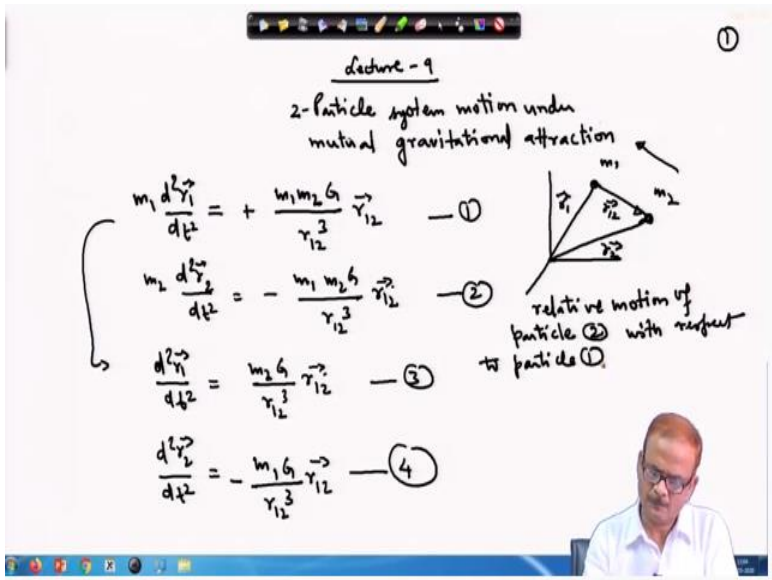
Space Flight Mechanics

Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology – Kharagpur

Lecture – 09
2-Particle System Motion under Mutual Gravitational Attraction

(Refer Slide Time: 00:15)



Okay welcome to the lecture number 9. So we have discussed about the 2-particle system. So in that discussion we were confined to deriving the constant of integral okay and in that way, we have been able to identify only 10 constants out of 12 okay. So, in this lecture we are not going to discuss further on that until unless we go into the three body system or the three particle system now we look into the relative motion of these 2-particles system.

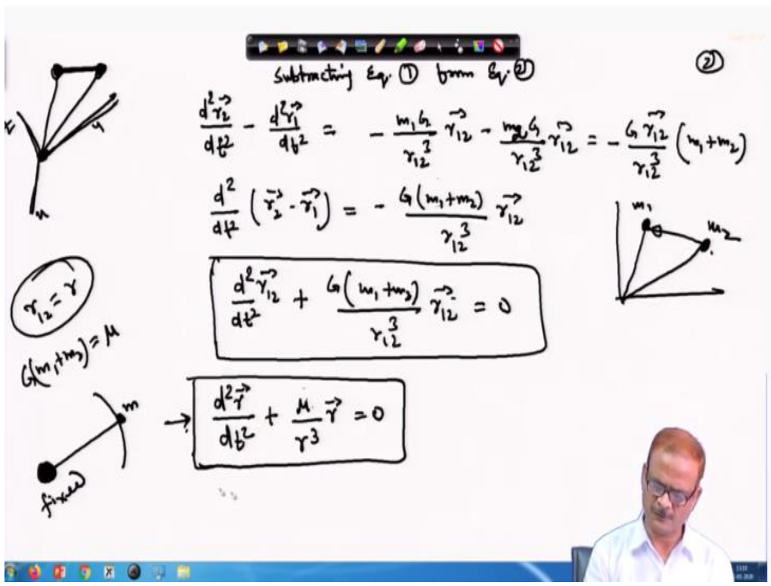
Just like the say the sun is there and the earth is there so how the sun and earth motion will appear okay as we look from the sun arises as look from the both way it can be described okay. So again the starting point is the same as we have written

$$m_1 \frac{d^2\overrightarrow{r_1}}{dt^2} = \frac{m_1 m_2 G}{\boldsymbol{r_{12}^3}} \overrightarrow{r_{12}}$$

I will draw the figure here again this is m_1 this is m_2 r_{12} this is r_2 this is r_1 because here on the particle to the force that is acting in this direction which is opposite of $\overrightarrow{r_{12}}$.

So therefore the minus sign so this can be re written square r_1 / m_1 we will eliminate because we are trying to look into the relative motion. So, this is m_2 G divided by $r_{12}^3 \overrightarrow{r_{12}}$ now we are looking whether the relative motion of 2 with respect to 1 so relative motion of particle 2 with respect to particle 1 if we do that.

