Aircraft Structures - 1

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Lecture No -24
Theory of Elasticity - Equilibrium

Welcome back to aircraft structures one course this is professor Anup Ghosh from Aerospace Engineering department, IIT Kharagpur. We are in the lecture of 5th week module 5 and we will see how the equilibrium equations are derived we are in the discussion of theory of elasticity which is very, very fundamental of solid mechanics if we say. And we have started in our last lecture about definition of stress.

In this lecture we will be considering the equilibrium condition. So, equilibrium has to maintain how mathematically we express the equilibrium in case of structural analysis or solid mechanics that we will learn without equilibrium probably nothing exist. In case of our analysis also we will see how the stress gets connected with the equilibrium or external surface loads or body force. So with that note let us proceed further.

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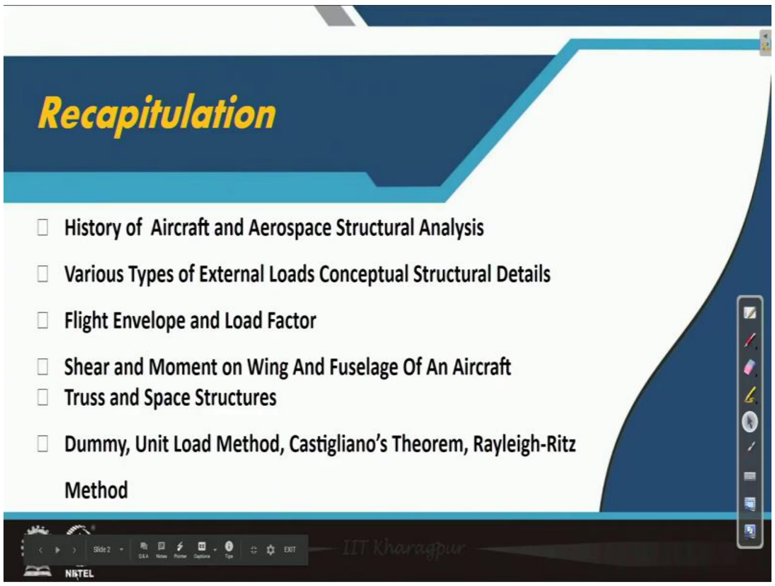

Before we proceed further as usual what we need to consider is that what we have learnt so far. In a quick way we will say that we have learned a history of aircraft and structural analysis. We have learned various types of external loads conceptual structural details flight envelope load factor shear and movement on wing and fuselage of an aircraft. Truss, three dimensional stress, space structure dummy or unit load method. We have considered we have also learned different energy methods to find out deflection.

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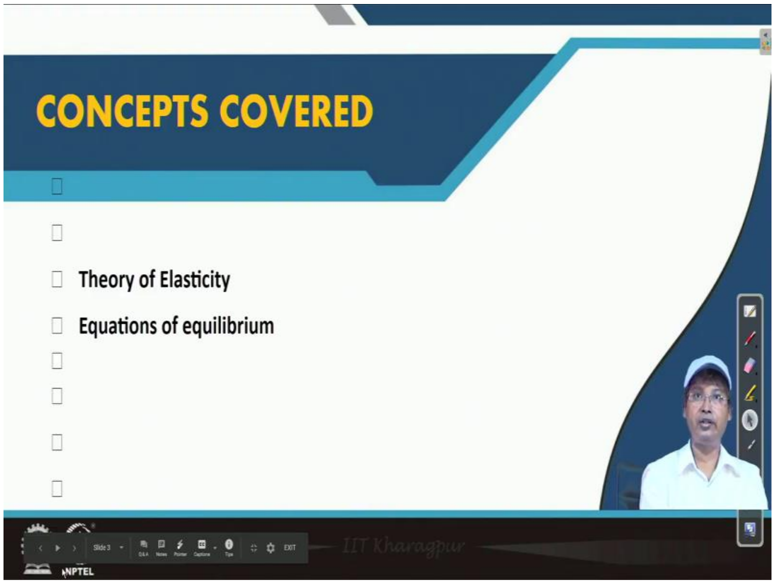

And then we move forward we have already learned the definition of stress how does it come and here we will be considering the equilibrium equation in this lecture.

