Space Flight Mechanics
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Lecture – 90
Patched Conic Section Method for Interplanetary Transfer

Welcome to lecture 90 in the previous lecture we finish the interplanetary mission, but that was only with respect to the Hohmann transfer. There we were trying to match the phase angle condition so that the we know what is the launch window when we can launch those things are calculated using the previous work we have done.

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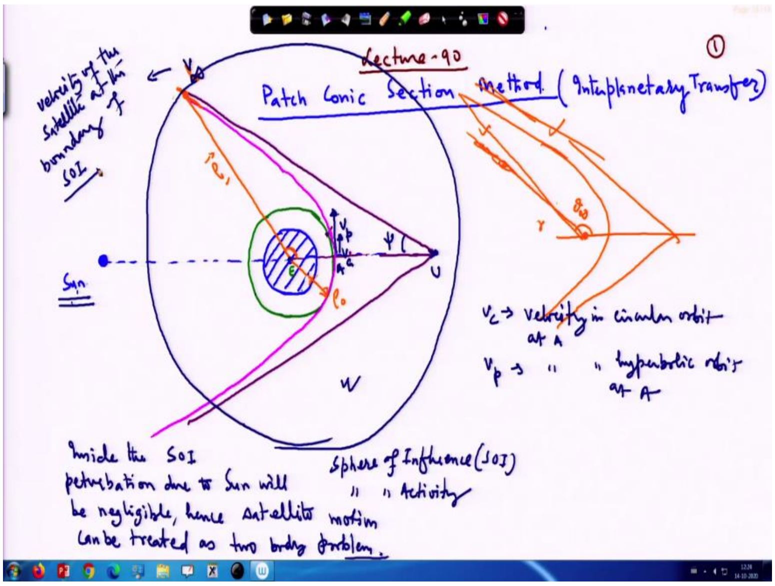

Now here in this we go for the actual transfer process which is involved. Say here in this case we have sun here and the Earth is located here in this place. This is your Earth, sun I have showed by small dot. Around the earth in a parking orbit there is a satellite. And we have to send the satellite from this parking orbit to Mars. Let us say this is earth and we have to send it to Mars. Now the problem is that going from unlike the previous problem.

There is only the Hohmann transfer that we have considered to escape out of the earth's gravitational field. We need to send the satellite in hyperbolic trajectory without this it will not be done. So, let us say that somewhere; this is the asymptotic of the hyperbola and hyperbolic trajectory it looks like this. This goes and becomes asymptotic. This is ρ. This distance we show as ρ_0, ρ_0 is the orbit of the satellite around the earth parking orbit radius.

This is your ρ_g, so here in this point your vehicle is located. This velocity is v indicate by V_∞. So as it becomes here asymptotic has been totally so you can see that this is your $\vec{\rho}$. So, if here little bit of distortion is there in this figure if you remember the hyperbola in the hyperbola case. For the asymptote the r is measured from the focus and then this is your θ_∞ and then r is real in this place.

So, it becomes wherever it is radius, vector is there. At infinity this will be parallel to this. So, though here it is not visible so here we do not say that this is going to infinity rather we do it by drawing a sphere around the earth like this, this we call as sphere of influence also called as sphere of activity. So this implies sun is at very large distance as compared to this radius of the sphere of influence.

This angle we write Ψ. So, inside the sphere of influence you can treat the motion of the Satellite has two body problems, why? Because the effect of gravitational attraction of the Earth or the perturbation; due to the effect of the perturbation due to the sun on the satellite will be negligible inside this sphere. So, inside the sphere of influence perturbation due to sun will be negligible and satellite motion can be treated as a two-body problem.

So let us assume this velocity we write here in the hyperbola this is the perigee and V circular we will indicate by V_C. So, V_C is velocity in circular orbit around Earth it can be electrical even in circular orbit for convenience writing circular orbit velocity in circular orbit at the point and V_P in the hyperbolic orbit. It is the point of intersection V_∞ this is referring to the velocity of the vehicle for the satellite at the boundary of a sphere of influence. These are the things given to us.