(Refer Slide Time: 03:24)



So we need to write it to turn like this subtracting equation (1) from equation (2) so from here we are subtracting this so on the right hand side then we will have m_1 G/ r_{12}^3times $\overrightarrow{r_{12}}$ this comes with the minus sign this is a minus sign and then subtracting from this and this part so this is minus m_1 G/ r_{12}^3this is m_2. So, G $\overrightarrow{r_{12}}$ divided by r_{12}^3this goes as common and we get here m_1 + m_2 with a minus sign.

And the left hand side we can write

$$m_1 \frac{d^2\,(\overrightarrow{r_2} - \overrightarrow{r_1})}{dt^2} = -\frac{(m_1 + m_2)G}{\boldsymbol{r_{12}^3}}\,\overrightarrow{r_{12}}$$

the minus sign is there. And $\boldsymbol{r_2} - \boldsymbol{r_1}$ is nothing but $\overrightarrow{r_{12}}$ so this is r_{12} + G m_1 + m_2 divided by r_{12}^3 okay this is the equation of relative motion of particle 2 with respect to particle 1. Okay remember that we have written the equation not motion of each of these two particles in inertial frame okay directly we have not taken this and this and with respect to this.

We are started writing the equation of motion if you do that will be absolutely wrong okay doing that approach we never do like this because we are trying to formulate the system equation in terms of yeah we are trying to write the equation of motion in inertial frame okay which may happen in certain cases that you write the equation as it turns out to be the relative motion with respect to another particle directly and it is turning out to be a right one but that happens only under various special circumstances.

So that I am not going to discuss that part you can refer to the Sims book engineering mechanics by Sims and go towards the end chapter on the rigid body dynamics. So there those things are discussed and also here in this attitude dynamics course on satellite attitude dynamics course here also I have discussed those topics so there so here I am not going to include all those things so the way we are proceeding we are writing the individual equation of motion in inertial frame an then subtracting them to get the description of how the particle 2 will move with respect to particle1.

So, here $\overrightarrow{r_{12}}$I will simply write as r for simplicity purpose so that I do not have to carry this subscript so whatever equation can then be written as here t^2+ G and m_1+ m_2 G m_1 + m_2 this we are going to write as μ. So, therefore this becomes

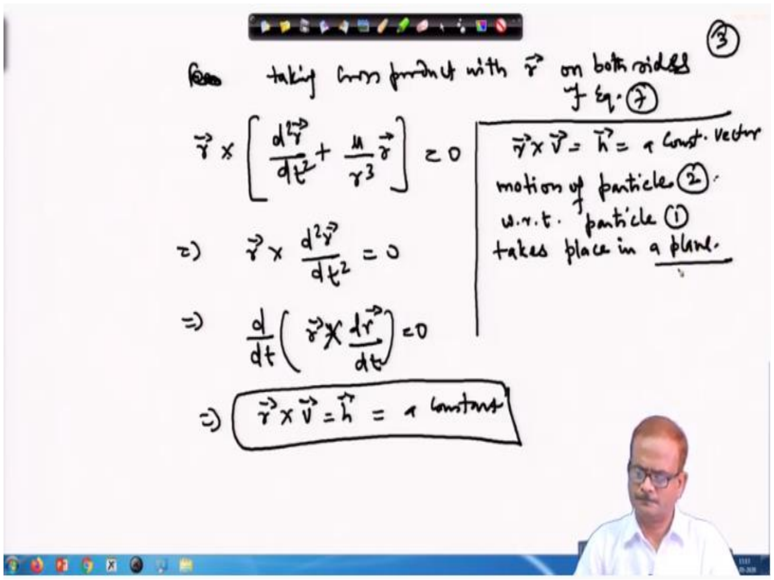
$$\frac{d^2\vec{r}}{dt^2} - \frac{\mu}{\boldsymbol{r^3}}\,\vec{r} = 0$$

and if you remember this kind of equation this looks like as the form of a central force motion under mutual under gravitational acceleration.

So a particle which is moving here big particle is fixed here okay on one another a small particle is moving here so under that circumstances we got this kind of equation okay so the relative motion it looks like the same thing like one particle is fixed another particle is moving around this. The same way if i reverse the sequence that if I have this equation and reverse it rather than subtracting writing in terms of, I am looking for then say the particle 1 motion with respect to 2.