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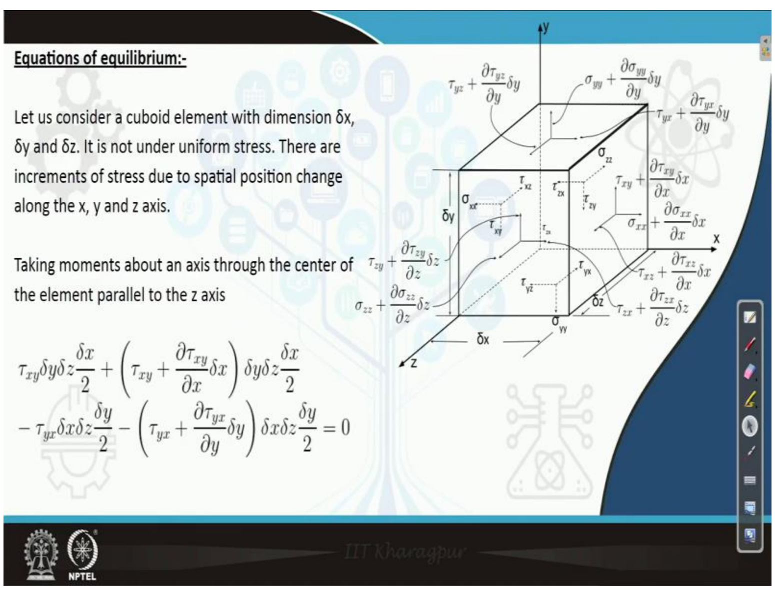

So as we have already mentioned that we will be learning equations of equilibrium or in a different word if we say that how the equilibrium is maintained while a structure is loaded that we will see. And as in the last lecture we have mentioned that the uniform stress condition is a very rare case it is difficult to find out. But in the other case which is most common that case in a generalized manner is shown in this particular slide.

The particular slide consists a huge thing on the figure the figure what is shown here this figure consists many thing. So, we will describe we will try to learn those things. And let us see what are the things how those things we need and how can we find out the equilibrium equation. Let us consider a cuboid element with dimensions dx dy and dz. So, this is dx along the x please note that this length is dx this is dx this is dx this is dx dy and dz.

Similarly dy is this length as it is mentioned once here dy dy dy or this is dy or dz dz is this length this length this length or this length. Now directly we are saying taking moment about the axis but before we take a moment about the axis it is better to note that here what are the things shown in this cuboid. This cuboid is not similar to the last cuboid what we have seen. In the last cuboid in both the x planes what was on the origin the plane on the origin.

This is the origin this is plane passing through the origin and this is a plane which is dx apart one more x plane. In the previous case we assume that since the stress considered was uniform in both the cases stresses were same but in this particular case these are not same. As I have mentioned or we have observed that its get it the stresses changes its value how we do not know how does it change but we can have a function which is a variable of xy z following that the sigma x or the all the 9 components of stresses changes its value.

So say in that all the stress components in the for the first example say sigma xx is a function of is a function of xy z. If it is a function of xy z for the increment along the length delta x we can find a gradient and this gradient if we multiply by the length dx this is the change for the length plane from this plane, the plane this plane to this plane for sigma x. Similarly for the that is what is shown here, this component is shown here sigma x plus this component.

Similarly tau xy if we consider tau xy on the plane x acting on the y direction change in the x direction that is why we are considering derivative with respect to delta x partial derivative with respect to x and for the length delta x this is the total value of change of tau xy that is added to tau xy that gives us the value in this plane. Similarly if we see this is delta tau xz del x del tau xz del x is the gradient and multiplied by delta x.

