Introduction to CFD
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Lecture - 27
Numerical Solution of Liner Wave Equation (Hyperbolic PDE)

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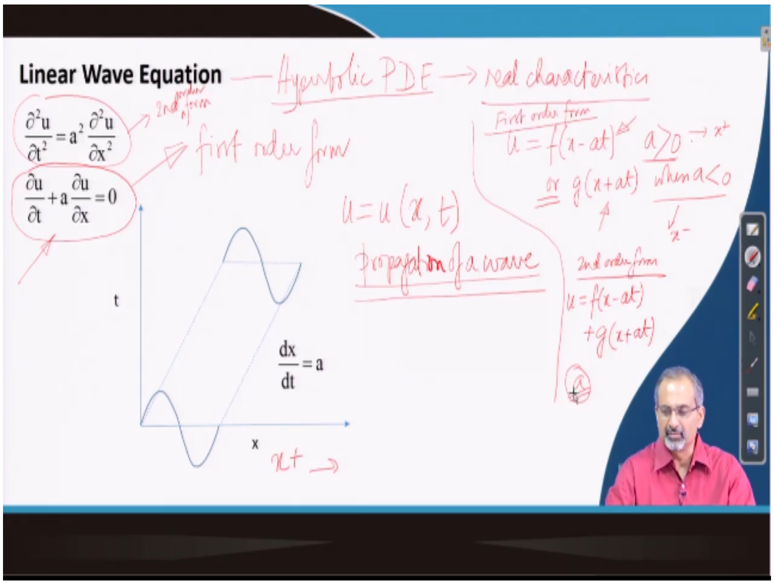

In this lecture, we will begin our discussion on linear wave equation. In earlier lectures where we had discussed about different governing equations, we had visited the linear wave equation, we are re-discussing it here and this time we are going to talk more about the different kinds of numerical schemes which could be used to solve linear wave equation. But before we go into numerical schemes, this could be a good time to recapitulate some of the issues related to the physics of linear wave propagation.

So, we can see that here the linear wave equation has been written in 2 forms. So, we more often would be discussing about this form which is often called as the first order form that means, the highest order of the derivatives that figure in this form of the linear wave equation is first order. So, as we can see, there are 2 derivatives of u. So, u is essentially a function of space and time.

So, there is one space dimension x and a time dimension. And most importantly here the physics that we are modeling is a kind of an offer wave form that means, the wave would move physically from one location to the other as time elapses. And this physics is quite different from the ones that we were trying to model in the elliptic and parabolic kind of partial differential equations where we were talking about diffusive mechanisms.

So, if you remember in elliptic partial differential equations, we talked about steady state or equilibrium solutions of problems, where there is diffusion phenomena, and again in parabolic partial differential equations, we talked about transients diffusion that means, we were tracking the time history of the diffusion phenomenon. Whereas, here when we discussed about linear wave equation, we of course, we recall that this type of equation falls in the category of hyperbolic partial differential equations.

We recall that they involve real characteristics. So, this was discussed earlier when we were talking about classification of partial differential equations. And therefore, there are specific directions along which information would propagate in the domain and this is a very dramatic feature of hyperbolic partial differential equations. And coming back to the equation that will most often discuss in this context.

It is the first order form of the linear wave equation. And whenever we talk about wave of course, we are most often talking about propagation of a wave through a certain medium until unless we are talking about standing waves. So, the wave moves through the domain with a certain velocity. Now, this equation could also be expressed in a second order form, where the partial derivatives in space and time are second order.

And it can be shown that the first order form would allow propagation of a family of waves which move along the positive x direction or negative x direction. Separately, while the second order form of the equation can allow the propagation of waves, both along positive x direction as well as negative x direction simultaneously. So, this is a very, very important feature of the problem and we can show that if the solution of the first order form of the equation is given by a functional form of this kind.