So, this is a reference frame this is m1 and m2 so then I can describe it like this okay. Particle m moving about particle 2 so if you do that again you are going to get the same sort of equation here format will not change okay? so this constitutes our basic equation and we know solution for this okay. So, this is our equation of relative motion okay.

(Refer Slide Time: 09:24)



And using this all the properties we have developed earlier so those can be developed here also so I will quickly do that without discussing much. So after replacing this so taking cross-product let us name this as a equation 1 yes you have 1, 2, 3, 4 this we will name is 5 this as 6 and this as 7 taking cross product with r on both sides of equation 7 okay this is with repetition of the previous thing but time therefore I am going to do it quickly okay.

So, here in this case $\frac{d^2\vec{r}}{dt^2} - \frac{\mu}{\boldsymbol{r}^3}\,\vec{r} = 0$ an what we see from this place this is

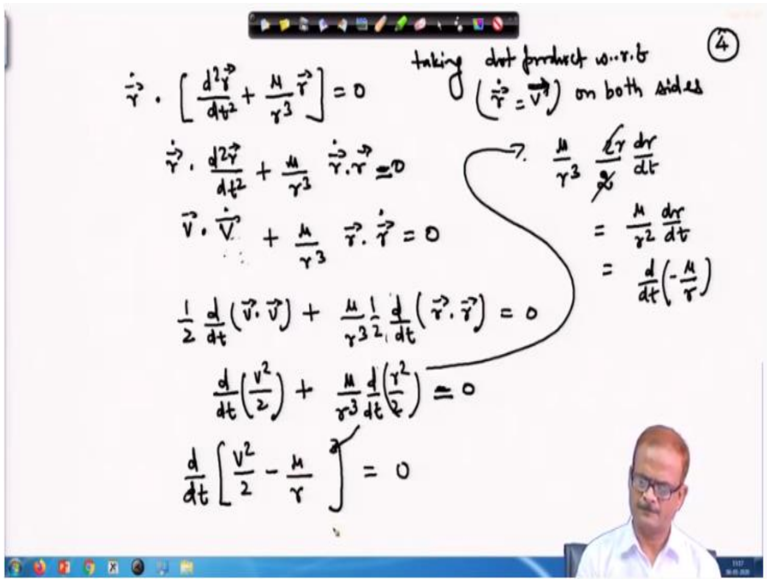
$$\vec{r} \times \left(\frac{d^2\vec{r}}{dt^2} - \frac{\mu}{\boldsymbol{r}^3}\,\vec{r}\right) = 0$$

because *r* cross *r* will vanish and remember that this can be written as *r* cross *dr/dt = 0* a this will replace

$$\vec{r} \times \vec{v} = \vec{h}$$

h is a constant. So, what is this $\vec{r} \times \vec{v} = \vec{h}$ a constant vector this says that relative motion of particle 2 or the motion of particle 2 with respect to particle 1 takes place in a plane. So motion of particle 2 with respect to particle 1 takes place in a plane. So this is the one property we will write it as A.

(Refer Slide Time: 12:34)



So, generally we take a dot product so as earlier we have done so we will get the energy is a constant so dot product with $\vec{r}$ dot $\frac{d^2\vec{r}}{dt^2}$ as we have done earlier this is again repeating this is a repetition of the previous thing sorry. Therefore, I am going quickly taking dot product on both sides product with respect to r dot which is nothing but v there is nothing but v with respect to r dot on both sides we are discussing the elementary things okay until unless we are aware of all these things we cannot go into their advanced topics.

So from here we have

$$\dot{\vec{r}}.\frac{d^2\vec{r}}{dt^2} + \frac{\mu}{r^3}\dot{\vec{r}}.\ \vec{r_1}\ = 0$$

okay so this this is your v in this format we can write it by *v* d/dt + μ/r^3 and here this we will write in this format $\dot{\vec{r}}$we will write it like this. So this part instead of writing it here like this let us write it in a simple format v v dot and we know that this quantity is nothing but *d/dt v dot v* and this quantity is also nothing but 1/2 d/dt $\dot{\vec{r}}.\ \vec{r}$right hand side 0 this can be written as