So similar way if we see, this plane and this plane will always have some incremental value according to the position of the plane that is the reason here all the three components are considered as partial derivative with respect to x multiplied by delta x. Here partial derivative with respect to y multiplied by delta y same. In this case partial derivative with delta z in multiplied by dz. So, all these three components we can see.

Now we have imbalance right which is a common case most general case now let us try to see whether all these 9 components what we have given by sigma ij are different or they have some relation in between. So, to consider that the first property what we will be trying to prove is the complementary stress equality. To do that what we need to do as it is mentioned here that we are taking a moment through the center of the element parallel to z axis.

So what is that parallel to z axis we are considering one axis like this and we are considering moment of all the forces this is the axis better we increase that makes better visual representation. About this axis we are considering moment what are the moment components will come we have already discussed a moment in three dimension in one example while we are solving truss problems.

Say let us start with the x plane, in this x plane this will not have any component because it is passing through this, this will not have any component whereas this is along the axis that is why you do not have any component whereas this will have some component because the perpendicular distance from this to this and it is acting in the transverse direction to that axis. So, if that is having a component similarly in the other plane this also will have a component that is what is written here tau xy multiplied by this area this area is delta y and delta z this is delta y this is delta z this is the area so it becomes force.

What is the momentum that is delta x by 2 this is delta x by 2 so that is the momentum. So, once we understand any one component of this equation we can understand all the components very easily that is the reason we are trying to do. So, this is a small mistake this distance is not this one actually the distance is this distance this is also delta x by 2. So, for this component this delta x by 2 is acting this component so that component is written here tau xy plus incremental value area makes it force and this is the momentum.

What else? What other components will come this is about x plane fine what will come from this z plane? Nothing will come because this component this component or this component or are acting in that same line. So, there will be no component coming from the z plane. Similarly this will not have any component but we will have a component from the y plane. This will have a component this will not have any component because it is along the line of that axis what we are talking about there is no difference.

This is also will not have any component because it is along this line whereas this will have a component this one. So, tau yx similarly this will have a component tau yz I think this is a mistake this should be yx this should be yz. So, in in the other figures also probably it is there we please correct it. So, it is acting in the x direction so it must be yx it is in the y plane. So, please correct in the future one if I remember I will also correct it is a simple copy paste error please ignore that.

So tau yx this tau yx multiplied by the area delta x delta z delta x delta z force and component as we have seen. This is; this component this is will come as delta y by 2 delta y by 2 and similarly the minus will come later let us see the components first and similarly this is the other component delta yx delta tau yx del y del y area and this is the moment r. Now why it is positive and why it is negative.

If we look at the components this is acting in the positive direction that means the right following the right hand screw system. So, it is rotating in following the right hand screw system so considering that as positive these components are considered this components are considered as positive and the other one which is acting this way it is following the left hand screw system if you look at so that is the reason this is having negative direction.

Now let us move to the next slide this slide has already become very, very dirty let us clean it and move it. So, please note that this is a small mistake you please ignore that and whatever corrected that value you can consider and proceed.

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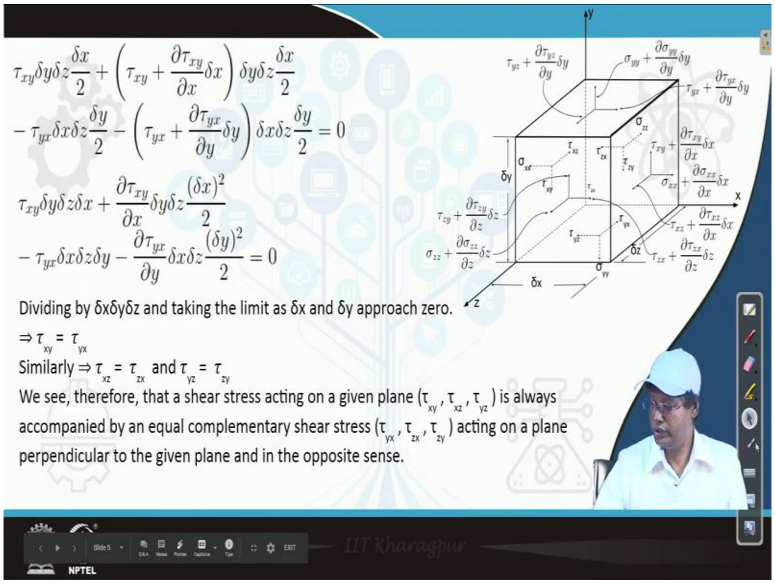

