Space Flight Mechanics
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Lecture – 99
Sphere of Influence

Welcome to lecture 99 so the last remaining topic which I thought of covering earlier, but I have not been able to do so. So I will cover it today. This is the sphere of influence. Already you looked into the interplanetary transfer that their sphere of influence definition we should know and the expression for that finding out the radius of a sphere of influence we have already worked out.

Not worked out but I have written it. So today I am going to work it out so that you understand how the sphere of influence it works.

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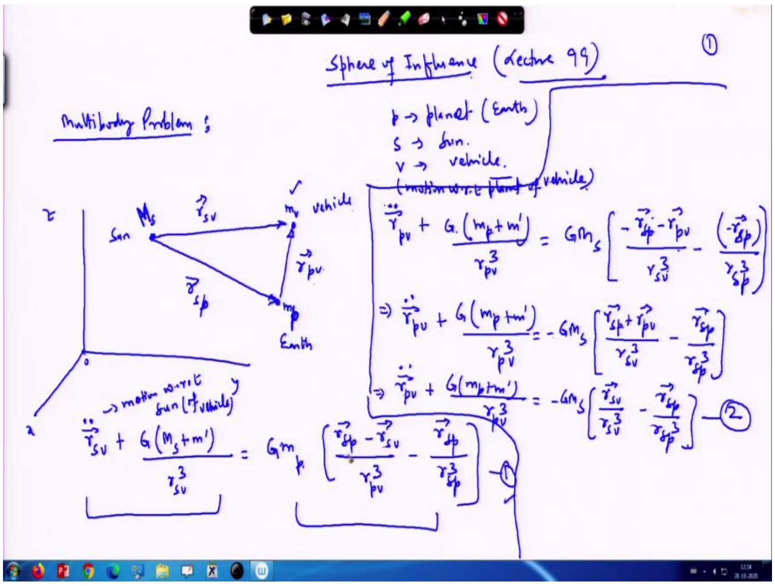

So for this we need to resort to the multibody problem. Say we have a inertial reference frame here. This is shown here, earth is here I will take the example with respect to the sun-earth system and here this is vehicle. Here p stands for planet this is our $\vec{r}_{pv}$ and $\vec{r}_{sp}$, p stands for planet. Here in this case this is earth. S stands for sun and v stands for vehicle. If you remember in the three body multi-body problem, we have written the expression for the motion of one particle with respect to the other one in the presence of many other particles.

So that expression we can utilise here. So I am not going to again rewrite it rather I will utilise it. So $\vec{r}_{sp}$ according to that you will have to refer back to the week 6, week 5 or 6 perhaps it was the three body problem. So there I have written these expressions utilised that expression to $\ddot{\vec{r}}_{sv}$ + G times and this mass we will assume to be m_p and this mass we write as m earth or m planet we can write.

So this is m_p and here we write as M sun planet sitting writing it is better because it can be earth it can be Mars or any other thing but mv is ok. m_v is the mass of the vehicle. So M sun, I will make it in capital letter M sun and rest others will keep it is small. So, motion of the vehicle with respect to the sun. This is very simple you know that this is the basic equation of the two-body problem on the left hand side what I have written r_{sp} whole cube.

And the other terms on the right-hand side, we will have only the perturbation term. So this is the basic 2 body problem and this is the perturbation term which is appearing here. Let us say this is equation 1. Similarly, for the motion of the vehicle with respect to the planet it can be written as $\ddot{\vec{r}}_{pv}$ this is the motion of the vehicle with respect to sun motion with respect to sun of vehicle.

$$\ddot{\vec{r}}_{sv} + \frac{G(M_s + m')}{r_{sv}^3} = Gm_p \left[\frac{\vec{r}_{sp} - \vec{r}_{sv}}{r_{pv}^3} - \frac{\vec{r}_{sp}}{r_{sp}^3}\right]$$

And here we are writing Motion with respect to planet of vehicle. So $\ddot{\vec{r}}_{pv}$ G times the two-body problem we have to write here m_p times m prime on the left hand side always it will be like that and r_{pv} whole cube. And on the right hand side the perturbation term goes so that will be due to the sun. So GM sun – r_{sp} look back into the derivation I have done earlier. I am not going to explain all these terms here.

