Introduction to CFD
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Lecture - 28
Numerical Solution of Liner Wave Equation (Hyperbolic PDE) (continued)

We have started discussion on numerical solution of the linear wave equation in the last lecture, we will continue with the discussion today.

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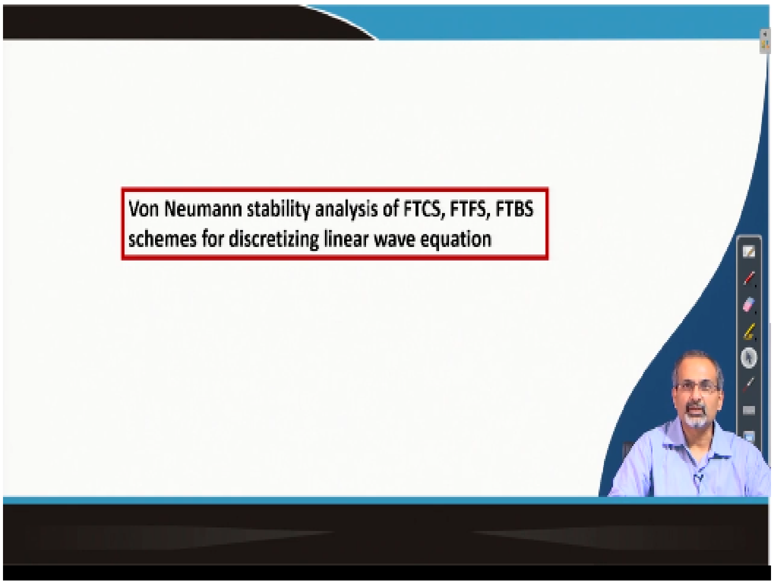

So, we were looking at the Von Neumann stability analysis for different possible discretization schemes for the linear wave equation. And to begin with, we had looked at the forward time central space discretization last time, since, we are familiar with that scheme and we had used that scheme for discretizing parabolic partial differential equation. But when we applied the Von Neumann stability to that scheme.

We saw that it did not work for the linear wave equation. We have thought about using two other possible ways of discretizing the linear wave equation. Let us go ahead to see first of all, whether such schemes are stable at all or not. And then whichever scheme we find to be stable could be further explored. So, we go ahead looking at the FTFS scheme, the forward time forward space scheme now.

As the subsequent scheme to be checked, for suitability for discretizing the linear wave equation. So, let us try to write down the discretized form for the FTFS scheme.

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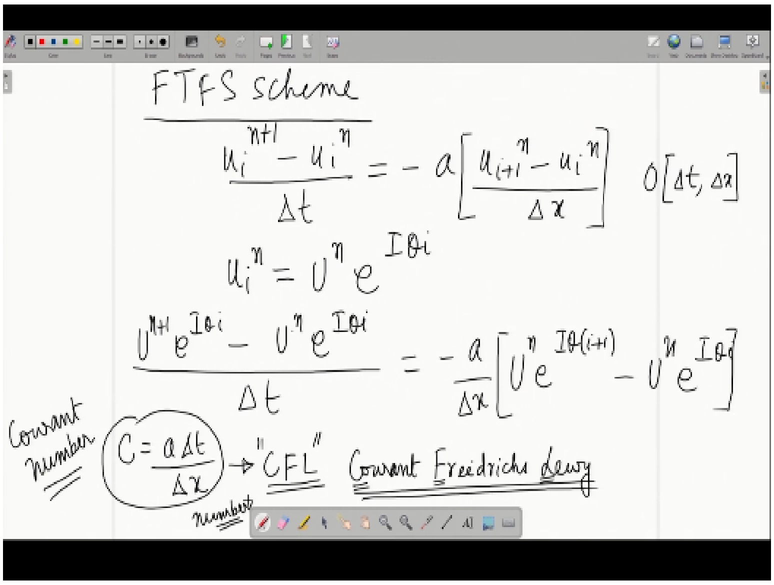

So, that is the time derivative on the left hand side and on the right hand side, we have the space derivative. Note that here we are using a one sided finite difference for the spatial derivative and that would reduce the order of accuracy of the scheme in terms of the spatial derivative to first order. So, it is first order accurate in time and space. We would now do the Von Neumann stability analysis by substituting the expressions for the different U terms in this equation.