So the next slide this equation is repeated the first equation just to have a reminder. As we have pointed out earlier this is a small correction this is yx and this is yz. So, what happens if we simplify this if we go for simplification we have components these two half, half will add up and give this component and we have the differential component separate where we have delta x square by 2.

Similarly here also we have that component this is half of makes it one and we have the other component. Now what we can do divide this equation by del x del y del z or the volume of that cuboid and taking the limit del x del y approaches to 0 what we have is that if you do you can easily see that that tau xy is equals to tau yx. So, this considering the limit the remaining dx and dy this part will go to 0 and will these two will remain these two are getting divided so that is not coming into picture you please solve it and you find that tau xy is equals to tau yx.

So what is happening three components are reducing from 9 we have 6 component of stresses because we can easily prove the same thing tau xz is equals to tau zx tau yz is equals to tau zy. So, that makes us that there are 6 components we see therefore that a shear stress acting on a given plane tau xy tau xz or tau yz is always accompanied by an equal complementary shear stress of tau yx tau zx tau zy acting on a plane perpendicular to the given plane and in the opposite sense.

So this is important point you should keep it in mind shear stress never comes alone it is always having a complementary part and we need to consider that whenever we are doing any analysis. So, with this node we move forward for equilibrium condition.

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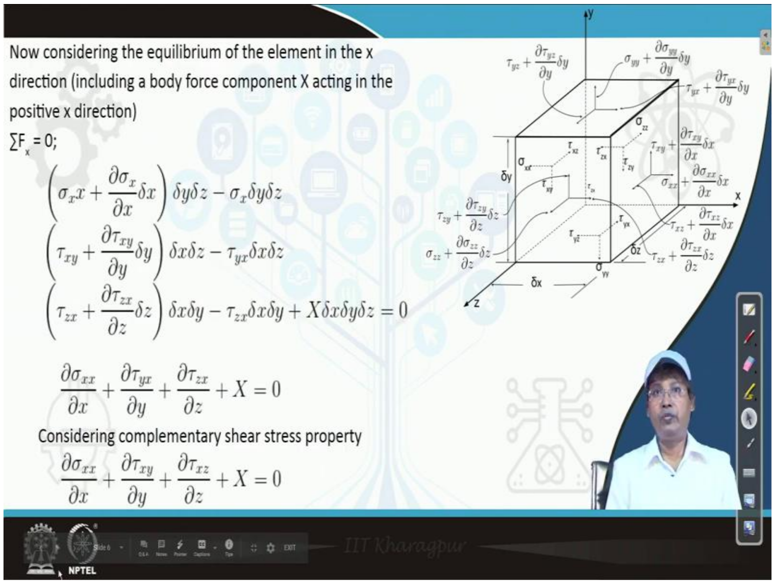

Now one more thing we need to establish that is the equilibrium condition we need to establish now considering the equilibrium of the element in x direction. Here a new thing comes in that is the body force. Now considering the equilibrium of the element in x direction including a body force component x acting in the positive x direction. Body force is the force which acts on the volume of the system here it is the structure.

It is gravitation is a very, very good example gravitational force is the very good example of body force electromagnetic force or some other forces may be visualized as body force. So, say if it is a body force is acting so inside the structures with the stress components how the body force keeps the equilibrium that is what we need to establish and we will try to establish in mathematical equations form.