$$\ddot{\vec{r}}_{pv} + \frac{G(M_p + m')}{r_{pv}^3} = GM_s \left[\frac{-\vec{r}_{sp} - \vec{r}_{pv}}{r_{sv}^3} - \frac{(-\vec{r}_{sp})}{r_{sp}^3}\right]$$

Why I am putting the minor sign after at the end, you should know all these things. So, this can be arranged as m planet plus m′ r_{pv} whole cube minus sign will take it outside r_s planet plus r_{pv} and little bit of effort we can further rewrite this $r_{sp} + r_{pv}$ this is nothing but r_{sp}. This is r_{sv} divided by r_{sv} whole cube minus r_{sp} by r_{sp} whole cube. So this we write as equation 2 and this is our equation 1 this can also be rewritten.

$$\Rightarrow \ddot{\vec{r}}_{pv} + \frac{G(M_p+m')}{r_{pv}^3} = -GM_s\left[\frac{\vec{r}_{sp}+\vec{r}_{pv}}{r_{sv}^3} - \frac{(\vec{r}_{sp})}{r_{sp}^3}\right]$$

$$\Rightarrow \ddot{\vec{r}}_{pv} + \frac{G(M_p+m')}{r_{pv}^3} = -GM_s\left[\frac{\vec{r}_{sv}}{r_{sv}^3} - \frac{(\vec{r}_{sp})}{r_{sp}^3}\right]$$

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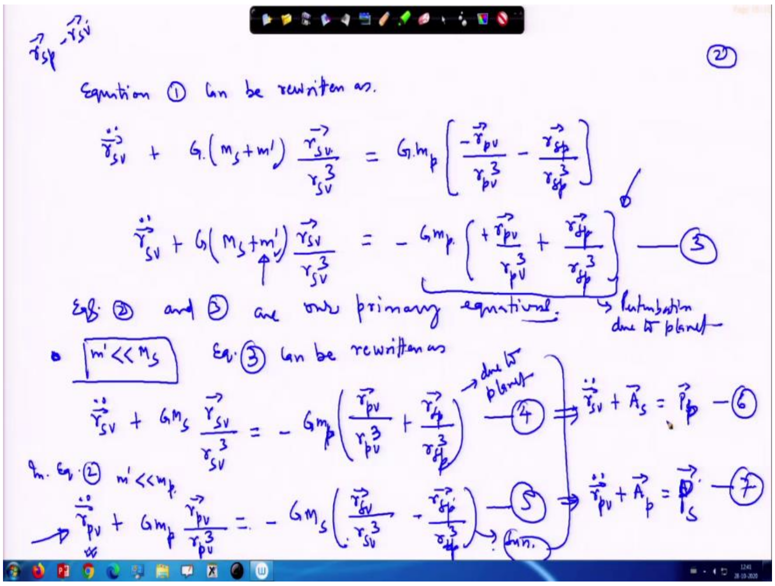

Equation one can be rewritten as $\ddot{\vec{r}}_{sv}$+ G times M sun + M Prime + r_{sv} divided by r_{sv} whole cube equal to G times M planet and rest of the things we have to rearrange. So if you look there the vector in the numerator was on the first term on the right hand side r_{sp} – r sv. So r_{sp} – r_{sv} this is the quantity – r_{pv} here this comes as r_{pv} with minus sign and divided by r_{pv} whole cube.

$$\ddot{\vec{r}}_{sv} + G(M_s + m')\,\frac{\vec{r}_{sv}}{r_{sv}^3} = Gm_p\left[-\frac{\vec{r}_{pv}}{r_{pv}^3} - \frac{\vec{r}_{sp}}{r_{sp}^3}\right]$$

$$\ddot{\vec{r}}_{sv} + G(M_s + m')\,\frac{\vec{r}_{sv}}{r_{sv}^3} = -Gm_p\left[\frac{\vec{r}_{pv}}{r_{pv}^3} + \frac{\vec{r}_{sp}}{r_{sp}^3}\right]$$