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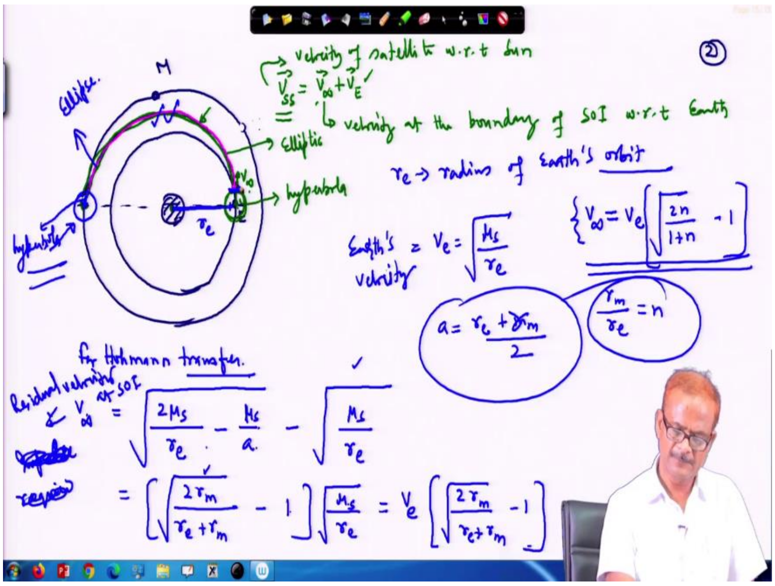

Now what is required? See here in the; inter interplanetary case suppose this is the case of the Sun, Earth and Mars. Here Sun is; earth in this plane and Mars here in this one. And if we have to go and catch the mars, so Mars maybe somewhere a little more ahead I will show it. Let us say see is this position of the Mars as we have done the calculation of earlier for the phase and all other things.

So, we do not have to worry about those issues. So what we have done that we have shown that trajectory to start like this and go to this point if we are launching here in this place. But whatever we have calculated if you remember, the gravitational field was neglected in the problem we have done. Gravitational field was ignored but here in the case of earth we cannot ignore it gravitational force is quite prominent.

So, inside the sphere of influence; let me draw around this sphere of influence and if we know the velocity here let us write this is the V_∞. We can assume that this V_∞ serves as the velocity in this elliptical orbit, transfer orbit. So V_∞ and plus the velocity of the Earth this makes you the V with respect to the sun with respect to Sun I will indicate the size of the satellite with respect to Sun.

$$\vec{V}_{SS} = \vec{V}_\infty + \vec{V}_E$$

So I will write this V_{SS} velocity of satellite with respect to Sun while this is going in this elliptical orbit it moving around the Sun, describing an orbit around the sun in an elliptical orbit. So, V_∞ which is the velocity with respect to the earth; velocity at the boundary of sphere of influence with respect to earth and to this then we had the velocity of the earth. So we get the total velocity because this is the velocity with respect to the earth.

So, the total velocity will be with respect to the sun and therefore we have written here like this. You can see the motion here as I have shown here in this previous page this is hyperbola. So, motion inside the sphere of influence. This is the conic section which is hyperbola while inside this hyperbola and this part is elliptic, I will show by another colour. So, elliptic is from this place to this place.

Now again once it goes to the moon. So, at the moon we have the sphere of influence. So the pink line I will remove from this part. So close this. This is ellipse which is with respect to the Sun once it goes inside the sphere of influence or this boundary. Then again, this will be treated as a two-body problem and here the trajectory will be then hyperbola. And for this reason, this is called the Heirs Conic section method because you are trying to patch elliptical and the hyperbolic orbit.

Just by giving an elliptical orbit condition you cannot go to another planet. You have to escape the gravity of the earth and for escaping minimum parabolic orbit is required. But we do the parabolic orbit around the boundary you will not get the necessary velocity required to go to the Mars. Ok and therefore it is sent in hyperbole orbit. So for Hohmann transfer what is required? So here in this case this is at the boundary we write this as V_∞ and V be the perigee velocity in the hyperbolic orbit already have written at A.