So, to do that we are what we are doing it is same cuboid please consider these two corrections better I do because I remember this is yx this is yz. So, we are considering the x direction sum of all forces equals to 0 here interesting point that we will have how many components 6 components because in this plane there are 6 planes and each plane we have one x component force.

So, considering the normal one first that is sigma xx this is also subscript sigma xx plus del sigma x del x dx this acting on the area this area minus this we are considering this way positive this way positive minus this this component similarly this will come xy del xy del y that means this component minus this component this component always the area is considered to make it force, stress to force.

And we have one more here this force tau xz del z this minus this now since we are we have talked about the body force that is acting on the volume that is del x del y del z of intensity x we put it we make it to 0 it is in equilibrium. Equilibrium has to be mentioned as we have done here and if we simple observation gives us the equation that del sigma is del x plus del tau x tau yx or xy del y plus tau tau del tau zx or xz by del x del z plus x is equals to 0.

So considering the complementary stress property as I was mentioning that we generally say it in this form that partial derivative of normal stress and two shear stress components in y and z plane plus the body force is equals to 0. So, this is considering on the on the x direction is not it this is considering x direction what about y? What about z? We will definitely have similar equation.

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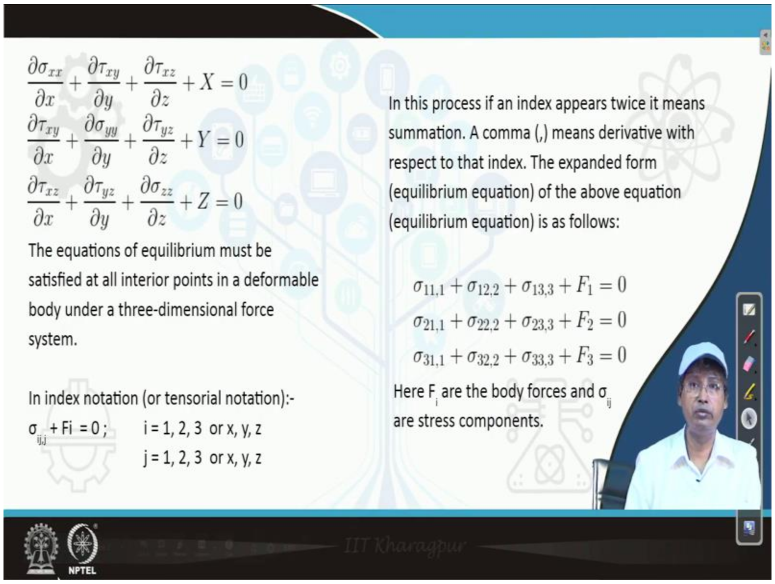

If we consider those directions equilibrium and we can write those without going into the same type of discussion that that similar equilibrium equations hold in all the three directions and we have the equations as mentioned here this is in the y direction. Please note that the normal stresses maintain the direction del sigma is del x del x this is the other two del tau xy del y delta xz del z.

Similarly this is delta del tau xy del x del sigma y by del y tau yz del z this is y because this is y and similarly this is z this is z and this is not difficult to memorize better you keep in mind this equation. These equations are good equations to remember and it helps in many, many way in many situation if you go for further study. The equations of equilibrium must be satisfied at all interior points in a deformable body under a three dimensional force system.

So under three dimensional force system this is what we have. One more way it may be written that is most popular to understand and that is generally followed in advanced books but that is the reason unless it is introduced now it will be difficult for you to follow those that so the tensorial notation or index notation as we have already introduced to we need to do. This F i is a subscript please note it down this is subscript so what we write that sigma ij, j plus+ F i is equals to 0 for i equals to 1 to 3 or xyz and j equals to 1 to 3 or xyz.

