Introduction to CFD
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Lecture - 25
Numerical Solution of Unsteady Heat Condition (Parabolic PDE) (continued)

We continue our discussion on numerical solution of parabolic partial differential equations. Here, we would talk about both unsteady heat conduction equation as well as the Stokes' first problem.

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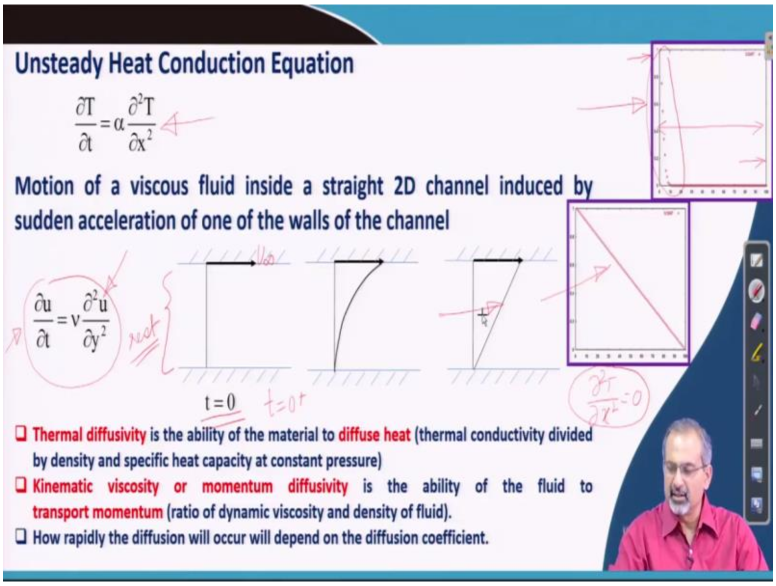

We will just discuss about the theoretical details once more to recollect that the unsteady heat conduction equation involves a time derivative term as well as a space derivative term. The time derivative term is the first derivative in time while as the space derivative term is the second order derivative, and there is a thermal diffusivity term which is associated with the second order spatial derivative.

And that is the term, which essentially decides the extent to which the diffusion of heat occurs. This problem could also be looked at from a fluid dynamic perspective. And then we have a slightly different placement of the problem, where we have 2 parallel flat plates. And there is an incompressible viscous fluid which is confined within the gap between the 2 plates. So, we initially say that this fluid which is confined within this gap.

It is at rest. That is, at t = 0, the fluid is at rest. And then at t = 0+, we make one of the plates to move here. Incidentally, we have shown the upper plate to be moving towards the right and then we try to find out how the fluid which is confined within the gap is set into motion by the movement of the upper plate, considering that this is a viscous fluid. So, then we see that there is momentum diffusivity, and then the governing partial differential equation works out like this, del u del t is equal to nu del 2 u del y square.

So, we have the kinematic viscosity coefficient associated with the second order derivative in space. And that is responsible for the diffusion of momentum. So, the unsteady heat conduction equation models diffusion of heat, while the reduced form of Navier Stokes equation which shows up in this form models the momentum diffusion in the gap where the fluid is confined between 2 parallel flat plates.

So, we recall that last time when we discussed about numerical solution of unsteady heat conduction equation using one of the numerical schemes, the FTCS scheme. Then we saw that in the time marching of the solution in the initial instance, where we had a one dimensional domain spanning a certain length with; the non dimensional temperature specified as unity, on the left hand side.

And as 0 on the right hand side, there was a sharp rise in temperature, close to the left hand in the initial instance of time steps when the computations were done. And then with gradual increase in time stepping, when we move the solution to very large number of time steps. Then there was a linear distribution of temperature being approached. And we also discussed that in such a situation, the governing partial differential equation essentially reduces to a Laplace equation.

When the time derivative term asymptotically limits to 0, so, we can expect that when we model momentum diffusion, with the lower equation here. Momentum would diffuse through the gap, and we would be able to reach a similar kind of linear distribution within the gap of velocity. Will quickly look back at the governing partial differential equation, how from the Navier Stokes equations.

We are able to reduce it to a form, much more simple, which is being used for modeling this problem. So here, we have placed the problem in this form that we have the 2 parallel plates.