And then rest of the things we have to copy from the previous one. So these becomes plus r_{pv} whole cube. So just we write as equation 3. So we have equation 2 and equation 3. These are equation 2 and 3, equations 2 and 3 are our primary equation which we need to use and solve the problem. Now this is because this is a satellite ok. So therefore, this small m prime this will be much, much smaller than the mass of the sun.

$$\ddot{\vec{r}}_{sv} + GM_s\,\frac{\vec{r}_{sv}}{r_{sv}^3} = -Gm_p\left[\frac{\vec{r}_{pv}}{r_{pv}^3} + \frac{\vec{r}_{sp}}{r_{sp}^3}\right]; \; m' << M_s$$

Therefore, this term can be neglected here. So equation 3 can be rewritten as r_{sv} double dot + G times M sun r_{sv} by r_{sv} whole cube - G times M planet we are not do the planet Mars is also much less than the sun mass but we are not neglecting it because if we neglect the right hand side will be zero. Ok, that means a perturbation term is gone and it is getting reduced to two body problem.

$$\ddot{\vec{r}}_{pv} + GM_p \frac{\vec{r}_{pv}}{r_{pv}^3} = -Gm_s \left[\frac{\vec{r}_{sv}}{r_{sv}^3} - \frac{\vec{r}_{sp}}{r_{sp}^3}\right]$$

But perturbation it depends on the distance from the sun where the vehicle is located on that the perturbation will depend due to the planet. And this right-hand term perturbation is due to the planet in equation 3. This is over equation 4. Similarly, the equation 2 can be rewritten episode 2 can be written here. The mass of the satellite with respect to the planet will be negligible. So therefore, here also the m' we can draw.

So in equation 2 m' is much less than M planet. So there are also we can draw the m prime and write this as G times m planet r_{pv} by r_{pv} whole cube under right hand side perturbation due to Sun r_{sp} by r_{sp} whole cube. So these are two equations we have got in a reduced format. And this equation we can rewrite as $\ddot{\vec{r}}_{sv}$ plus acceleration due to Sun this is equal to perturbation due to planet p.

$$\Rightarrow \ddot{\vec{r}}_{sv} + \vec{A}_s = \vec{P}_p$$

This is perturbation due to planet and this is due to Sun because this is the motion we have written about the planet motion of the vehicle about the planet and where sun is acting as a perturbating agent. Similarly this one can be written as $\ddot{\vec{r}}_{pv}$ plus e acceleration due to the planet and on the right hand side acceleration perturbation due to Sun. So this is our sixth equation and 7th equation.

$$\Rightarrow \ddot{\vec{r}}_{pv} + \vec{A}_p = \vec{P}_s$$

So the ratio so I will write it on the next page. So the ratio P_p divided by A_s and P_s divided by A_p these two they so decide about the sun or about the planet. We can consider the motion of this vehicle to be a two-body problem. Ok and that will go to the next page and look into that.

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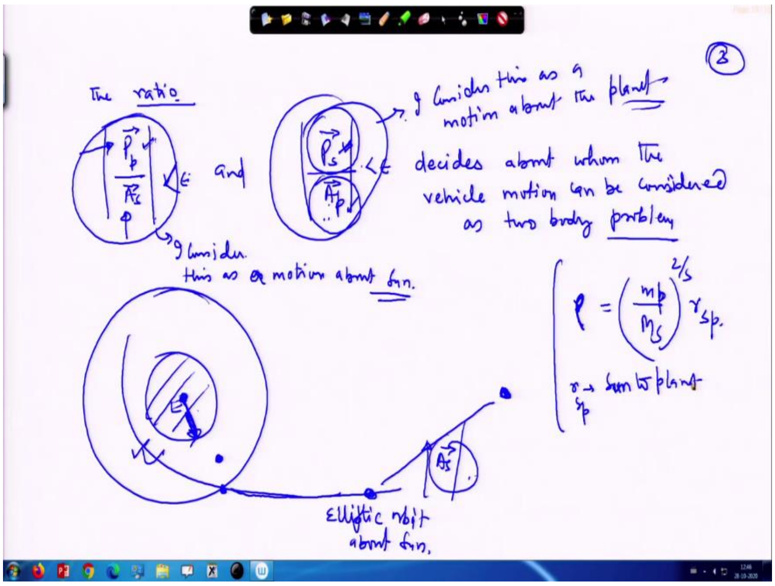

So the ratio P_P by A_S and P_S by A_P decides about whom the vehicle motion can be considered as 2 body problem. Now why I did not consider this part in the three-body problem you can appreciate. Once we have done that can just rejected transfer problem, we have looked into you know that this is the Earth here and this is a sphere of influence and once the satellite is going out of this.