Or as we have seen in the previous lecture i, j equals to 1 2 3. So, there is a comma here what does that mean in index notation there is a repetition of j here what does that mean in index notation? How this notation expresses these three equations at a time. This notation at a time expresses all these three equations that we need to note. So, if anyone says this actually he is talking about these three equations how?

In this process if an index operator appears sorry index appears twice in it means summation and a comma j is twice here that that will yield these plus signs this is and the comma means derivative with respect to that index comma j means this sigma ij is per taken a partial derivative with respect to j so this since it it is a comma j it is partial derivative that is the reason here it is; say the first term if we talk about i equals to 1 j equals to 1 that gives me that del sigma 11 since comma is there del x plus is there.

So that is the reason we have del then j will change keeping one constant j is changing what will happen del sigma 12 divided by del y better to say I think I say it in xy that will make it easier to understand del sigma xx this is x this is xyz like that del sigma del x plus del sigma xy del y second value of this plus del third one sigma xz divided by del z this plus x 1 this the first subscript remains same is that is the reason x here it is F i this is fF i or if 1 is equals to 0.

So if I expand this similarly if we consider two next time and then again this this this if we consider we get the second equation and similarly we can have the third equation also. So, that is the reason that is the reason it is stated in this here in a different form this is the way it is generally expressed sigma 11 with derivative 1 this is nothing but this, this is this, this is this and this is this. So, with this notation we move forward to the next slide.

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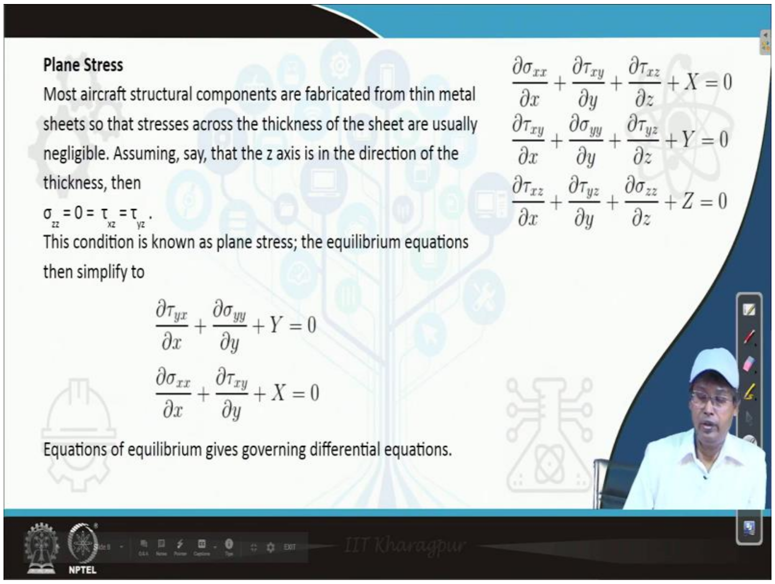

In this scenario it is a good opportunity to introduce the plane stress it is a special case where of three dimensional stress analysis. Most aircraft structures structural components are fabricated from thin metal sheets so that stresses across the thickness of the sheet are usually negligible. Assuming say that the z axis is in the direction of the thickness what we can assume that sigma z equals to 0 tau xz equals 0 tau yz equals to 0.

So if we sigma zz if we put all the z direction stresses equals to 0 this column vanishes as well as since this and this are the same this also vanishes. So, the remaining thing is that these 2 and this this 2 and this so the stress consideration becomes tau I think this may be the first equation this is second or maybe the other way del sigma xx del x plus delta tau xy del y plus x is equals to 0 and the second equation is this one.

So again generally convention is to write this as the first equation this as the second equation but while copy pasting it has reversed sorry please note that. So, with that the particular condition what we prevail in plane stress condition is that the z direction stress components are 0 and that induces the equilibrium equation in a simplified form.