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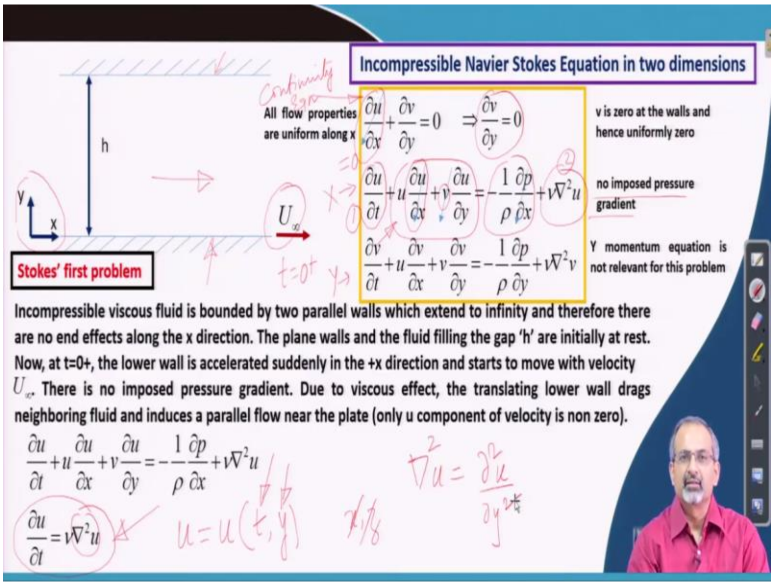

The lower plate is moving with a certain velocity and that movement starts at t = 0+. Before that both the plates are the walls, as well as the intermediate fluid are at rest. So, it is an incompressible viscous fluid, and therefore, we invoke the incompressible Navier Stokes equation. Here, we invoke it in two dimensions, because we can very clearly see two dimensions here.

The x direction and y direction, x is along the length of the walls of the plates and y is normal to them. And we have a gap of h separating the 2 walls. And this problem is known as the Stokes' first problem. So, when we look at the governing partial differential equations, we see continuity equation. So the first equation is continuity equation. Since, we have these 2 walls stretching infinitely to the negative and the positive x directions, they remain parallel.

Therefore, all the flow properties would remain uniform along x. Therefore, the x derivative would go to zero. And therefore, the outcome is that we cannot change along the y direction. Now, incidentally, if we are next to the 2 walls, then u and v are both zero. Next to the walls, and therefore that essentially means that v is uniformly 0 everywhere in the field. With this outcome, if we go to the x momentum equation. We have a time derivative of u.

We have the so called convective derivatives. And then we have the pressure gradient on the right hand side, and the viscous term on the right hand side. Now, as far as the time derivative term is concerned, that is non zero, because, as the one of the walls start moving. The u component of velocity would certainly change with time, most strongly in the initial instance, and then the chain will become weaker and weaker with time.

Now, coming to the convective derivatives, u del u del x, if you look at this term, things are not changing along x and therefore this gradient goes to zero. When you come to the next term, the v we have already seen it to be zero, and therefore the convective derivatives vanish. We are not imposing any pressure gradient on the flow that means we are not pushing or pulling the flow from any side. Therefore, the pressure term will also go to zero.

So, we are just left with the first term on the left hand side and the second term on the right hand side from this momentum equation. And that is essentially what is figuring over here at the bottom of the slide. If you go to the y momentum equation, then you see that we have already found v to be vanishing, and therefore, this equation essentially becomes irrelevant for us.

So if you were to write down a z momentum equation, considering a three dimensional flow, even that would become irrelevant. That means, the properties will not change along that direction. So finally, we have a governing partial differential equation, which it looks much simpler than the Navier Stokes equations, and we can understand that. From this equation, it is very clear that u is a function of both space and time.

And therefore, here, when it comes to time, we write t and then as far as space is concerned, there is only dependence on y, because we have already discussed, it cannot depend on x or z. So, we are just having 2 independent variables t and y, on which u depends. And therefore, essentially this Laplacian would reduce to del 2 u del y square. So, this equation and the unsteady heat conduction equation are very similar to each other.