So, this is what is possible in a first order form that means, to allow a solution of either this kind u is equal to f of x – a t when a is greater than 0 that means, the wave is propagating along the positive x direction or you can accommodate a solution of this kind g of x + a t when the wave propagates along the negative x direction. So, it can allow any one solution at a time while the second order form would allow a solution of this kind.

f of x – a t + g of x + a t, simultaneously. So, there will be a waveform moving along the positive x direction and other waveform moving along the negative x direction and the speed of propagation of the waveform along both directions would be in terms of magnitude. It will be a. So, it will be a positive a along the; for the form which moves along the positive x direction and a negative a, for the form which moves along the negative x direction. So, what is this wave all about?

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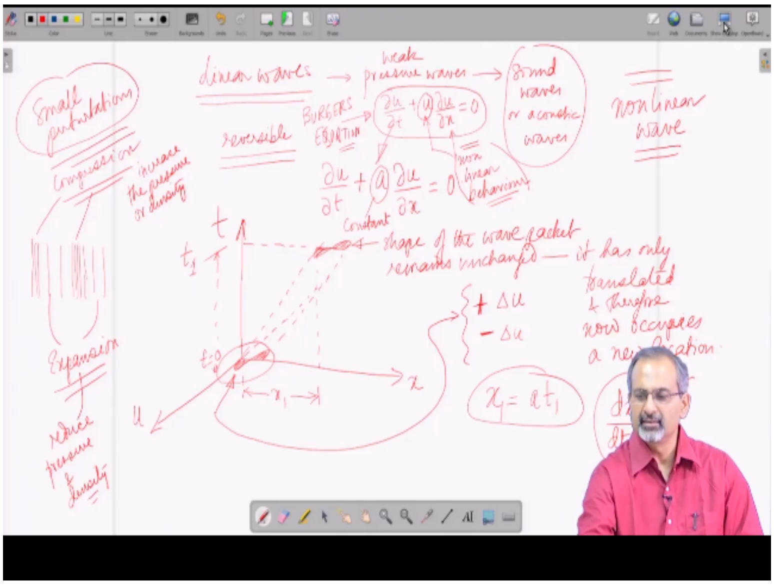

So, we may try to physically look at the problem like this that when we talk about linear waves or propagation of linear waves, a good example of linear waves could be pressure waves which are weak. So, weak pressure waves the kind which we are very familiar with sound waves or acoustic waves. So, these kinds of waves produce mild compression and expansion in the medium through which they propagate.

As long as these changes are reversible, that means, there is a mild compression or a mind expansion and all this is happening in a reversible manner thermodynamically. Then you can show that such wave propagation can be governed by the linear wave equation. So, if we look back at the linear wave equation in its first order form. This equation models the physics of this kind that if you look at how the wave propagates through the domain.

You try to look at it in the x t plane. And let us take the other direction to represent what that mild wave looks like in terms of say u. So, at t = 0, it may be looking like a mild ripple of this kind which means there is a portion where u is slightly positive; there is a portion where it is slightly negative. So, a positive portion would mean that it is a small change in u, which is positive in nature and the other one would be a small change in the negative direction.

So, we would like to see how this small change which we see at the initial instance. That is a t = 0 propagates through the domain. So, in order to see that we try to look at it at a later time. And then we will see that the very same wave packet in terms of appearance has now translated to a new location. So, the shape of the wave packet remains unchanged. It has only translated and therefore, now occupies a new location.

So, if you have allowed it a certain time, let us call this time interval. So, from t = 0 to let us say t 1 you have allowed it so much time to propagate then it would have traversed a certain distance, we call that distance as say x 1. So, you allowed the wave to move for time t 1 and you are now finding it in a new location x 1 from the origin. Then you will find that it follows this equation.

That x 1 is equal to a times t 1 and you can very well understand that the wave is essentially propagating with a speed a. Therefore, is governed by dx dt is equal to a. Now, if this wave was not a weak wave of the kind like sound waves or acoustic waves and a more stronger wave, then this might become a nonlinear wave. And a dominant feature of a nonlinear wave is that as it propagates through the medium it also changes in shape.