From here V_∞ which is required to go from this place to this place it will be indicated on the solar scale this quantity sphere of influence is quite small and therefore we will be able to do this approximation here. So V_∞ this will be equal to 2 μ sun divided by radius This is say here the earth's orbit radius -1 by r μ sun by k which is the semi major axis of this orbit under root as we have done in the Hohmann transfer – μ sun divided by the r radius.

$$V_\infty = \sqrt{\frac{2\mu_s}{r_e} - \frac{\mu_s}{a}} - \sqrt{\frac{\mu_s}{r_e}}$$

$$V_e = \sqrt{\frac{\mu_s}{r_e}}$$

So the radius ra is the radius of Earth's orbit. So earth is moving around this with velocity V earth equal to this is the Earth what is velocity? V_S is equal to μ sun divided by r e under root. For this we are subtracting with the velocity required in this elliptical orbit and that velocity how much we are writing? We are writing as V_∞. This is the extra impulse required which we have written as $\Delta V r_e$.

$$a = \frac{r_e + r_n}{2} \; ; \; \frac{r_m}{r_e} = n$$

And a is here r earth + r mars divided by 2, if we insert here in this place this a and rewrite this so this will turn out to be $2r_m$ this you can check yourself. I am just skipping that particular part - 1 times μ by r_e under root means this is the earth times 2 r_m divided by r_e + r_m under root -1 and if we use our earlier notation. So V_∞ this will be the rm by r_e we write as n this quantity then becomes 2n divided by 1 + n -1.

$$= \left[\sqrt{\frac{2r_m}{r_e + r_n}} - \right]\sqrt{\frac{\mu_s}{r_e}} = V_e\left[\sqrt{\frac{2r_m}{r_e + r_m}} - 1\right]$$

$$V_\infty = V_e\left[\sqrt{\frac{2n}{1+n}} - 1\right]$$

This is in the Hohmann transfer the same kind of notation we have used your r_m by r earth is equal to n. Ok so here at the sphere of influence how much velocity is required how much schedule speed is required because once it is coming out of the hyperbola. If we see here in this place it started with V_P velocity and here it is V_∞. So that I will not say this is impulse because the impulse will provide here in this place.

So that this is not impulse required but it is rather the residual velocity at SOI sphere of influence. Impulse we are giving in this place. Here in the $V_{circular}$ + ΔV this is the impulse given that makes it V_P and then V_P it starts with V_P and goes here at the V_∞. From here we have detected that how much V_∞ is required ok for doing the interplanetary transfer. Now V_∞ is known so we will be able to calculate V_P and that we do as follows.

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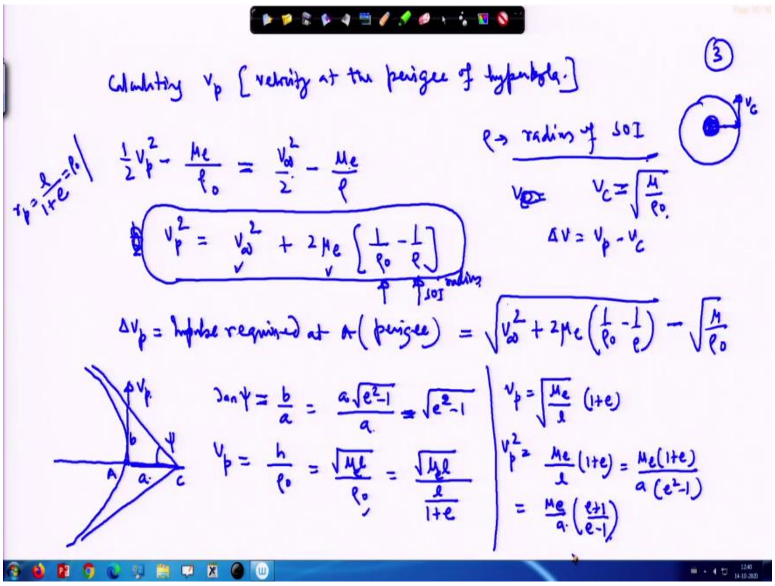

$$\frac{1}{2}V_P^2 - \frac{\mu_e}{\rho_o} = \frac{V_\infty^2}{2} - \frac{\mu_e}{\rho}$$

Calculating V_P that is velocity at the perigee of hyperbola, the technical detail I will come later on let us first work out this part 1 by 2 we apply the conservation of energy for the two body problem 1 by 2 V_P^2 – μ earth by ρ_o, which is the parking orbit radius as we have taken on earlier here this is your ρ_o from here to here. And this should be equal to V_∞^2 divided by 2 at the square of influence - μ by earth divided by ρ.