So, this equation would be relevant whenever we are looking at the Stokes' first problem, where u essentially means the x component of the velocity, while the unsteady heat conduction would help us solve for the heat conduction through conducting medium, as a function of time and space. We are interested to know whether we could look for an analytical solution for this problem.

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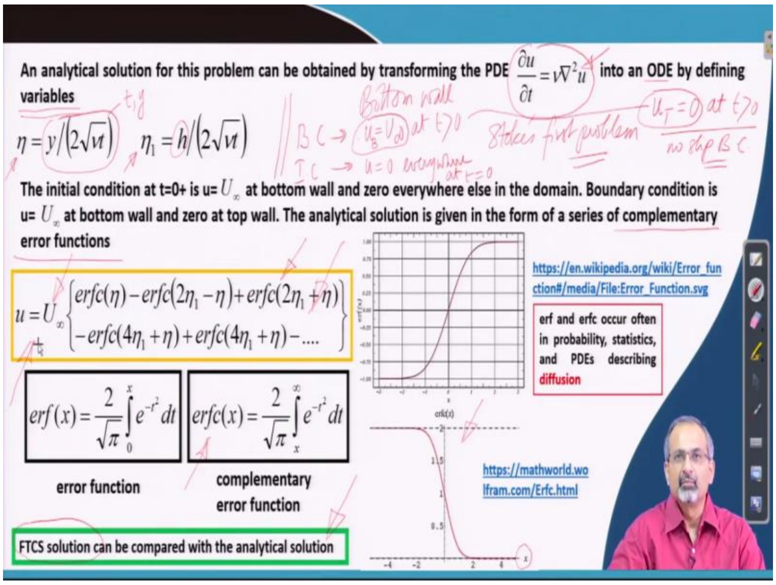

So we have our partial differential equation in this form for the fluid dynamic problem. The Stokes' first problem with the initial and boundary conditions, that is, there that means at the bottom wall. We have u is equal to u infinity, at t > 0 and we have initial condition as u = 0, everywhere in the field. So, this is the initial condition and this is the boundary condition. Additionally, we have to impose the boundary condition that on the upper wall.

So this, if we call as the bottom wall, then additionally, the boundary condition on the top wall would be that u top is equal to 0 at t > 0. So, that is what we call as the no slip boundary condition, so like we are imposing the condition that the bottom wall is equal to U infinity, and the fluid layer adjacent to the bottom wall will move with this velocity. Similarly, the top wall will remain static and the fluid layers adjacent to the top wall immediately adjacent to the top wall would remain static.

They will not move at any instant of time. So, this is how the whole problem gets posed. So, you have the governing partial differential equation here. And we also have the initial and the boundary conditions posed. So, this essentially gives it a well posed nature. And then if we want to obtain an analytical solution, we convert it into an ODE by defining some variables, define eta and we define eta 1.

So, both eta and eta 1, or other eta contains both the independent variables t and y. And in eta 1, we replace y by the gap length, h. Once we do that we can show that the solution comes out in this form. So, u as a function of the distance from the bottom wall would be expressed in this form which involves the so called complimentary error functions as an infinite series.

And if we briefly look at these functions, we have this complimentary error function expressed as this integral and if we want to see the nature of the complimentary error function. This is how it looks as a function of x. So, as you vary x from a negative through 0 to the positive direction, this is how the value of the complimentary error function shows up. So, that is what would contribute to the solution over here.

The series, forming the complimentary error functions. And once, we have this analytical solution, we should be in a position to compare the numerical solution that we generated through the first scheme that we learned for discretizing or only partial differential equation, that is, the FTCS solution that we discussed at length in the previous lecture. The solution can now be compared with the analytical solutions that we have over here.

And that would quantify the numerical error that we are committing. This could be taken up as a good homework problem for which a small computer program may be written. And then a comparison made. We need to remember that the error function or the complimentary error function. They are of extremely high importance in different domains like probability, statistics, partial differential equations especially when we are discussing diffusion phenomena.

So, when we are talking about the parabolic partial differential equations, we are talking about transient diffusion phenomena, when we talked earlier about elliptic partial differential equations. We talked about steady state or equilibrium solutions for diffusion phenomena, so, both elliptic as well as partial differential equations.