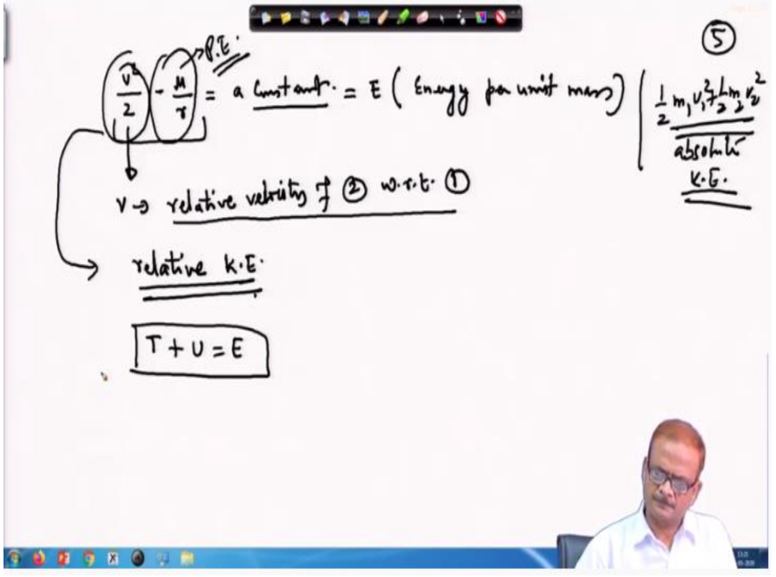
$$\frac{d}{dt}\left(\frac{v^2}{2}\right) + \frac{\mu}{r^3}\ \frac{d}{dt}\left(\frac{r^2}{2}\right) = 0$$

So, can we bring this term together like this? so if you look into this term this is $\frac{\mu}{r^3}$and this gets reduced to *2r dr/dt / 2* this 2,2 cancels out and we get here $\mu\, r^2$dr/dt. So, this is nothing but

$$= \frac{d}{dt}\left(-\frac{\mu}{r}\right)$$

If you differentiate it, you are going to get this quantity so this says that I can write this term as $\frac{\mu}{r}$ and once we integrate this so this implies $\frac{v^2}{2} - \frac{\mu}{r}$ this is a constant.

(Refer Slide Time: 17:02)

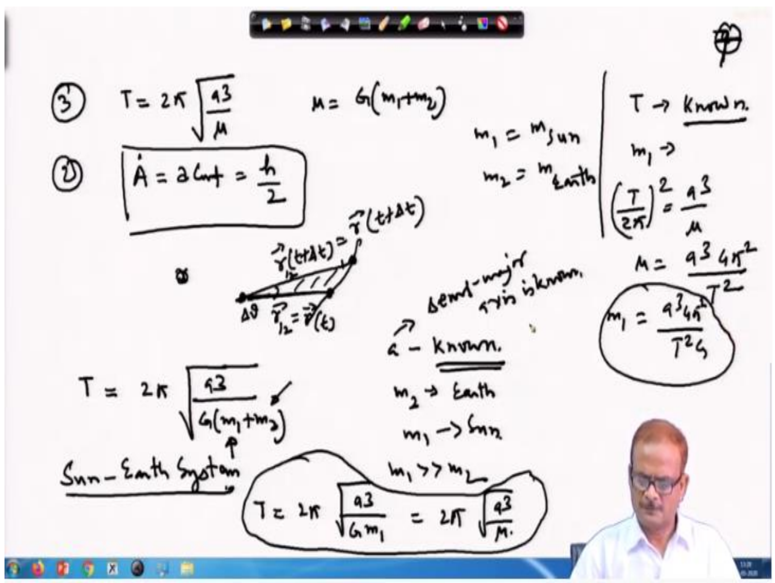


So what we have got $\frac{v^2}{2} - \frac{\mu}{r}$this is a constant and we can write this as E prime the symbol, we reuse it. So this is a energy per unit mass so in what sense it is a very important to consider what exactly they are implying. So this is the v here it is appearing this is the relative velocity of 2 with respect to 1 and therefore this is indicating this term is indicating relative kinetic energy this is not absolute kinetic energy relative kinetic energy.