Here ρ is the radius of sphere of influence, so from we write V_P^2 this is will be equal to again V_∞^2 + 2 μ e times 1 by ρ_o – 1 by ρ. We can see that we have been able to calculate the velocity in the hyperbola. This is known to us. Here if you have the parking orbit here, and it is a circular orbit, which is known to how much will be the velocity in this parking orbit.

$$V_P^2 = V_\infty^2 + 2\,\mu_e\left[\frac{1}{\rho_o} - \frac{1}{\rho}\right]$$

So this will be μ by ρ_o under root. So, therefore ΔV will be $V_P - V_C$ from this quantity we have to subtract this quantity. Therefore ΔV_P equal impulse required at A which is perigee here. This is equal to V_∞^2 + 2 μ e 1 by ρ_o – 1 by ρ under root -1 by ρ_o. Ok once we have worked out this the rest other things can also be calculated.

$$V_C = \sqrt{\mu/\rho_o}$$

$$\Delta V = V_P - V_C$$

$$\Delta V_P = \sqrt{V_\infty^2 + 2\,\mu_e\left[\frac{1}{\rho_o} - \frac{1}{\rho}\right]} - \sqrt{\frac{\mu}{\rho_o}}$$

Now look into this hyperbola for the hyperbolic trajectory this is the centre and this distance is a this is b this we have shown as Ψ. So tan ψ this is equal to plus minus or I can remove this minus sign just keeping it plus b by a. So how much b is? b is a times e^2– 1 under root for hyperbola, this is divided by a. There after we require anyway V_∞ is coming from our Hohmann Transfer condition that is known to us.

$$\tan\psi = \frac{b}{a} = \frac{a\sqrt{e^2-1}}{a} = \sqrt{e^2-1}$$

So here in this case this quantity is known to us μ earth this is known to us ρ_o is known to us and the sphere of radius of a sphere of influence this is also known to us. SOI radius of a sphere of influence and ρ_o is the radius of the parking orbit. So, all these quantities are known to us. Now V perigee; velocity here in this point this is the V perigee this is the point A we have marked here.

$$V_P = \frac{h}{\rho_o} = \frac{\sqrt{\mu_e l}}{\rho_o} = \frac{\sqrt{\mu_e l}}{\frac{l}{1+e}}$$

$$V_P = \sqrt{\frac{\mu_e}{l}(1+e)}$$

This can be written as h by ρ_o and conservation of angular momentum equation is basically this is μ times l under root divided by ρ_o and μ times l under root divided by ρ_o because this is perigee position. So this we can write as l by 1+ e for the hyperbola, r perigee equal to l by 1 + a and which is nothing but here in this case ρ_o. And here this is μ earth. And rest other also we can work out here. So μ earth divided by 1 under root times 1 + e a times e^2- 1 go to the next page.

$$V_P^2 = \frac{\mu_e}{l}(1+e) = \frac{\mu_e(1+e)}{a(e^2-1)}$$

$$= \frac{\mu_e}{a}\frac{e+1}{e-1}$$

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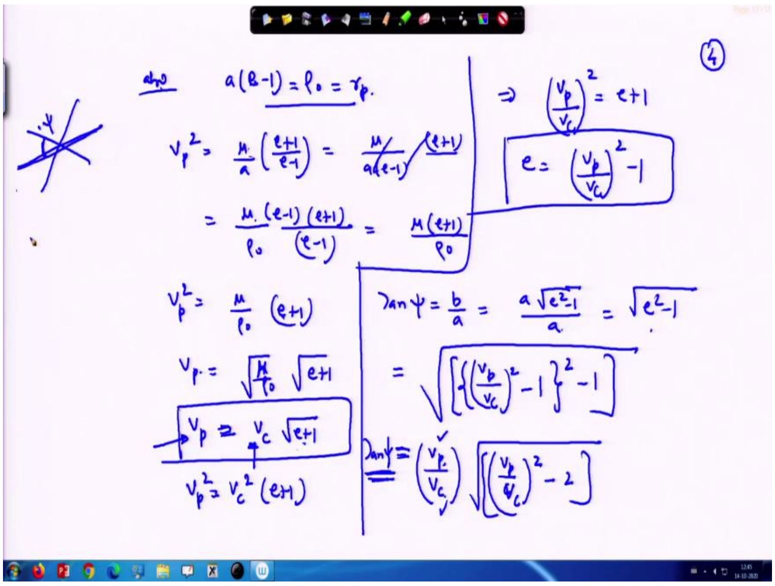