So, let us do that. So, by now, we have been doing this process for several cases and we must have become quite familiar with it. We had defined this term last time briefly as Courant Number. Incidentally, this is having a more formal nomenclature, which is called as CFL in the name of three outstanding scientists Courant Friedrichs and Lewy. So, in abbreviated form CFL.

So, CFL number which happens to be a very, very important number in wave equations. So, we can write down this equation in terms of C and further divided all through.

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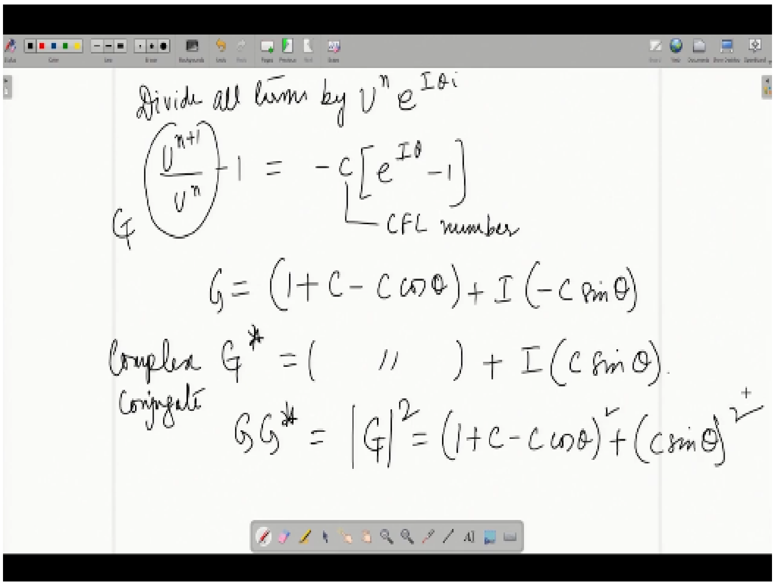

So, we divide all terms by U n e to the power of I theta i; capital I followed by small i. And then once you do that you will get an expression which looks like this. So, this is the CFL number. That we are talking about and this is our amplification factor. So, we get an expression for the amplification factor, if you rearrange these terms and use the Euler formula for e to the power of I theta. Finally, you will get an expression which looks like this.

So, it is a complex form. We saw last time that in that case, we would define its complex conjugate in order to find out the expression for the modulus of G. So, complex conjugate will be the same term here come from the real side while the sin of the imaginary part gets flipped. And then if we multiply G with G star we get modulus of G whole square and that is nothing but square of the real part plus square of the imaginary part.

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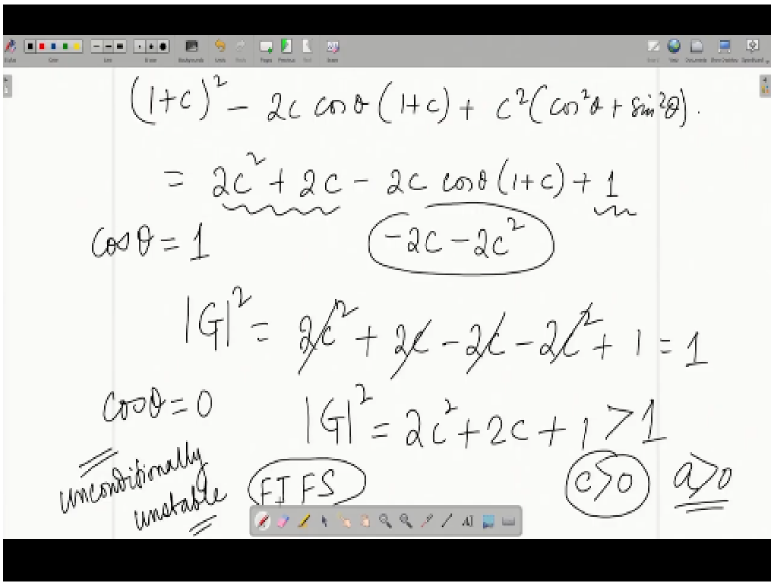

If you expand it, you will get and then further. So, as you know theta is the face angle and so, it can vary.So, we can check the value of this expression based on some specific values of cos theta. Let us say, we assume cos theta is equal to 1. If we do so, then we can have highest possible contribution coming from the negative term and in that case, the negative terms will end up contributing say – 2C – 2C square.