Similarly, this mu/r term which is appearing here so remember that this r is nothing but r_{12} we have replace r/r_{12} while discussing the in this place the equation of motion. So, this is the relative kinetic energy in a relative sense this is not in an absolute sense okay if the previously if you remember that we have written $1/2\, m_1\, v_1^2 \; + \; 1/2\, m_2\, v_2^2$ in the two body problem. So, there this was the absolute kinetic energy okay but here this is appearing as relative kinetic energy, and this is the corresponding this particular term the potential energy term with respect to the first body okay.

So therefore this is says that $T + U = E$ the constant again the same kind of conclusion we are deriving what we have done for the one body problem. The two body problem also if you look into the system equation it is a relative motion, so this gets reduced into the same format okay? So if a Kepler's law also be derived using the equation using this equation, we have derived all the 3 Kepler's law okay. So that means this is valid whatever we have done earlier so that is also valid here in this case okay.

(Refer Slide Time: 20:33)



So, therefore whatever we have done in the previous lectures so the same thing is applicable here and we can just summarize it rather than working out the whole thing we can summarize. So, time period here in this case this be

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

where μ is G times $m_1 + m_2$ and while writing here h if you look here in this part so this of course we know that this can be reduced to $r^2\,\dot{\theta} = h$.

So, r indicates the distance of the r is nothing but = r_{12} so distance of particle 2 from the particle 1. So this appears as angular momentum of the particle 2 as if the particle 1 is at rest because we are taking here it in the relative sense so this way we are describing. So, here v is not the absolute velocity this is the relative velocity and therefore h is not the absolute angular momentum of particle 2 because this is a relative velocity and therefore this is also relative angular momentum.

So, all these terms we remain very much clear about what we are writing and what we are doing so this was your the third Kepler law and in the second what we had A dot this is a constant which we have written as $h/2$. So, that is also valid here in this case if you take this equation and work out this particular part you get the same solution okay that is what we have done this is particle 1 particle 2 is here this is moving from this place to this place so this is r this is nothing but r_{12} okay this angle is $\Delta\theta$.

This is time t this is time t + Δt r_{12} = r_t + Δt and from there then taking this area and at a rate of change we have derived this part. So, this part will come so I am not going to work out this you can do yourself these are very simple things to work out. Now look into this

$$T = 2\pi\sqrt{\frac{a^3}{G(m_1 + m_2)}}$$

Suppose we consider the Sun-Earth system so in the Sun-Earth system a is known it is a measure distances of the stars we have written using the parallax method you can find in book of astronomy or in B.Sc. level any astronomy book if you pick up so you will find it.

So this semi-major axis is known I can assume that m_2 is the mass of the earth and m_1 is the mass of the sun. So, if m1 is greater than $m_2$time of equation will reduce to this format a^3/Gm_1 which is nothing but

$$T = 2\pi\sqrt{\frac{a^3}{Gm_1}}$$

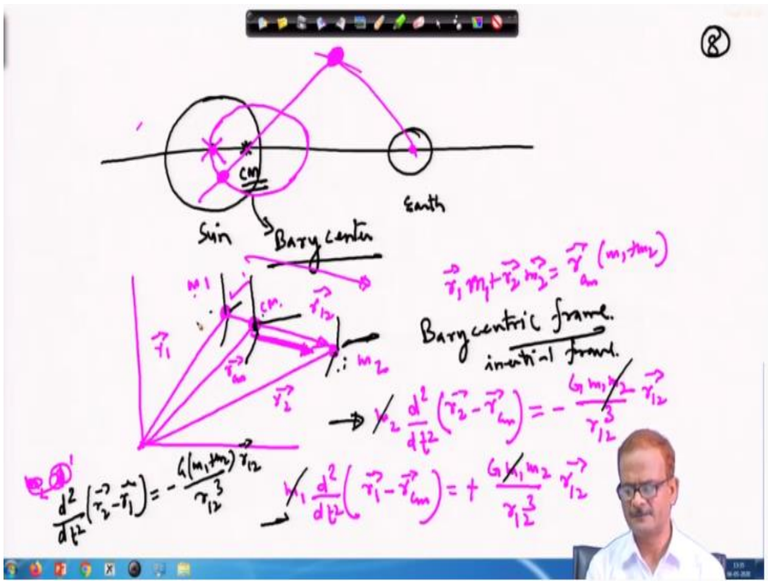
where μ = Gm; m_1 is nothing but the mass of the sun. m earth. So, you can see that it appears like if the system had worked with respect to which you were trying to describe the motion it is a very a heavy one.