This is your sphere of influence so you remember that within this we have considered as the two body problem. And this we have considered as the hyperbolic orbit. And thereafter from here we are considering as the elliptic orbit about sun. So you can appreciate now if the perturbation due to the planet and here perturbation due to the sun. If the perturbation due to the sun divided by acceleration due to the planet considered this term.

If this is very small if perturbation due to the sun and perturbation due to the acceleration due to the planet if this is very small means this quantity in the denominator its magnitude becomes very high as compared to perturbation from the Sun. So in that condition you can consider that your planet this vehicle is moving about the earth and you can consider this as a two body problem. On the other hand, if acceleration due to the sun is very high as compared to the perturbation due to the planet in some where your say the vehicle is lying here.

So if the vehicle is lying here the perturbation due to the sun will be a small as compared to the acceleration due to this planet. On the other hand, if your vehicle is lying here sun is here. So the acceleration due to the sun is this will be much larger, its magnitude will be much larger than the perturbation due to the earth because sun is very massive as compared to the Earth or Mars or Jupiter whatever ok.

So if this quantity is very small general considered motion about the; so if this quantity is less than certain quantity say epsilon so I can consider this is a fraction number. So can I consider this as a motion about Sun and if this quantity is less than certain quantity epsilon so I considered this motion about the planet. So after being the interplanetary transfer you are able to appreciate why we are going through this without this you would not have appreciated it.

So therefore what we will derive now that the radius of influence which we have shown by Rho in the interplanetary transfer is we are shown by rho about the planet this can be given by M planet divided by M sun to the power to 2 by 5 times r_{sp} sun to planet distance, r_{sp} is sun to planet distance. This is what we are going to work out.

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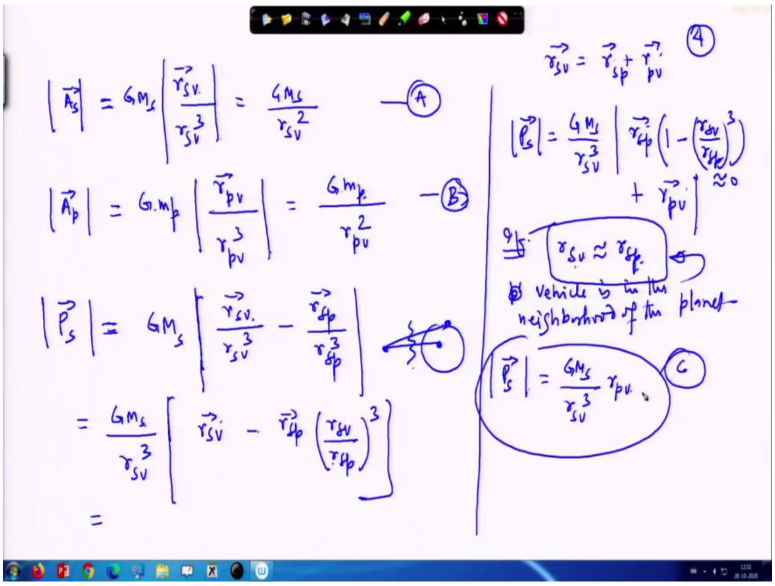

So A_S is the quantity. This is the acceleration due to the sun r_{sv} its magnitude we are interested in, I will number this as A. Similarly A planet its magnitude will be G m planet and r planet vehicle divided by r planet to vehicle distance whole cube. This is B. Now perturbation part we write. Perturbation due to the sun GM times this part we are using perturbation due to sun this particular part –G times M_s r_{sv} magnitude r_{sv} – r_{sv} whole cube – r_{sp} by r_{sp} whole cube minus sign got eliminated because we are taking magnitude.