They deal with diffusion phenomena in one case we talk about steady state or equilibrium solution that is in the case of elliptic equations, while in the other case; we talk about transient solution that means how the time dependent diffusion takes place that is model through parabolic partial differential equation. And we now know that diffusion could happen for heat diffusion can happen for momentum.

And therefore, the governing partial differential equations may look slightly different to model the different phenomena, but inherently they also have a lot of analogy, and they broadly belong to the family of parabolic partial differential equations.

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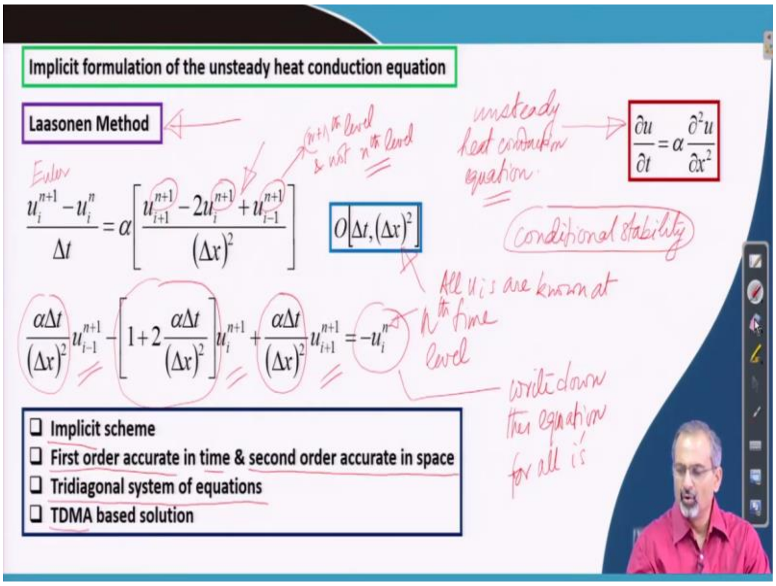

The FTCS solution was an explicit solution of the unsteady heat conduction equation will now discuss about a possible implicit formulation. We remember that when we discussed about the FTCS scheme, we came up with a so called conditional stability. So, there was a certain condition that we derived, which would need to be followed. In order to keep the computations stable and finally give us meaningful results.

So, we are interested to know that if we switch from explicit schemes to implicit schemes, or implicit formulations, whether that would have any impact on stability. In order to do a test of that kind, we introduce a method which is called as the Laasonen method. When we say a method, we also often refer to it as a scheme so the words method scheme, this need to be interpreted in a very synonymous manner.

Remember that here. When we are showing the discretization, we are talking about the unsteady heat conduction equation. So, we are making use of the thermal diffusivity. So, if you look at the Laasonen method, what has been done is that you have the Euler first order discretization for the time derivative term and you have a second order central discretization for this space derivative.

But the interesting thing is that the time level of all these terms are at the n + 1th level and not nth level. So, if it was nth level that would have given you back the FTCS scheme but here we are introduced a new scheme which is called as the Laasonen scheme or the Laasonen method. And here, we would like to keep these terms at the n + 1th time level. Now, if you follow this discretization.

The formal order of accuracy of the scheme that you will have is first order in time, and second order in space. That can be shown through the Taylor series approach. If you rearrange the terms a bit, you would see that you would have coefficients for the 3 variables for this equation. So these are the 3 variables that you have. And this is the known term because it comes from the nth time level.

So, we know all u i's are known at nth time level, and therefore, what do you have on the right hand side is a known term, a known value and what you have on the left hand side are 3 unknowns. And therefore, this gives you a tridiagonal system of equations, when you write down this equation for all the i's. So that would give you a tridiagonal system of equations. And we have discussed earlier how to solve them.

So, we use the TDMA method for solving solution. So, we now have an implicit scheme, we have first order accuracy in time, second order accuracy in space. And we would like to know, how this formulation might affect the stability.