So, which can actually have the maximum negativing effect on the positive terms which are these. So, if you do that then what does mod G square come to. So, it will be 2C square + 2C – 2C – 2C square + 1. So, the C square and C terms get cancelled, you are left with 1. So, for this particular value of theta which is say either 0 or 2pi. You will get a modulus of G = 1, but if you have say cos theta value of 0 then what do you have.

You have mod G square = 2C square + 2C + one and C anyway is greater than 0 because a is greater than 0 which means that this term will become greater than 1 and then in having stability. Now, that is going to happen for a large number of values of theta and therefore, that means that this scheme is unconditionally unstable. So, we now find that even FTFS does not work for linear wave equation.

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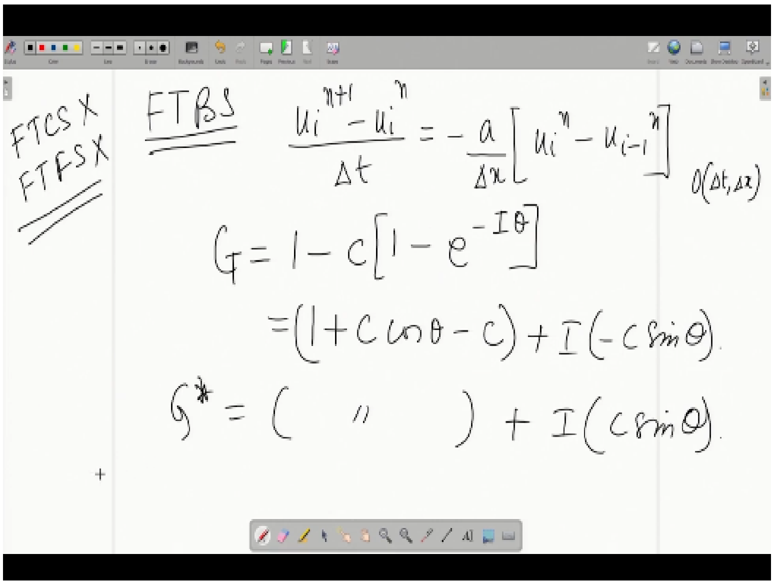

So, let us check with FTBS, the forward time backward scheme. So, we are running out of options. We have found that both FTCS and FTFS have not worked. We now need to check whether FTBS works. Let us do the discretization. Again remember that if you are looking at forward or backward with a 2 point stencil, then the order of accuracy will become first order like it happened for FTFS.

So, first order in time and space and then the usual procedure for the Von Neumann stability analysis, which will give you an expression for G = 1 - C into 1 - e to the power of - i theta. And then if you the Euler formula and do the necessary simplifications. We will come up with this expression for G, again the complex expression, so, therefore have to define the complex conjugate of G.

So, it is again the same real part and the imaginary part with the sin flipped. So, that brings us closer to mod G.

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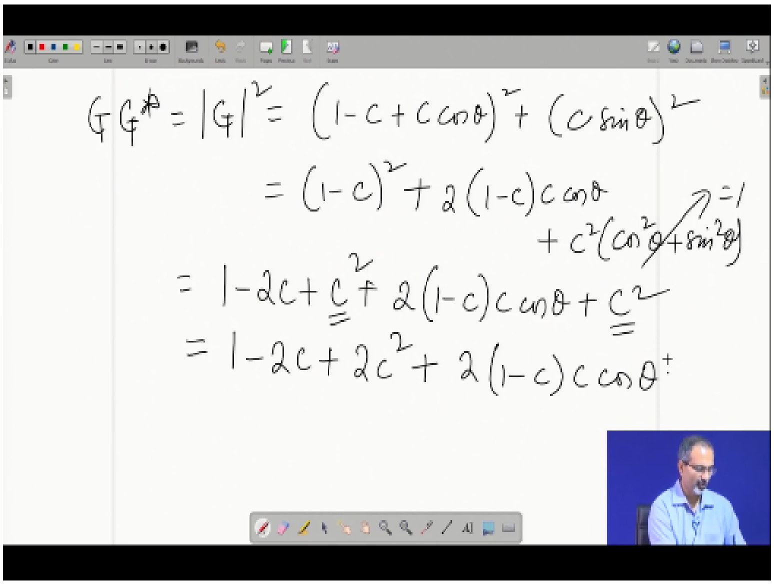

So, we have 2C square terms. So, we can collect them. And we therefore have the final expression. Again, it is not very easy to check what this expression will work out to be for different values of theta. So, we use the same procedure like what we have done in earlier cases and check the value for a few specific value of cos theta.