$$a(e-1) = \rho_o = r_p$$

$$V_p^2 = \frac{\mu}{a}\frac{e+1}{e-1} = \frac{\mu}{\rho_o}\frac{(e+1)(e-1)}{(e-1)} = \frac{\mu(e+1)}{\rho_o}$$

$$V_p^2 = \frac{\mu}{\rho_o}(e+1)$$

$$V_p = \sqrt{\frac{\mu}{\rho_o}}\sqrt{e+1}$$

$$V_p = V_c\sqrt{e+1}$$

Also a times e - 1 this quantity is ρ_o because this is the perigee position therefore it becomes V_p^2 equal to μ by a times e + 1 divided by a - 1 as we have written here. This is μ earth this is not fixed eccentricity some where it may be confusing so we can drop it for the time being. We can drop this μ e so that it does not appear to save eccentricity. We have to be careful in writing this. μ son already we have used to I will use here just mu.

$$V_p^2 = V_c^2(e+1)$$

$$\frac{V_p^2}{V_c^2} = (e+1)$$

$$e = \frac{V_p^2}{V_c^2} - 1$$

Now using this a times e -1 if we are replacing by; actually, be we have to replace the say μ a equal to ρ_0 times e - 1 times e + 1 divided by e – 1. So this is the velocity required in the hyperbolic orbit but this is known to us. This is known to us. But what we are interested at? We are interested in finding out e. So, rather than doing this I have to find out e therefore V^2 both side and write it like this V_c^2 e + 1.

And this implies V_P by V_c^2 equal to e + 1 e equal to V_P by V_c^2 -1 so this way eccentricity of the orbit is also calculated. So, these are the terms that we required for vehicle transfer and this way we have calculated it. And now tan ψ we have written as b by a equal to a times e^2 - 1 under root divided by a^2– 1 under root and e from this place we can write here V_P by V_C is square -1 and this whole square – 1 under root.

$$\text{Tan } \psi = \frac{b}{a} = \frac{a\sqrt{e^2-1}}{a} = \sqrt{e^2-1}$$

$$= \sqrt{\left[\left(\frac{V_p}{V_c}\right)^2 - 1\right]^2 - 1}$$

$$\text{Tan } \psi = \left(\frac{V_p}{V_c}\right)\sqrt{\left(\frac{V_p}{V_c}\right)^2 - 2}$$

And if we break it and open expand it and then write it so this gets reduced to this format. So these tan ψ so this angle is also known. So we can see that V_P is known to us from the infinity. So V_∞ comes from the Hohmann condition and transfer condition, which is known to us. So this way ψ is also known to us and what is the ψ? ψ is the angle of the asymptote. This is the ψ from the axis line. Ok this part we are calculated then other things are also required which we need to work out.

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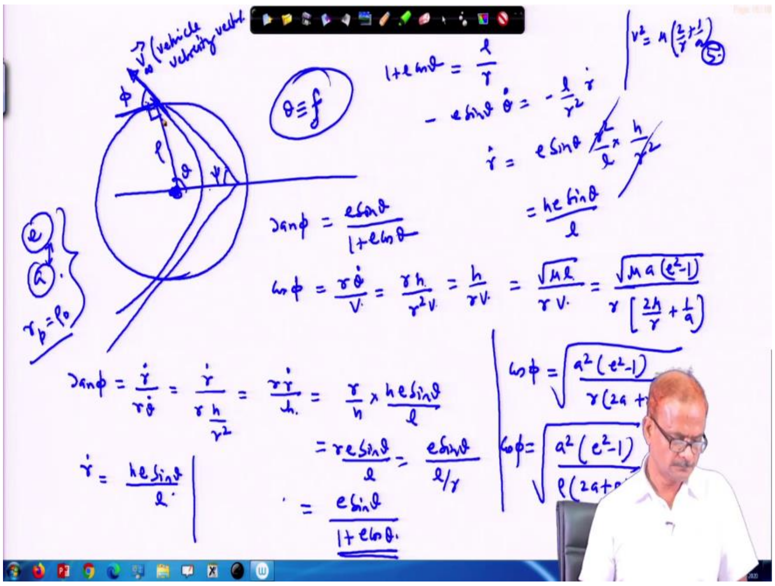

Before I show you the all the things, so we need to write one more thing. Let us say this is the earth and at the boundary here hyperbola we have shown here is like this and at this boundary V is the velocity vector. So, this is your angle ϕ this is vehicle velocity vector we write rather as V_∞ because we are writing it with respect to this angle ψ we have shown. This angle we write as θ and this is your flight path angle.

