Space Flight Mechanics

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Lecture – 98
Interception in non-coplanar orbit (Contd.,)

In the last lecture we are finished the non co-planar manoeuvre, but sometimes in the same thing sync context you need to propagate the orbit. Already we have looked at this Kepler's equation for the elliptic and hyperbolic orbit we have derived and that can be used to propagate orbit. But besides this f - g is a function also used for propagating. Sometimes it may be much more convenient to work with that. So let us look into that.

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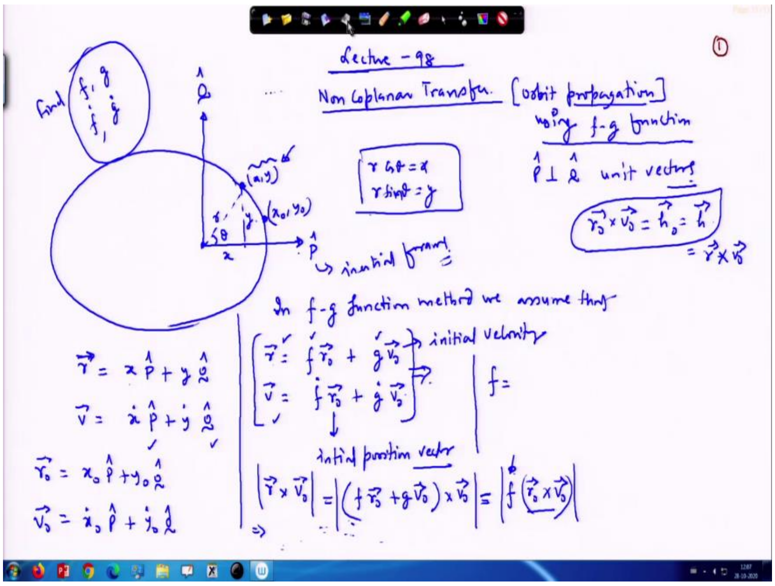

$$\vec{v} = \dot{x}\hat{P} + \dot{y}\hat{Q} ;$$

So I have written here the orbit propagation you see using the f - g function. So what does this mean? Let us look into this figure which shows here as an ellipse ok. However, it may be anything. This is the co-ordinate XY this is r and this angle is e cos θ. So, r cos θ is equal to X and then r sin θ is equal to Y. $\hat{P}$ and $\hat{Q}$ these are the unit vectors along the; basically, in the Plane of the orbit the co-ordinate axis.

These are the unit vector $\hat{P}$ and $\hat{Q}$ mutually perpendicular unit vector. Now let us assume that the position of the satellite at any other time can we hire expressed as the coordinator at any other point. Initial point may be somewhere let us say it was here x_0 y_0 ok and this is a μ point.

So this makes it convenient to write with. So X times $\hat{P}$ plus Y times $\hat{Q}$ this very simple this is basic representation we have been using from the beginning.

$$rCos\theta = x \; ; \; \overrightarrow{r_0} = x_0\hat{P} + y_0\hat{Q}$$

$$rSin\theta = y$$

$$\vec{r} = x\hat{P} + y\,\hat{Q}$$

And the corresponding velocity vector then this can be written as $\hat{P}$ y_0 times $\hat{Q}$. These are the unit vectors they are fixed they are not changing. They are strictly constituting initial frame. In the f - g series method a f - g function method f - g series another method which is pertaining to the numerical propagation. Here we are not doing numerical but analytical propagation. So in the f - g function method we assume that r is a function of initial position of the satellite and initial velocity of the satellite.

$$\overrightarrow{v_0} = \dot{x}_0\hat{P} \dot{+} \dot{y}_0\hat{Q}$$

So f and g this 2 are appearing as functions. Similarly, the velocity vector at any time can be written as $\dot{f}$ r_0 + $\dot{g}$ V_0 why, r_0 and v_0 they are the initial velocity and r_0 is initial position vector. So these are fixed these are known and these are fixed. We do not need to differentiate this. So the velocity and the position can we expressed during this to be equation.

