see that what the result will be. This accounts for the great time in terms of mathematics So I will suggest doing the problem the way I have done and to that you can always at this quantity plus minus $2N\pi$.

Just like if you remember I have subtracted 360° that means I am subtracting – 2π -n I have taken here. So many times you are not able to visualise then you can just take help of the equation and then work it out. So to wind up this particular part what we have done. We have taken the planet one. The planet one in the wait time it will move by and one-time state T_w and then during the flight time it will move by n_1 times T.

So I have added this and to this plus minus $2N\pi$ I have added because it is the planet is faster so it can go many times. The faster it is the most number of times it can go. This quantity then if it is going many times immediately you can see that you are to subtract. If it is slower n_1 is slower as compared to n_2 then you have to add plus $2N\pi$. Here subtraction is required. On the left-hand side what we have done ϕ_f is the initial phase difference between 1 and the 2.

During the wait time further phase difference is created. So this comes here and then to this we add this Π angle which is the angle from this place to this place so that it matches with the position of the earth. In here in this case this is the earth this is your earth and this is Mars. So this way the angle can be matched. So the same problem but I have tried using the mathematics the purely the numerical style now here theoretically this is done. So we stop here and continue in the next lecture.