As we have done this tan ϕ this will be equal to $\dot{r}$ divided by r times $\dot{\theta}$ as we have done in the case of the orbit determination problem r times h by r^2 this becomes $r\dot{r}$ divided by h. And $\dot{r}$ expression also we have derived this is h e sin θ divided by l and how we are driving $\dot{y}$ equation. As I can think here this we have repeating here as l by r equal to 1 + e cos θ and differentiating this is e sin θ times $\dot{\theta}$ we got as -l by r^2 by $\dot{r}$ and then $\dot{r}$ equal to e sin θ times r^2 divided by l times $\dot{\theta}$.

$$Tan\phi = \frac{\dot{r}}{r\dot{\theta}} = \frac{\dot{r}}{r\frac{h}{r^2}} = \frac{r}{h} \times \frac{he\sin\theta}{l}\,; \dot{r} = \frac{he\sin\theta}{l}$$

$$= \frac{r\,e\sin\theta}{l} = \frac{esin\,\theta}{\frac{l}{r}}$$

$$= \frac{esin\,\theta}{1+ecos\,\theta}$$

So $\dot{\theta}$ is equal to h by r^2 this cancels out we get e sin θ times h divided by l which is sin θ divided by l we have written here in this place. So therefore this becomes r by h times e sin θ divided by l this quantity gets reduced to h cancels out r e sin θ divided by l and we can write this is e sin θ divided by l y r e θ divided by 1 + e cos θ. So instead of using θ I will later on I will use the θ identical with f.

$$1 + ecos\theta = \frac{l}{r}$$

$$-e\,sin\theta\dot{\theta} == -\frac{l}{r^2}\,\dot{r}$$

$$\dot{r} \;= e\,sin\,\theta\frac{r^2}{l} \times \frac{h}{r^2}$$

$$= \frac{hesin\theta}{l}$$

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So θ I will use for some other symbol. This expression we are using here. So using this we have tan ϕ equal to e sin θ divided by 1 + e cos θ and similarly cos ϕ we can write as r $\dot{\theta}$ divided by V. So r times $\dot{\theta}$ h by r^2 this is h by r V and h is μ times h under root divided by r V μ times a times $e^2 - 1$ under root times r and V is 2 μ by r. We calling the expression V^2 equal to μ times 2 by r + 1 by a for the hyperbola so 1 + 1 by a.

$$Tan\phi = \frac{eSin\theta}{1+eCos\theta}$$

$$Cos\,\phi = \frac{r\dot{\theta}}{V} = \frac{rh}{r^2V} = \frac{h}{rV} = \frac{\sqrt{\mu l}}{rV} = \frac{\sqrt{\mu a\,(e^2-1)}}{r\left[\frac{2h}{r}+\frac{1}{a}\right]}$$

And if we rearrange this cos ϕ will turn out to be that you can check yourself that cos ϕ is equal to a^2 times e^2 - 1 divided by r times 2a + r under root. So here in this case r we have to replace why this ρ this becomes a^2 $e^2 - 1$ ρ times 2a + ρ under root anyway because e is known to us and therefore e is also known to us because r perigee is $\dot{\rho}$, ρ_0. So, from there ρ is also not solved, so all the quantities are known here.

$$\cos\phi = \sqrt{\frac{a^2(e^2-1)}{r\,(2a+\rho)}}$$

$$\cos\phi = \sqrt{\frac{a^2(e^2-1)}{\rho\,(2a+\rho)}}$$

So therefore this flight path angle of the vehicle is can also be calculated which we always compute with the local horizontal.