$$\vec{r} = f\vec{r}_o + g\overrightarrow{v_0}$$

$$\vec{v} = \dot{f}\vec{r}_o + \dot{g}\vec{v}_o$$

Now writing in $\vec{r} \times \vec{v}_0$ and this gives us f is equal to $\vec{r} \times \vec{v}_0$ divided by $\vec{r}_0 \times \vec{v}_0$. And $\vec{r}_0 \times \vec{v}_0$ this is because it is a central force motion so this is conserved for this is control angular momentum per unit mass this is conserved. And once we expand it or we write in terms of x + y PQ and $\vec{v}_0$ similarly we will write; how we will write r_0? R_0 we can write as x_0 $\hat{P}$ + y_0 $\hat{Q}$.

$$|\vec{r} \times \vec{v}_0| = |(f\vec{r}_o + g\vec{v}_0) \times \overrightarrow{v_0}| = |f(\vec{r}_o \times \overrightarrow{v_0})|$$

Similarly $\overrightarrow{v_0}$ we can write as a x_0 $\hat{P}$ + y_0 $\hat{Q}$. The left and right-hand side they are vectors so directly we cannot divide it. We need to take the magnitude but f is a component. So f is a function of time. So in what format it will emerge this is what we are looking for. So we are looking for find f-g and $\dot{f}$ $\dot{g}$. And f-g is known as you can see that using this to immediately r and V can be calculated.

So this is our aim to find out f - g and $\dot{f}$ and $\dot{g}$. So, if we put r_0 and v_0 in terms of this and expand it, so this will get reduced to or either you take the magnitude on the both side if we write it like this.

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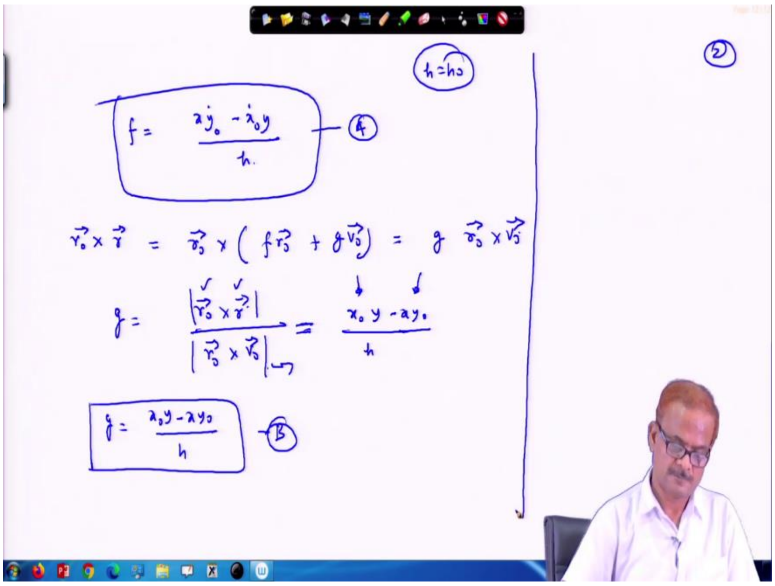

This can be reduced to f can be written as some of the steps I am taking short cut you can check it h is equal to h0 has no difference because the angular momentum remains conserved. So we write this has A. Similarly taking cross product with r_0 with r, so magnitude wise g can be written as $r_0 \times r$ magnitude divided by $r_0 \times v_0$ and this quantity is nothing but h. So we write here in the denominator is h and $r_0 \times r$ this yields us x_0y - xy_0 both of them are of same dimension no differential sign is appearing here. Thus, our g is $x_0y - xy_0$ divided by h this is a equation B.