We will discuss more of this in a later lecture, but for now, we just tried to understand it this way that for a linear way, the different portions of the waveform would propagate at the same speed a. And therefore, as the waveform or a wave packet moves through the domain, there will be no change in its shape, because different portions of the wave packet are propagating at the same speed. If that was not the case, then the waveform would have changed in shape.

So, that is the basic idea behind linear wave propagation. Another thing that we have to keep in mind is that when we try to show a ripple of this kind, which we try to explain through small ripple of delta u in the positive and negative sense, you could physically try to look at it this way. That when sound propagates through mediums, it does so, in the form of longitudinal ways. So, there may be a region of compression followed by regions of expansion.

So, there are packets of compression again packets of expansion juxtaposed or adjacent to each other and the compression packets would locally increase the pressure or density. And the expansion packets will reduce pressure and density. Also, temperature or velocities locally will change by very small amounts. These are all delta amounts. And as long as they remain weak; they are called as small perturbations.

And all such small perturbations give rise to linear wave formation and propagation. The moment the perturbations become large and strong, they would lead to formation of nonlinear waves which are strong waves. And the nonlinear waves as they propagate through a domain would distort as they propagate. So, it is a very important thing to understand this physics before we try modeling the numerical aspect of linear waves.

Remember that one way of modifying this equation by replacing the term a could be that you bring in the dependent variable in place of a. The a here happens to be a constant; the constant which is the wave speed. So, each portion of the wave is propagating with the same speed a. The moment you change the governing equation to one which looks like this, which incidentally is called as Burgers equation.

This term no longer is a constant, it is now the dependent variable itself and that brings in non linearity. So, the dependent variable being multiplied by its spatial derivative brings in nonlinear behavior in the wave. And therefore, the waveform here once it propagates through the domain would distort as it does so. So, with this introduction, let us start looking at possible ways of numerical discretization of linear wave equation in its first order form.

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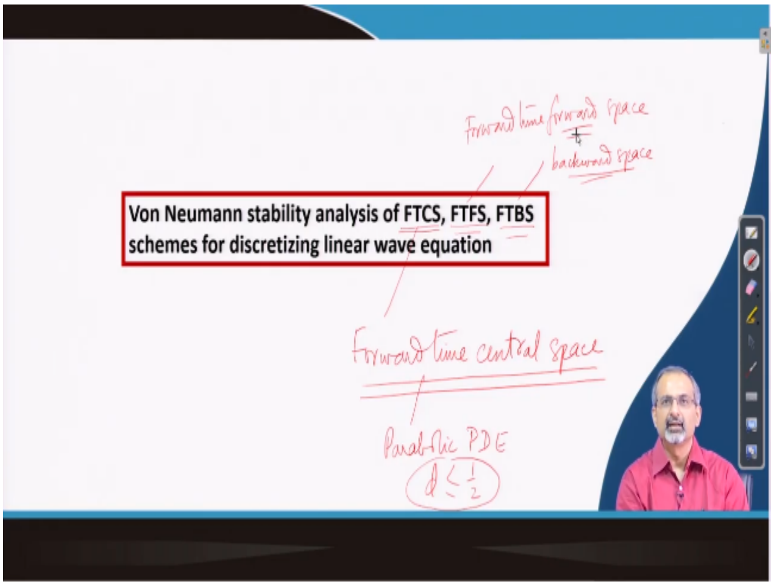

We would look at a few possible ways of discretization of the wave equation for doing that will use clues of some numerical schemes that we have looked at earlier say in the context of parabolic partial differential equation. Why we would like to look at parabolic partial differential equation is because that had a transition term that means a time dependent term as well as a space dependent term.

Now, incidentally wave equation also has a time dependent term and a space dependent term. So, and intuition could be that why do not we try some of the schemes that worked for parabolic partial differential equation. In the case of linear wave equation, let us see whether they work as well. But one thing that we have to be careful about is that from our past experience, we know that explicit schemes could become unstable.

So, they could either be unconditionally unstable or at best conditionally stable, but unconditional stability from explicit schemes is rather rare, if not impossible. Whereas, implicit schemes are in general stable schemes. So, keeping this in mind, we would first like to look at some possible ways of discretizing the linear wave equation and then try to check for the stability through the Von Neumann stability analysis.