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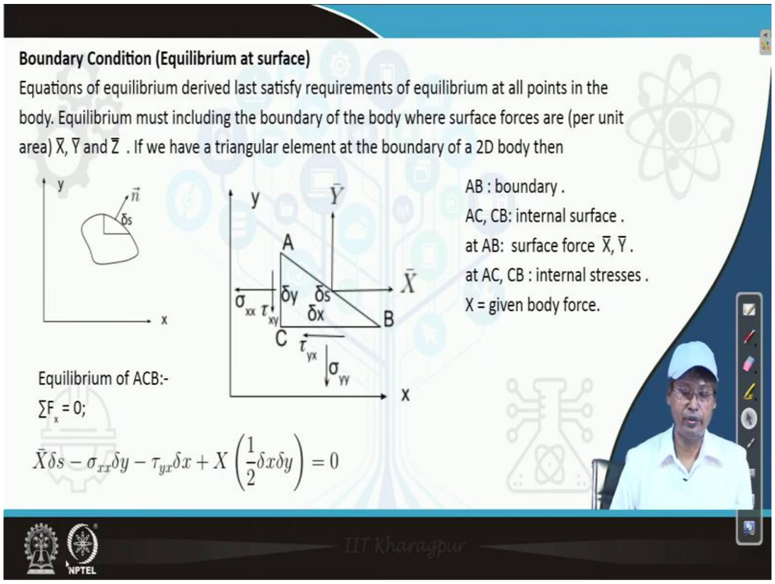

So, the boundary condition continues we will be considering the bound we have considered equations with respect to the body forces inside the structures but we need to consider the equilibrium also if there are some surface forces acting. So, with respect to the surface forces what are the things equations we get and how do they look like let us see it is not very difficult since the most difficult understanding in cuboid all those stress distributions you have already come across.

So with that we proceed further. Equations of equilibrium derived last satisfy equilibrium sorry requirement of equilibrium at all points in the body or inside the body. Equilibrium must include equilibrium must be including the boundary of the body where surface forces are x bar y bar and z bar. So, it has also to be mentioned or maintained in the surface and per unit area where the surface forces are x bar y bar and z bar.

If we have a triangular element at the boundary of a 2d body then it is it looks like this, this particular portion will be covering from 2 dimensional to 3 dimension. I some books it is it is started from a tetrahedral from three dimensional consideration but there are some lacuna I do not want to mention those but it is easy to understand and follow from the two dimension. So, what we are considering let us proceed we are considering this element and that element in this is in a body with a normal end and that element is simplified way shown here.

And what we see in this element is uh that AB is the boundary AC CB the internal surface this is inside the body this, this, this and this. And at AB the surface forces are x bar and y bar x bar and y bar at AC and CB internal stresses are acting that is since it is in x direction sigma xx tau xy is acting this way and in the y plane we have sigma yy and tau yx acting this way x is given body force. Here it is not shown x multiplied by the volume is the body force this multiplied by the surface area this is the surface force.

So there are two types of forces we need to consider those we need to have an equilibrium. So, what we are trying to do we are considering similar way the summation of x equals to 0. So, if we consider summation of x equals to 0 this as I said this is surface force delta s we are considering unit width on the transverse direction on the on the depth side of the element so ds remains makes the area so x bar multiplied by ds this is acting this way sigma x dy is acting this way.

This one is acting here tau yx dx is acting in this direction and then the other body force coming here that is capital X multiplied by half of delta x delta y is the volume. Unit width is considered that is the reason we say in that tetrahedron derivation this part is not shown it is said it may be done. So, what will happen let us see.