$$\left|\vec{A}_s\right| = GM_s\left|\frac{\vec{r}_{sv}}{r_{sv}^3}\right| = \frac{GM_s}{r_{sv}^2}$$

$$\left|\vec{A}_p\right| = G.m_p \left|\frac{\vec{r}_{pv}}{r_{pv}^3}\right| = \frac{GM_p}{r_{pv}^2}$$

G M_s by r_{sv} taken outside so this gets reduced to r_{sv} and r_{sv} we are aware of r_{sv} is nothing but r_{sp} and plus r_{pv}. So if we use this expansion here. Perturbation due to the sun then gets reduced to G M_s divided by r_{sv} the whole cube r_{sp} we can taken outside. In one step I am doing it skipping one step here. You can check here r_{sv} divided by r_{sp} whole cube and times r_{pv} magnitude.

$$\left|\vec{P}_s\right| = GM_s \left|\frac{\vec{r}_{sv}}{r_{sv}^3} - \frac{\vec{r}_{sp}}{r_{sp}^3}\right|$$

$$= \frac{GM_s}{r_{sv}^3}\left|\vec{r}_{sv} - \vec{r}_{sp}\left(\frac{r_{sv}}{r_{sp}}\right)^3\right|$$

So if r_{sv} this magnitude is nearly equal to r_{sp} ok that means the planet is the vehicle is in the neighbourhood of the planet. Then this is valid. Here sun is here in this place this is vehicle, planet is here. So see that these distances are comparable. So these will be nearly equal to quantity in the bracket nearly equal to 0 because this is 1-sv by sp.

$$\vec{r}_{sv} = \vec{r}_{sp} + \vec{r}_{pv}$$

$$\left|\overrightarrow{P_s}\right| = \frac{GM_s}{r_{sv}^3}\left|\vec{r}_{sp}\left(1 - \left(\frac{r_{sv}}{r_{sp}}\right)^3\right) + \vec{r}_{pv}\right|$$

$$\text{If } r_{sv} \approx r_{sp}$$

$$\left|\overrightarrow{P_s}\right| = \frac{GM_s}{r_{sv}^3}\, r_{pv}$$

And sv by sp we are writing equal to; so your P sun is equal to perturbation due to Sun this gets reduced to G M_s by r_{sv} whole cube times r_{pv}. So this is equation C.

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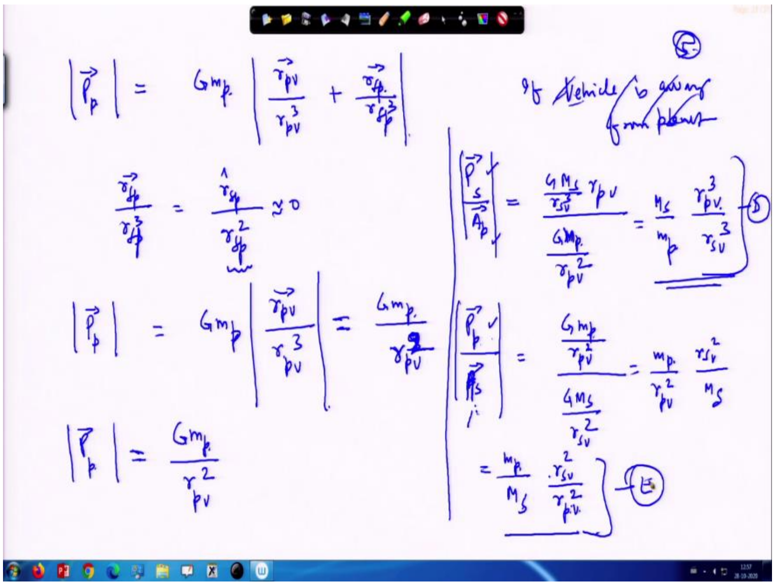