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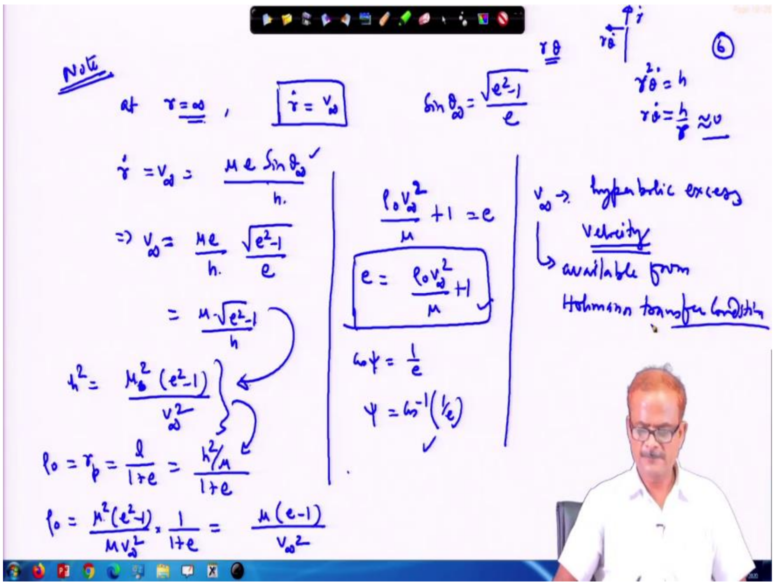

One more thing we can note down that at once r tends to very large value that is r is equal to infinity ok so at that time your $\dot{r}$ will be equal to V_∞ y. Here you see in the case r times $\dot{\theta}$ this one is radial velocity, which is $\dot{r}$ and here you have the r times $\dot{\theta}$. So r times r^2 $\dot{\theta}$ is equal to h so r times $\dot{\theta}$ is h by r and once r becomes very large. So this almost tends to 0.

$$\dot{r} = V_\infty = \frac{\mu e\,\mathrm{Sin}\theta_\infty}{h}$$

Therefore this is the approximation we can write $\dot{r}$ equal to V_∞. And therefore from this place $\dot{r}$ equal to V_∞equal to μ e sin θ_∞ here in this case where θ_∞ is referring to this. This becomes asymptotic ok once θ becomes θ equal to θ_∞ this is becomes asymptotic. So μ e sin θ $\dot{r}$ value we are inserting here divided by h and this implies V_∞ equal to already V_∞ we have calculated but here also you can look from this place μ e divided by h.

And sin θ this quantity sin θ_∞ earlier in the conic section equation we have derived it. This is quantity e^2- 1 divided by h this e^2– 1 under root divided by e u. So μ times e^2- 1 divided by h and h is the quantity already we have derived this is μ e^2- 1 or μ we would delete here this V_∞^2. So this we arrange it and write here in this way.

$$V_\infty = \frac{\mu e}{h}\frac{\sqrt{e^2-1}}{e} = \frac{\mu\sqrt{e^2-1}}{h}$$

$$h^2 = \frac{\mu^2(e^2-1)}{V_\infty^2}$$

$$\rho_o = r_p = \frac{l}{1+e} = \frac{\frac{h^2}{\mu}}{1+e}$$

After arranging we have written it. Earlier also we have derived its relationship. We can see the same thing comes from two different phase. So ρ_0 is equal to r_p is l by 1 + e equal to h^2 divided by μ 1 + r ρ_0 this then and gets reduced to h^2 we can insert from this place then insert here. This is μ^2 e^2 – 1 divided by μ times V_∞^2 divided by 1 + e.

$$\rho_o = \frac{\mu^2(e^2-1)}{\mu V_\infty^2} \times \frac{1}{1+e} = \frac{\mu(e-1)}{V_\infty^2}$$

$$e = \frac{\rho_o V_\infty^2}{\mu} + 1$$

So this gets reduced to μ times e - 1 divided by V_∞^2. So we have ρ_0 times V_∞^2 divided by μ + 1 is equal to e. So here see another expression we are getting and also cos ψ was equal to 1 by e again referring to conic Section and therefore ψ is equal to $\text{Cos}^{-1}\left(\frac{1}{e}\right)$ is known cos ψ is known. We can see the multiple ways we have tried to evaluate this ψ. So once we have calculated this V_∞ this happens to be our hyperbolic excess velocity which is the label from Hohmann, transfer condition.

$$\text{Cos}\,\psi = \frac{1}{e}$$

$$\psi = \text{Cos}^{-1}\left(\frac{1}{e}\right)$$

Now once we have done this then we have to exit from the sphere of influence and there after the normal question will take place along the elliptic. And then again, it goes and enters the sphere of influence of the Mars so at that time the entry condition is there. So we need to just worked out this exit and entry condition and rest other things are done. So it does not require much Mathematics now only thing on the geometry I have to show the rest of the things.

How to calculate the angles? What are the angles involved? So I will do that in the next lecture. Thank you very much we continue in the next section.