$$f = \frac{x\dot{y}_0 - \dot{x}_0 y}{h}$$

$$\vec{r_0} \times \vec{r} = \vec{r}_0 \times (f\vec{r}_0 + g\vec{v}_0) = g\ \vec{r}_0 \times \vec{v}_0$$

$$g = \frac{|\vec{r_0} \times \vec{r}|}{|\vec{r}_0 \times \vec{v}_0|} = \frac{x_0y - xy_0}{h}$$

$$g = \frac{x_0y - xy_0}{h}$$

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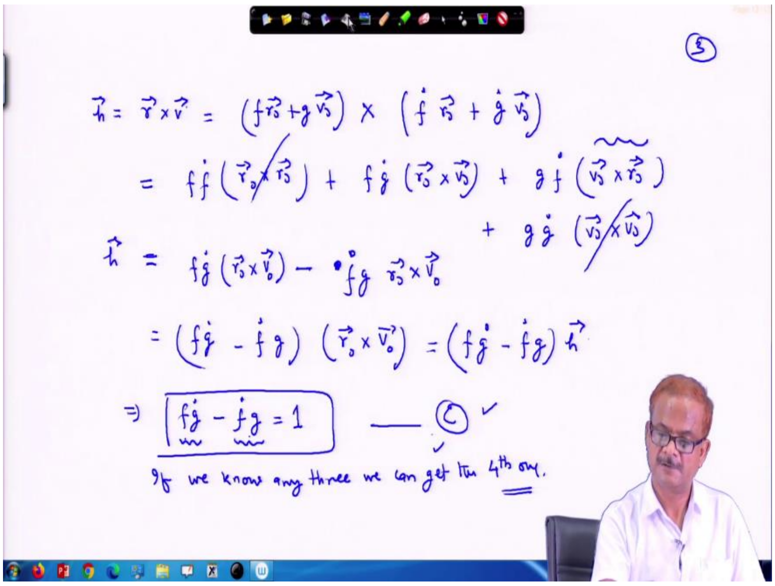

Next we write h is equal to r × v g times v_0 cross $\dot{f}$ times r0 + $\dot{g}$ times v_0. In this immediately we can see that these 2 terms will dropout and we are left with $f\dot{g}\,(\vec{r}_0 \times \vec{v}_0)$ and here $g\dot{f}\,(\vec{r}_o \times \vec{v}_o)$, minus sign I have introduced this is h, $\vec{r}_0 \times \vec{v}_0$. So this implies both side h and therefore $f\dot{g} - \dot{f}g$ this gets reduced to 1 this is equation C. So we have got f equation and we have got g question.

$$\vec{h} = \vec{r} \times \vec{v} = (f\vec{r}_0 + g\vec{v}_0) \times (\dot{f}\vec{r}_o + \dot{g}\vec{v}_o)$$

$$= f\dot{f}(\vec{r}_0 \times \vec{r}_0) + f\dot{g}\,(\vec{r}_0 \times \vec{v}_0) + g\dot{f}\,(\vec{r}_o \times \vec{v}_o) + g\dot{g}(\vec{v}_0 \times \vec{v}_0)$$

$$\vec{h} = f\dot{g}\,(\vec{r}_0 \times \vec{v}_0) - \dot{f}g\,(\vec{r}_o \times \vec{v}_o)$$

$$= (f\dot{g} - \dot{f}g)(\vec{r}_o \times \vec{v}_o) = (f\dot{g} - \dot{f}g)(\vec{h})$$

$$f\dot{g} - \dot{f}g = 1$$

And here we have relation combining both of them. Now from here from here this place $\dot{f}$ can be written as x y_0 – x_0 y this is the initial velocity and part of initial velocity therefore, we are not differentiating this, these are fixed quantities initial condition. And from here this we write as A1 and from this place similarly we can write $\dot{g}$ is equal to f_0 $\dot{y}_0$ divided by h this we write B1.

So we have A, A1, B and B1 and besides we have C which is available here. So from this what we can look at that if we have if we know any three there are four quantities involved here, so if we know any three we can get the fourth one. Once we have done this now, we need to find out the quantities because all these things are written in terms $\dot{y}$, $\dot{x}$ etc. So we need to find out these quantities.

So, initial position velocity is the argument. You know $\dot{x}_0$, $\dot{y}_0$and this is planar case and similarly x_0 and y_0 are known to us. So we do not have to worry about this.