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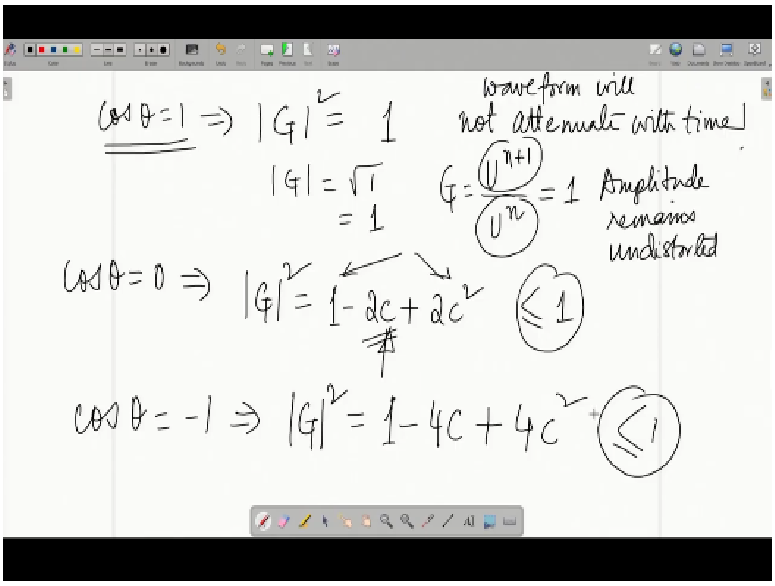

So, when we take cos theta = 1, we see that mod G square comes out to be exactly 1. That means there is no amplification of the signal. In the sense that the signal will not attenuate with time or say the waveform will not attenuate with time, which is a good feature because, by definition, we know G is equal to U n + 1 by U n. So, when G is equal to 1, it means that the amplitude remains undistorted.

So, amplitude remaining undistorted is certainly a favorable feature. However, remember that for this scheme, when cos theta is equal to 1, we are able to achieve more G is equal to 1, of course, mod G is nothing but root over 1 and you just extract the positive root because the negative root does not make sense. For specific values of theta, this can be achieved but if you go to other values of theta most often this will not be achieved.

So, let us try out other values of theta 7 cos theta is equal to 0. You will have an expression looking like this, which means it gives you a nonzero amplification factor and if the scheme needs to be stable then of course, this needs to be less than 1. So, we are not sure at this point whether it will actually do so, whether it will actually satisfy that condition at all or not. Now, it may satisfy it.

Because you have positive contributions from these 2 terms while a negative contribution from here and therefore, if they just balance each other incidentally. Then we might be able to satisfy this condition under certain conditions. So, let us try one more value of cos theta. Let us say it is - 1. And then what does mod G square work out to be? We find it is working out to be 1 – 4C + 4C square appears somewhat analog us or similar to what we saw for cos theta = 0.

So, there may be a possibility of achieving less than equal to 1, even in this case, we will have to explore further to see that under what conditions they may be less than 1.

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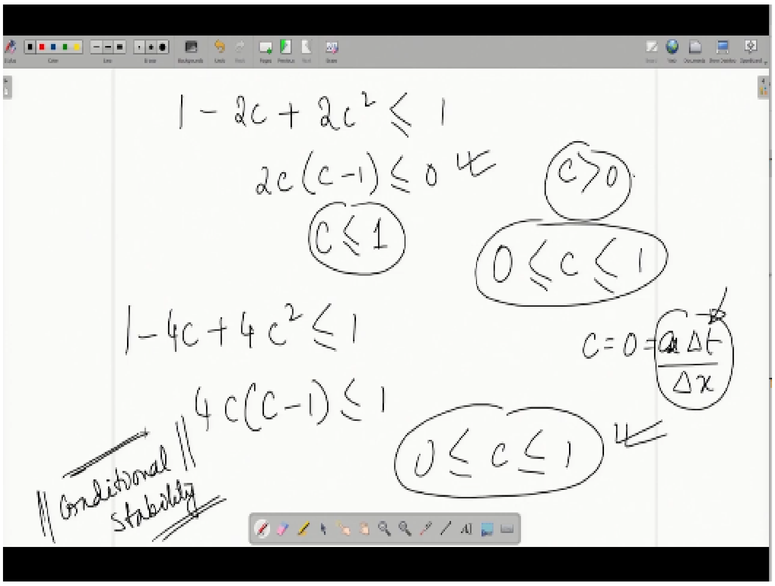

So, taking the first condition, that is when 1 – 2C + 2C square is less than equal to 1. We can just rewrite this inequality like this. And then we know that C is greater than 0. Additionally, C has to be less than or equal to 1 in order to satisfy this inequality. So, C is greater than 0 and as well is less than equal to 1 will satisfy this inequality. Of course, even if C is equal to 0, it would end up satisfying.