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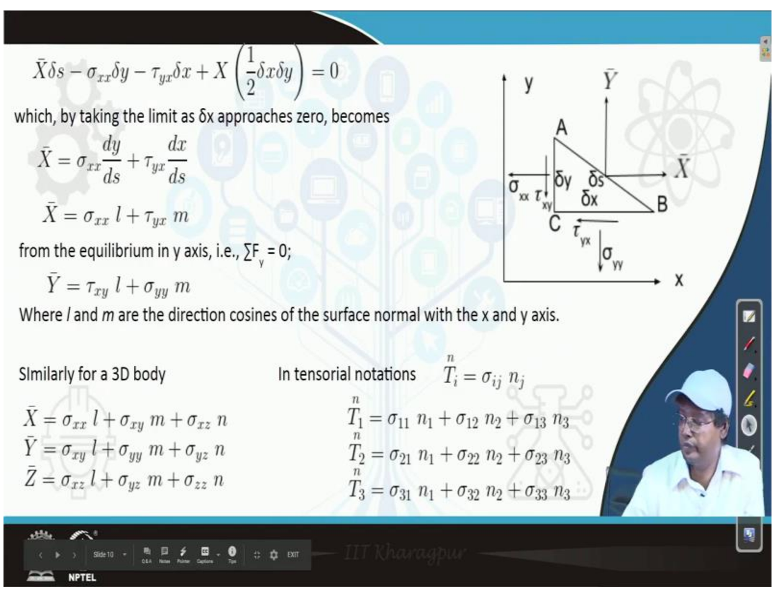

The same equation is repeated in this slide by taking the limit as delta x approaches 0, we can have that x bar this terms vanishes, so that gives us that x bar is equals to actually this is becoming delta y by delta s then we are considering limit so we are considering a derivative or we can say that that becomes the direction cosine. So, that direction cosine component comes this term vanishes because delta x tends to 0 approaches 0 and we have the body forces as x bar equals to sigma x xl + tau yx m.

So this is for two dimension definitely in the y direction we can have from the y equals to 0 the same way that y bar is equals to tau xy l plus sigma y by m is equals to y bar. Similarly for a three dimensional case we can have with the surface forces the equations sigma x sigma xy sigma xz l m n and here also similar to that there it was a derivative here it is only direction cosines are coming there is no derivative is considered and we have the equilibrium of surface forces.

And in tonsorial real notation as we have described I would ask you to do this homework for the tonsorial notation it is better to learn the tonsorial notation or index notation j is repeated please note that is the reason plus sign has come 2 2 3 3 1 remains same j is changing so T 1, T 2, T 3 this again will come in a very different way. In many times this equation will come we will use this equation many times so better you memorize this equation this or this or this anyway is the most simple way to remember that whole equation we generally remember that.

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So with that note we move forward and it is similar the book references are same elasticity books and other books are used here.

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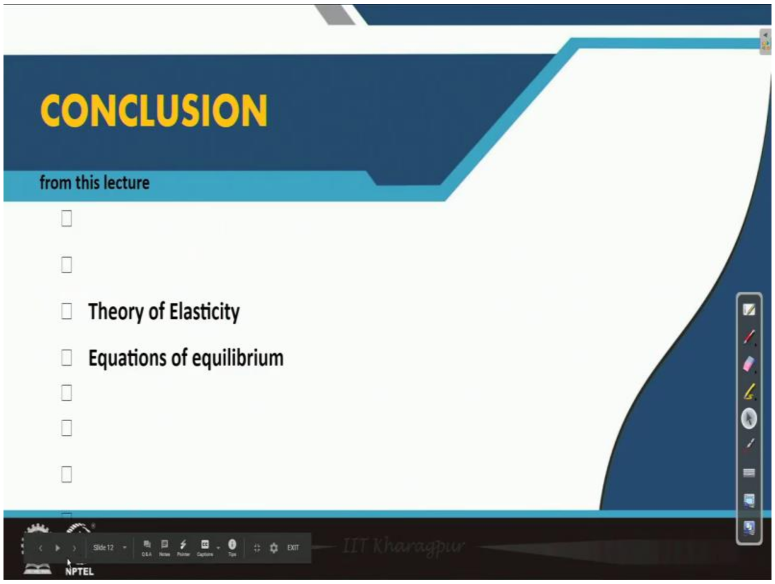

We have learned equilibrium from theory of elasticity point of view and thank you for attending this lecture.