Remember that here we have both space and time and therefore, Von Neumann stability analysis like the way we did for parabolic partial differential equation also applies over here. And again we are handling a linear partial differential equation. So, we are satisfying some of the basic requirements which are mandatory for Von Neumann stability analysis. The linearity being the major issue. So, let us go ahead and try out some possibilities.

So, here we have proposed a few possible schemes by which we can discretize the linear wave equation. So, one of them is already familiar to us that is the forward time central space. This is a scheme which gave us conditional stability in the case of parabolic partial differential equation. You remember that for one dimensional parabolic partial differential equation.

We were able to show that the diffusion number is less than equal to half is the stability condition of the criteria for keeping the calculations stable. So, we would like to see how FTCS does in this case. Additionally, we are trying to explore the impact of biasing the spatial derivative calculation through one sided finite difference. So, we are proposing forward time as usual, but forward space and then again forward time.

But here backward space as the other possible ways of discretizing the linear wave equation. So, we will see that how these schemes do in terms of stability. So, let us begin the exercise by looking at the forward time central space scheme.

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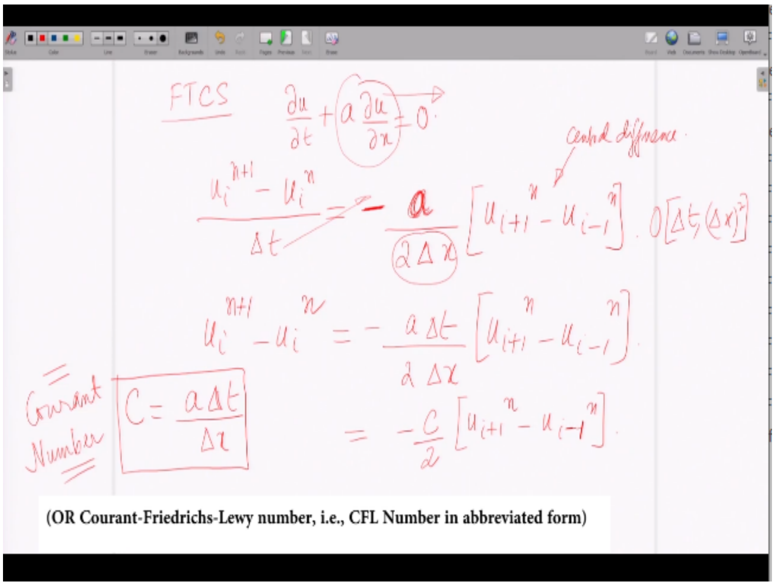

We again write down the governing partial differential equations and then try to discretize it using the forward time central space scheme. And then try out the Von Neumann stability analysis. So, what we have done is, we have first of all transpose this equation transpose this term to the right hand side and thereby there is a negative sign here, the coefficient a is here in the numerator, which is the coefficient of the derivative and this comes essentially from the denominator of the finite difference for del u del x using central differencing.

Now, if you look at the formal order of accuracy of this scheme, this would be first order accurate in time and second order accurate in space. Now, in order to carry out the Von Neumann stability analysis, we follow the usual procedure. But before we go ahead doing that let us try to collect all these terms together to define a particular number which is very, very important in the wave equation world.

So, we will take the delta t to the other side; it gives us - a delta t by 2 times delta x. And therefore, the left hand side would now have to be written this way. Now, incidentally the collection of these terms a times delta t by delta x is represented as C which is called as the Courant Number. Keeping that in mind, this can be written as - C by 2 u i + 1 n – u i – 1 n.