This we have got perturbation due to the sun. Similarly, for perturbation due to the planet we can compute. G m planet and we go back and look into the expression. Here we have written. This one the 4th perturbation due to planets, the right hand side we are considering. So –G mp so the magnitude we have to take care of r_{pv} by r_{pv} cube. So r_{pv} by r_{pv} whole cube + r_{sp} by r_{sp} whole cube.

$$|\vec{P}_p| = GM_p \left| \frac{\vec{r}_{pv}}{r_{pv}^3} + \frac{\vec{r}_{sp}}{r_{sp}^3} \right|$$

$$\frac{\vec{r}_{sp}}{r_{sp}^3} = \frac{\hat{r}_{sp}}{r_{sp}^3} \approx 0$$

So here in this case if vehicle is away from planet or other way let say this is r_{sp} divided r_{sp} whole cube. So this we can write as r_{sp} cap is unit vector by r_{sp} square. So sun to planet distance it will be always very large. So this quantity can be neglected and therefore perturbation due to planet it primarily depend on G m_p divided by r_{pv} multiplied by this and r_{pv} whole cube.

$$|\vec{P}_p| = GM_p \left| \frac{\vec{r}_{pv}}{r_{pv}^3} \right| = \frac{GM_p}{r_{pv}^2}$$

$$|\vec{P}_p| = \frac{GM_p}{r_{pv}^2}$$

This is nothing but G mp divided by r_{pv} whole square this equation D. So P p the perturbation due to the planet this turns out to be G times mass of the planet by r_{pv} square. Once we have done this now we can compute all the related quantities. So first of all we calculate P s by A p perturbation due to the sun divided by acceleration due to the planet. Perturbation due to sun we have calculated already this is the quantity your G ms by r_{sv} whole cube.

G times M sun by r_{sv} whole cube times r_{pv} and in the denominator A P acceleration due to the planet which is; we have written somewhere this is in this part G m_p by r_{pv} square G m_p by r_{pv} square. So if we rewrite this, this gets reduced to M_s by M planet times r_{pv} by r_{sv} whole cube. Similarly, we can compute P planet by perturbation due to Sun whole square and perturbation due to Sun.

$$\left|\frac{\overrightarrow{P_s}}{\overrightarrow{A_s}}\right| = \frac{\frac{GM_S\, r_{pv}}{r_{sv}^3}}{\frac{Gm_p}{r_{pv}^2}} = \frac{M_S}{m_p}\ \frac{r_{pv}^3}{r_{sv}^3}$$

Perturbation due to sun we have written G M_s by r_{sv} whole cube. We have to be careful in writing this is perturbation due to Sun and acceleration due to planet. This is perturbation due to Sun and acceleration due to sun. So now we have to look for A s. A s we have written here in this place. So, G M_s by r_{sv} whole square and this quantity can be written as G G will cancel out m_p by r_{pv} whole square times r_{sp} whole square divided by M_s.

$$\left|\frac{\overrightarrow{P_p}}{\overrightarrow{P_s}}\right| = \frac{\frac{Gm_p}{r_{pv}^2}}{\frac{GM_S}{r_{sv}^2}} = \frac{m_p}{r_{pv}^2}\ \frac{r_{sv}^2}{M_S} = \frac{m_p}{M_S}.\frac{r_{sv}^2}{r_{pv}^2}$$

This is m_p by M_s times r_{sv} square by r_{pv} square here. So, this 2 equations A, B, C and we write this as D and this is E. So from D and E the sphere of influence so suppose this is the Earth suppose and this is a sphere of influence of the earth and this is sun here. So on this sphere of influence the perturbation due to the sun and acceleration of the planet this 2 so they will become equal. D and E will become equal in magnitude. So this gives us the sphere of influence of that particular planet.