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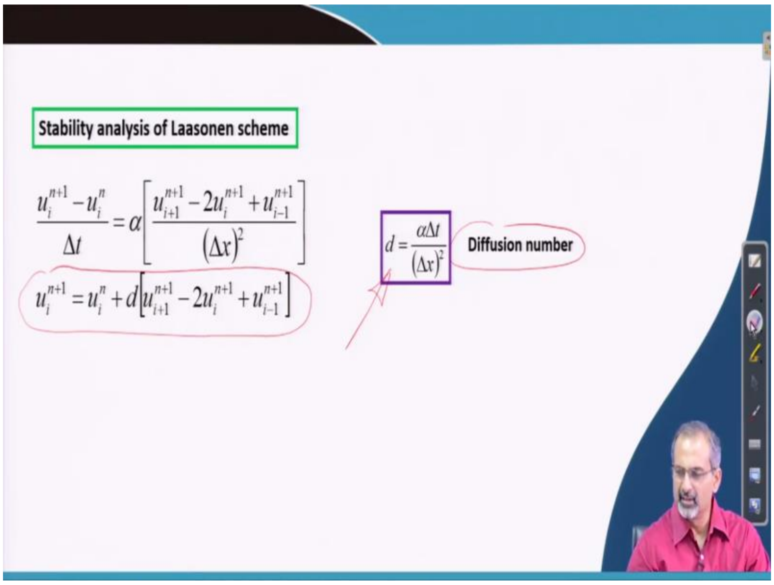

For analyzing the stability of the Laasonen scheme, we just rearrange the scheme and write it down like this. We know that the d is what is called as the diffusion number. And now, let us try to see how to work out the stability analysis using the Von Neumann stability approach.

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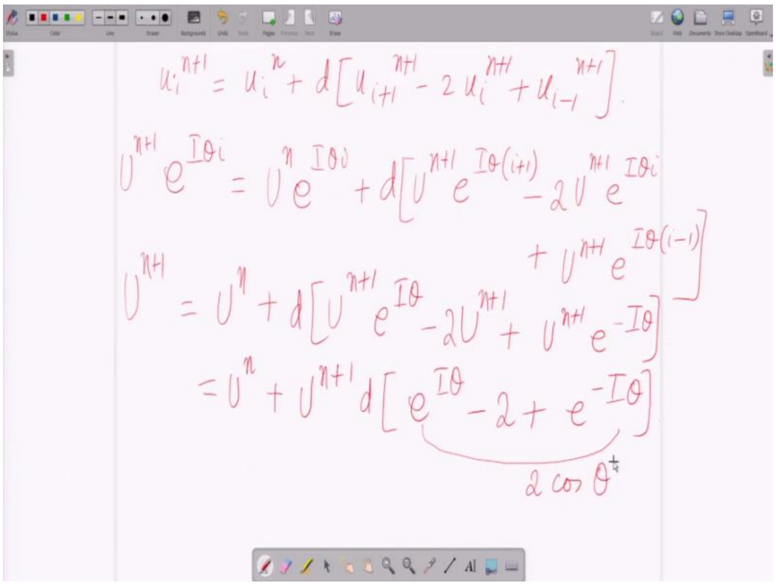

So, according to the Von Neumann stability approach, we will replace all these terms by Fourier terms. So, let us do that. We would write this as U n + 1, e to the power of capital I theta small i. And then likewise, for the remaining terms. So, we can take e to the power of I theta i common. So, we should be able to get this form. Once we have that we can additionally take that U n + 1 out of the bracket, and we are left with. Now, if you sum up these 2 terms, then you will be left with 2 cos theta by the Euler formula.