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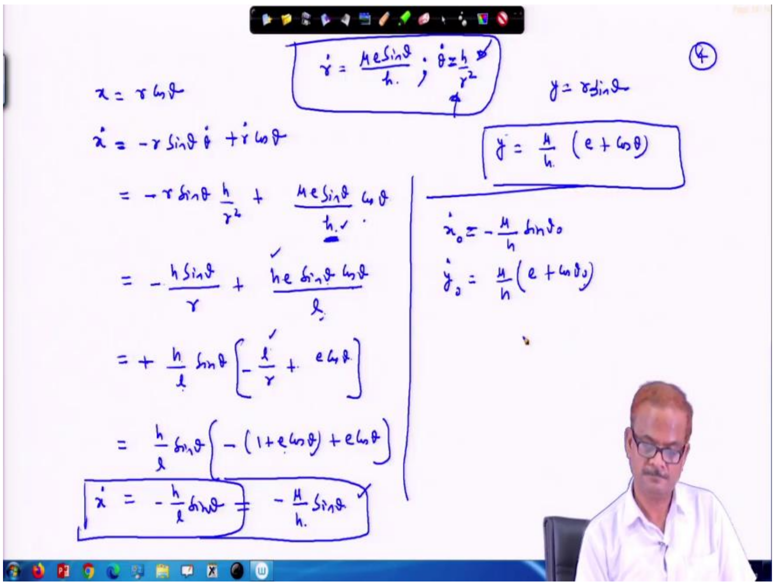
$$\dot{x} = -\frac{h}{l}\,\text{Sin}\,\theta$$



But we have to calculate x and y at any other time $\dot{x}$ and $\dot{y}$ which is appearing here $\dot{x}$and y dot. So we need to work out this quantity. So x equal to r cos θ and therefore $\dot{x}$ can be obtained from this place and similarly y also we have y equal to r sin θ. If we differentiate this will be r and sin θ $\dot{\theta}$ + r times cos θ and $\dot{r}$ we are aware of this quantity from our previous lecture $\dot{r}$ is equal to sin θ divided by h.

$$x = r\text{Cos}\,\theta$$

$$\dot{x} = -r\,\text{Sin}\,\theta\,\dot{\theta} + \dot{r}\text{Cos}\,\theta$$

So we can utilise this ok and $\dot{\theta}$ also we are aware of $\dot{\theta}$ equal to h by r^2 in the central force motion or the gravitational force what we have inverse2 field we are for the central for itself. Irrespective of inverse of square whatever this is always valid. So we can insert this value is here so this is -r sin θ h by r^2 and plus $\dot{r}$ sin θ divided by h times cos θ, -h sin θ divided by r and this can be simplified as h sin θ cos θ divided by l.

$$= -r\,\text{Sin}\,\theta\frac{h}{r^2} + \frac{\mu\,e\text{Sin}\,\theta\,\text{Cos}\,\theta}{h}$$

$$= -\frac{h\,\text{Sin}\theta}{r} + \frac{he\,\text{Sin}\,\theta\,\text{Cos}\,\theta}{h}$$

So we multiply the numerator and denominator by h so this part will become h square. Ok h^2 equal to μ l, so μ from the numerator denominator will cancel out an l will remain in the denominator and numerator you get it. So this you can check it yourself. So this we can write as + h by l sin θ - l by r h sin θ will go from this place and l also we are taking from this place comes out and appear in the numerator.

$$= \frac{h}{l}\,\text{Sin}\,\theta\left[-\frac{l}{r} + e\,\text{Cos}\,\theta\right]$$

$$= \frac{h}{l}\,\text{Sin}\,\theta\,[-(1 + e\text{Cos}\,\theta) + e\text{Cos}\theta]$$

And from this place h by l goes and sin θ go so this is e cos θ -1 l by r is -1 + e cos θ plus and -e cos θ cos θ will cancel out and you get $\dot{x}$ is equal to this quantity. This is nothing but - μ by h sin θ. Similarly, you can derive for y and this will turn out to μ by h e + cos θ. And therefore $\dot{f}_0$ because $\dot{f}$ g this equation can be satisfied at any place so $\dot{f}_0$ we can write as μ by h sin θ_0. And $\dot{y}_0$ is equal to μ by h e + cos θ_0.