So, you can assume it to be the fixed okay and the second mass moving around that and the time period then it is given by simply this equation. So if the time period is now time period is known say in the case of the sun earth system this time period is known and therefore from there m_1 can computed. So you can see that this will be $(T/2\pi)^3 = a^3/\mu$ or

$$\mu = a^3 \frac{4\pi^2}{T^2}$$

So, from here the mass of the sun will be known so this is one way of describing the motion of the 2 particle system.

(Refer Slide Time: 27:27)



Now one very important thing I would like to point out see in the case of the sun-earth system this is not only the sun-earth, but the other planets are also present but let us forget about the other planet in the sun-earth system the center of mass lies somewhere inside the sun. So, this is your sun, and this is sun and this is earth. So, both this the earth and the sun they are moving about the mutual center of mass which we call as the Bary center.

Okay so sun is also the center of mass this is the center of mass of the sun this will go in a circle like this circle or eclipse whatever you tell, and this will also move about this point okay? so and they will always maintain same orientation. So, this will come to this point and from here this will go to this point. But if you are looking in the relative manner means you are looking with respect to the center of this sun how the earth is moving.

So whatever we have done on the in this on this previous cases business all these derivations we have done see this all of these things are applicable okay relative motion basically we have described okay this is the relative motion. So this is applicable once you are describing the motion with respect to the sun, but it is possible and it is also prudent that you can write the equation of motion say this is m_1, m_2 this is r_{12} this is r_2 this is r_1 okay.

So since we are in Bary-center there is a line here in this place, so this is your center of mass and this location you know this is r_{cm}. Because r_1 times m_1 an r_2 times m_2 so this is r_{cm} m_1+m_2. So, rather than describing the motion with respect to this m_1 also I could describe the motion with respect to the center of mass. So in that case what we need to do we proceed in the same way we write here m_2 times square.

Now here I will be describing with respect to the center of mass okay? So if I described with respect to the center of mass, we have to take this distance. So, this distance how much this will be d square/dt square. So, how this particle is changing with respect to the center of mass so this will be r_2-r_{cm} and then the force acting on the system what is this? this is because of the mass 1. So, this is – Gm_1 m_2 this is acting in the opposite direction divided byr_{12}^3 and unit vector in the r_{12} direction.

This is the way of the writing similarly if you want to write equation of motion 1 so this will be written as

$$\frac{d^2}{dt^2}(\overrightarrow{r_1} - \overrightarrow{r_{cm}}) = -\frac{m_1 m_2 G}{\boldsymbol{r_{12}^3}} \overrightarrow{r_{12}}$$

now we are in this case r_{12}vector we are taking here in this direction okay this is r_{12} vector direction. So, you can see that I will write it r_{12}, but this will come with a positive sign. So, right hand side does not change it because this is the force term.

So force term does not change what is changing in the even in the absolute motion once we were considering so at that time also on the right hand side, we got the same term okay. Here we are taking with respect to the center of mass. So, the same thing is appearing okay nothing different and if you add them so you can see that what is going to happen adding or. If you subtract so you will get the same relative motion here in this case we write as m_2, m_2 we can divide on both sides.

So, here if we see if we include if we divide on this side and m_2 we divide on this side m_1 we divide on this and divide on this what this will be. The relative motion then will appear as d/dt square r_2- r_1 if we subtract these two and on the right hand side similarly, we have we are subtracting from this one so they will add up and on the right hand then you get -Gm_1+ m_2/ r_{12}^3 cube times $\overrightarrow{r_{12}}$.

So, we recover the same equation using the with respect to the center of mass because center of mass and why we can write equation with respect to the center of mass directly because center of mass is non-accelerated here in this case. So, if I fix a frame here this will constitute an inertial reference frame and therefore, I can write a equation of motion with respect to the center of mass or either in the center of mass frame which we call as the barycentric frame.

So I can write directly the equation of motion in Bary-centric frame without hesitating because this constituents an inertial frame. But if you fix a frame here so or either do fix a frame here in this place so this is not constituting an inertial frame this is not constituting an inertial frame itself. So we will not take that approach we go through a very right method so that we do not do any mistake at any stage. So I will stop here and we will continue in the next lecture thank you very much.