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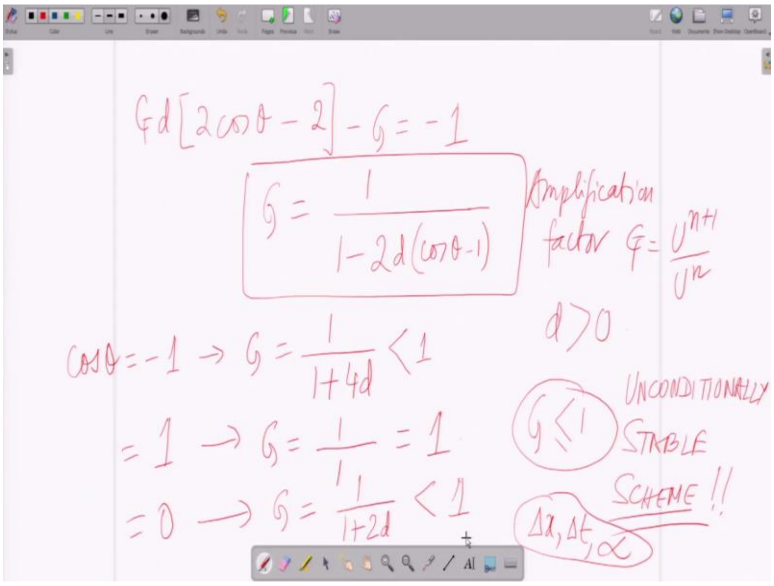

Just rearrange the terms and you have a form like this. We remember G is our amplification factor, which means it is a ratio of U n + 1 by U n. Now, we need to figure out that for various values of cos theta how this term will work out. Remember that d, of course, is greater than 0, it is a positive number. Let us try out a few possible values of cos theta let us say if cos theta is equal to -1, then what happens to G.

We have G is equal to 1 by 1 + 4d, which is, of course, always less than 1. Because, d > 0. If you take a value of cos theta equal to 1, let us say. So in that case, what is G, G is going to be 1 by 1. And then you have G exactly equal to 1. When it is 0, then cos theta is equal to 0, then you have G equals 1 by 1 + 2d, which again is less than 1. Which means, irrespective of the value of cos theta, we are able to maintain the condition that G is less than or equal to 1.

And that essentially satisfies the condition of stability that we have discussed. And therefore, this gives you a so called unconditionally stable scheme and that is a big advantage. As we know from our experience that irrespective of your choice of spatial and temporal steps, that means, how you choose delta x and delta t with the given value of alpha. There would not be any kind of instabilities arising out of your computations. And this is a very big advantage as we can understand.

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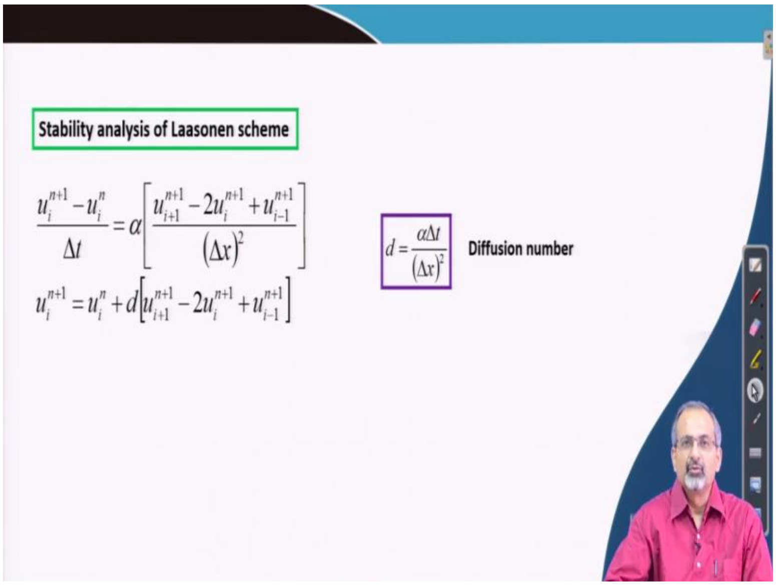

So, we now understand that the stability analysis that we did for Laasonen scheme produces an unconditionally stable scheme. If we look, ahead into other schemes which can help us; discretize partial differential equations of the parabolic type that we are discussing currently. We would be able to look at schemes which can be even better than the Laasonen scheme in certain respects. One could be that it improves the order of accuracy.

So, in a later lecture, we will discuss about certain other schemes which can help us improve the order of accuracy, keeping the unconditional stability, which we saw for the Laasonen scheme. We will also discuss schemes, where we could have an advantage over explicit schemes, in the sense that when we apply them to multi dimensional problems, then there will be no severe restrictions on the choice of time steps. So, these would be discussed in later lectures. Thank you.