$$\dot{r} = \frac{\mu\,e\text{Sin}\,\theta}{h}\;;\;\;\dot{\theta} = \frac{h}{r^2}$$

$$y = r\text{Sin}\,\theta$$

$$y = \frac{\mu}{h}\,(e + \text{Cos}\,\theta)$$

$$\dot{x}_0 = -\frac{\mu}{h}\,\text{Sin}\,\theta_0$$

$$\dot{y}_0 = \frac{\mu}{h}\,(e + \text{Cos}\,\theta_0)$$

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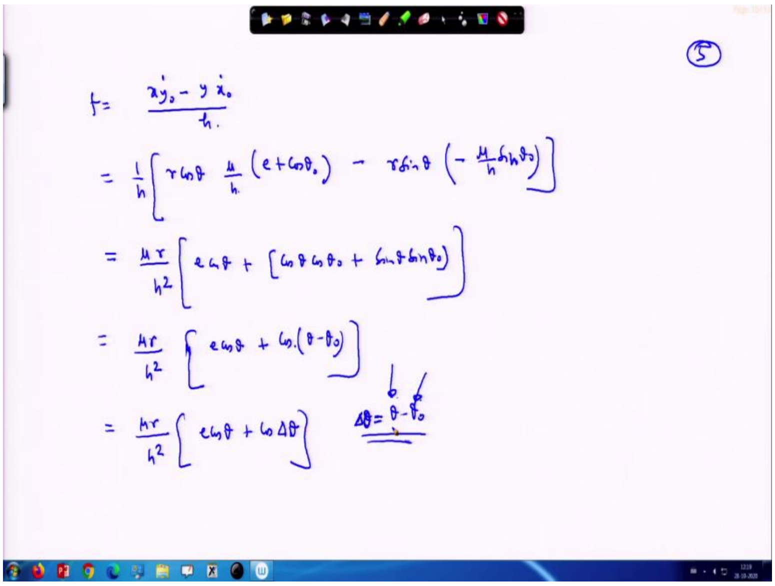

So once we have got these quantities therefore f can be written as insert all the values y equal to r sin θ and θ_0 we have to insert here and θ_0. So h we take outside r we take outside this becomes μ r by h^2 and we get here, e cos θ + cos θ times cos θ_0 + sin θ times sin θ_0. So, cos Δθ here Δ θ equal to θ minus θ_0 so this is the initial true anomaly and this is a final true anomaly. So how much your change in anomaly has taken place it can be written like this.

$$f = \frac{x\dot{y}_0 - \dot{x}_0 y}{h}$$

$$= \frac{1}{h}\left[r\mathrm{Cos}\theta \frac{\mu}{h}\,(r + \mathrm{Cos}\theta_0) - r\mathrm{Sin}\theta\left(-\frac{\mu}{h}\,\mathrm{Sin}\theta_0\right)\right]$$

$$= \frac{\mu r}{h^2}\left[e\mathrm{Cos}\theta + [\mathrm{Cos}\theta\,\mathrm{Cos}\theta_0 + \mathrm{Sin}\theta\,\mathrm{Sin}\theta_0]\right]$$

$$= \frac{\mu r}{h^2}\,[e\,\mathrm{Cos}\,\theta + \mathrm{Cos}\,(\theta - \theta_0)]$$

$$= \frac{\mu r}{h^2}\,[e\,\mathrm{Cos}\theta + \mathrm{Cos}\,\Delta\theta],\ \text{Where}\ \Delta\theta = \theta - \theta_0$$