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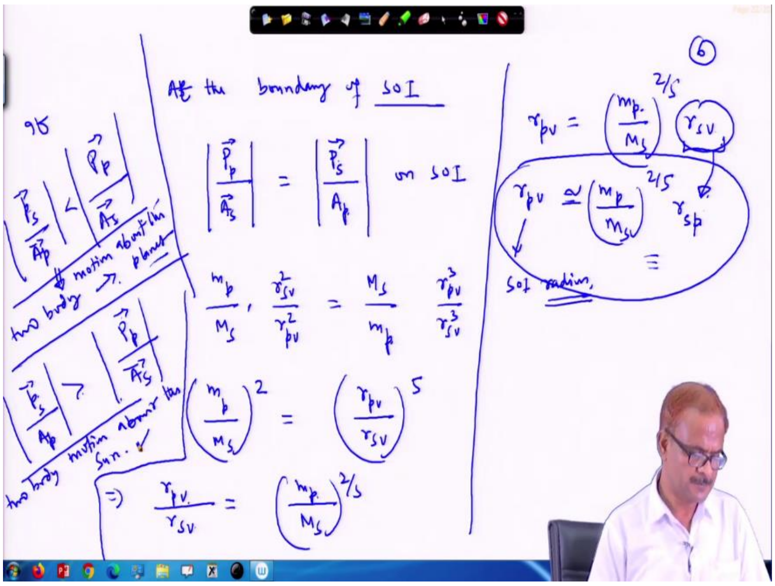

So at D boundary of sphere of influence P planet divided by A sun this will be equal to parturition due to the sun divided by acceleration due to planet. These 2 will become equal and already I have discussed what does mean by these 2 results? On the other hand this is on SOI. Immediately we can write the value of P_p by A_s is m_p by M_s times r_{sv} by r_{pv} square, square and then the P sun this part M_s by m_p.

$$\left|\frac{\overrightarrow{P_p}}{\overrightarrow{A_s}}\right| = \left|\frac{\overrightarrow{P_s}}{\overrightarrow{A_p}}\right|$$

$$\frac{m_p}{M_s} \cdot \frac{r_{sv}^2}{r_{pv}^2} = \frac{M_s}{m_p} \frac{r_{pv}^3}{r_{sv}^3}$$

And then r_{pv} by r_{sv} whole cube. So from here what we get if we try to rewrite it m_p by M_s square this will be equal to r_{pv} by r_{sv} to the power 5. This implies r_{pv} by r_{sv} M planet by M_s to the power to 2 by 5 and r_{pv} will be equal to m_p by M_s to the power to 2 by 5 times r_{sv}. So this is sun to vehicle distance ok this is sun to vehicle distance. So sphere of influence if it is in the influence of the planet.

$$\left(\frac{m_p}{M_s}\right)^2 = \left(\frac{r_{pv}}{r_{sv}}\right)^5$$

$$\frac{\mathrm{r_{pv}}}{\mathrm{r_{sv}}} = \left(\frac{\mathrm{m_p}}{\mathrm{M_s}}\right)^{\left(\frac{2}{5}\right)}$$

$$\mathrm{r_{pv}} = \left(\frac{\mathrm{m_p}}{\mathrm{M_s}}\right)^{\left(\frac{2}{5}\right)} \mathrm{r_{sv}}$$

So this quantity can be written nearly equal to r_{sp} $_{sv}$ can be replaced by$_{sp}$ this we are replacing. So this gives you then the radius of influence, sphere of influence radius. If on the other hand

P s by; so thus we have calculated the sphere of the equation. I have written earlier that this is a equation expression for the sphere of influence. This is the way we derive it.

$$r_{pv} \approx \left(\frac{m_p}{M_s}\right)^{\left(\frac{2}{5}\right)} r_{sp}$$

Now one more thing I would like to write here. So if P_s by A_p this magnitude if this is less than parturition due to planet divided by acceleration due to the sun then we will we can consider this is the motion about the planet. This interior to the SOI, so it is a two-body problem we can consider about the planet. On the other hand if we have P_s by A_p this is greater than perturbation due to a planet divided by A_s then we can consider this to be the two body motion about the sun.

That means the of gravity of the sun is more effective in that case that means it is out of sphere of influence of the earth. So we conclude it here whatever the basic concepts required to discuss it. I have covered it. So thank you very much and thus the course comes to an end rest we will skip discussing the problem. So I will keep posting now that some of the solution to some problem except that tutorial one.

But I will give you some extra material also to solve problem so that you get to know how to work out various problems. There was no time here to discuss many problems in the class because we have covered lectures in going over to so many lectures. So thank you very much for being with me. All the best; for your exam.