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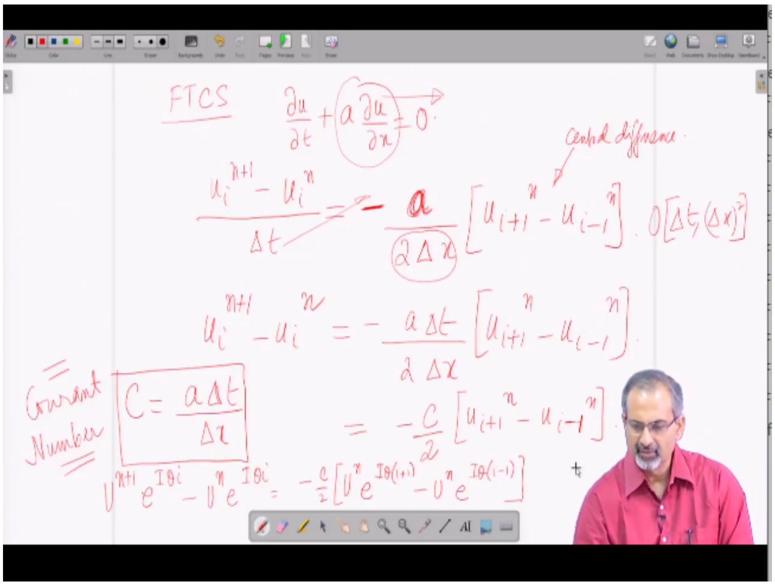

So, now, if you go ahead doing the Von Neumann stability analysis, what would you substitute for each one of the terms and then what we usually do is we try to extract common factors.

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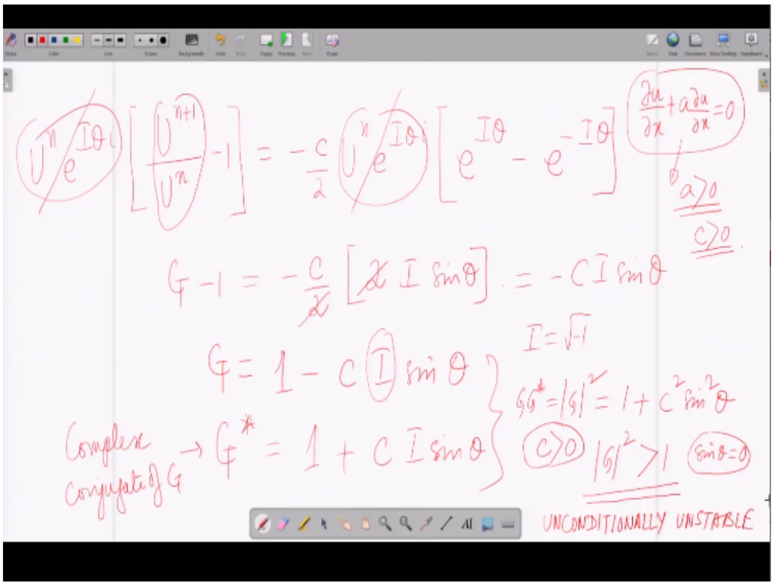

So, it can be written like. So, we recall that this is the amplification factor G and before we do that this and this; the common terms; they cancel out. So, we are left with G – 1 on the left hand side and you will have – c by 2 into this would give you 2 I sin theta by using the Euler formula. So, these 2 cancel out, so, we have – c I sin theta. So, that gives us G is equal to 1 - c I sin theta.

Now, this happens to be complex expression because of the Capital I. So, in order to obtain the mod G, we would invoke the complex conjugate of G which is written as G star. So, that will become 1 + c I sin theta. So, if you multiply these 2 terms you get G, G star is mod G square which is nothing but 1 + c square sin square theta. So, anything other than sin theta = 0 would produce a mod G square greater than 1 which means this is unconditionally unstable.

And this is something which is different from what we found for the parabolic partial differential equation because FTCS scheme had worked for parabolic partial differential equation conditionally. So, there was a conditional stability clause whereas, here we find that it is unconditionally unstable that means, forward time central space scheme is not suitable at all for handling linear wave propagation.

So, we would stop here with this lecture, we will look at the possibility of using the other schemes that we talked about that means, whether the forward time forward space or forward time backward space kind of schemes work for linear wave equation. We will try to explore this in the next lecture. Thank you.