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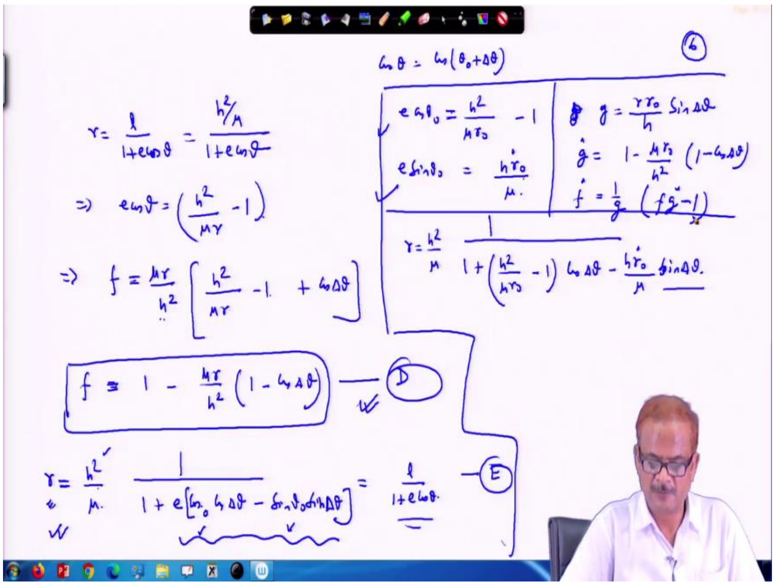

Thereafter we can also use this equation l by 1 + e cos θ equal to h^2 divided by μ h^2 by μ r – 1. So, e cos θ on the previous page this quantity we replace this part will cancel out h^2 μ r h^2 μ r if we break the bracket this becomes 1 minus, we can write this as μ r by h^2 1 - cos Δ θ. Now f is expressed in a proper way we name this as; we write this as h is equal to this quantity.

$$r = \frac{l}{1 + e\,s\mathrm{Cos}\theta} = \frac{\frac{h^2}{\mu}}{1 + e\mathrm{Cos}\theta}$$

$$\Rightarrow e\,\mathrm{Cos}\theta = \left(\frac{h^2}{\mu r} - 1\right)$$

$$\Rightarrow f = \frac{\mu r}{h^2}\left[\frac{h^2}{\mu r} - 1 + \cos\Delta\theta\right]$$

$$f = \ 1 - \frac{\mu r}{h^2}\,(1 - \cos\Delta\theta)$$

Here this is the equation in this place as D, r also can be written as h2 divided by μ which is nothing but l. So, l by 1 + e cos θ is equal to cos θ_0 + Δ θ sin Δ θ we can write it this way. So this is our usual equation l by 1 + e cos θ where e cos θ_0 is the quantity h^2 by μ r_0 -1 and e sin θ_0 is the quantity is h $\dot{r}_0$ divided by μ. This we have worked out earlier you can check from the earlier lecture.

$$r = \frac{h^2}{\mu}\frac{1}{1+e\,[\cos\theta_0\cos\Delta\theta - \sin\theta_0\sin\Delta\theta\,]} = \frac{l}{1+e\cos\theta}$$

Once we have written this substituting all these things here in this equation. So we know e cosθ here in this place and e cos θ_0 sin θ_0 is also available from this place. This quantity will be known h^2 which is already known because initial position velocity is a given position vector and velocity vector. So the r can be computed from this place. So finally, to conclude the r can be written as h^2 divided by μ times 1 + h^2 by μ r_0 - 1 cos Δ θ.

$$e\cos\theta_0 = \frac{h^2}{\mu r_0} - 1\,;\, e\sin\theta_0 = \frac{h\dot{r}_0}{\mu}$$

$$g = \frac{r\,r_0\sin\Delta\theta}{h}\ ;\ \dot{g} = 1 - \frac{\mu r_0}{h^2}\,(1 - \cos\Delta\theta)$$

$$\dot{f} = \frac{1}{g}\,(f\dot{g} - 1)$$

And then you can go and check that your f is available from this place from the equation D and rest others can be written as g equal to r r_0 divided by h sin Δ θ and I have taken some shortcut you can verify all these quantities. So using all this information you will be able to propagate the orbit find out the position at different position somewhere else. Now what we do the problem related problem I will not solve here.

But rather I will give you the solution at the time was this week starts so that you are able to solve the other problems. Ok so we stop here and we conclude this lecture here. Thank you very much.