But then C = 0 means a delta t by delta x is equal to 0, which means either the wave is a stationary wave, you do not have a moving way problem, or your time step is limiting to 0 that means you are not marching in time. So either way, it does not give you a meaningful physical situation. So, we will look at the other case 1 – 4C+ 4C square less than equal to 1. So, it gives you an inequality which looks like this, and the outcome remains the same.

So, we have reached a condition at least for the few values of cos theta that we tried that if C is defined in this manner, then you can have a so called conditional stability. At least we could come up with a scheme which seems to be working for a certain specified value of C; a range of values of C and therefore, we have a case of conditional stability.

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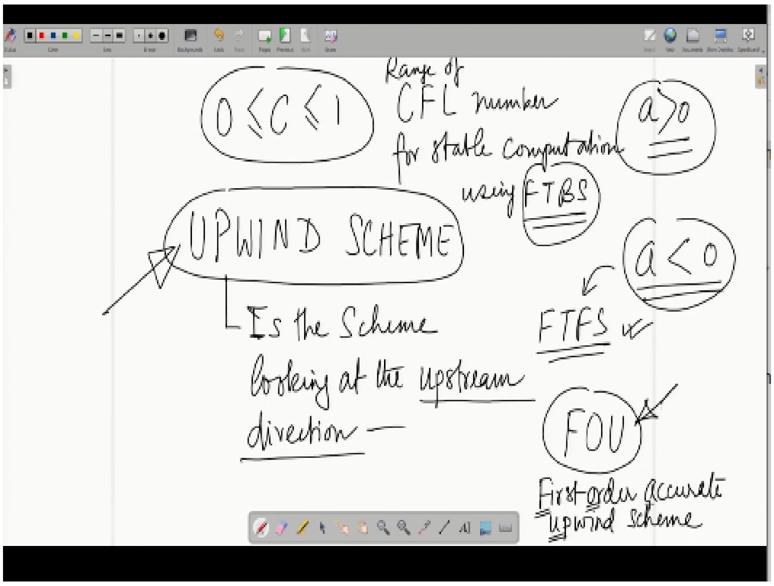

The condition that 0 less than equal to C less than equal to 1 is essentially giving you the range of CFL number for stable computation using forward time backwards space. You remember that this works for a > 0, the discretization that you had used works for a > 0. Now, the backward differencing in space that you have used in this case, how would it work, if a incidentally is less than 0, you would then find that backward differencing fails to work.

And then it is forward differencing, which will work. So, forward time forward space would work for a < 0. We will discuss more about why this works that way. But for now, we take it as a known fact. So, we are seeing that forward difference works for a < 0 while backward difference was for a > 0. So, can there be a generalized structure in that case. So, the generalized structure is what is called as an upwind scheme.

That means is the scheme looking at the upstream direction which basically means the direction from which the wave is moving towards you at any given point of time. So that would be considered as an upstream direction. So, let us say, you have a small ripple like what we had discussed in the previous lecture. The ripple is propagating through the domain. So, if you look at the direction from which the ripple is propagating towards you, that is essentially the upstream direction.

So, when we talk about a numerical scheme, which caters to or gives importance to the upstream direction and accordingly does the spatial differencing that would be called as an upwind scheme. So, that essentially encompasses both backward as well as forward differencing depending on what is appropriate for a given value of a. So, when a is positive, then we see that a is moving from left to right.

Let us say around the positive x direction and then you need to look towards your left in order to see the wave propagation. So, that is why backward differencing works in that case, while if the a is less than 0, then it is essentially propagating from the positive x to negative x direction. So, it is like right to left and then until unless you are looking to you towards your right, you will not see the ripple coming towards you.

So, the differencing scheme essentially is looking at the upstream direction in the sense that the direction from which the ripple is reaching you. And if you are doing that you are essentially following a so called upwind scheme. So, backward or forward has now become more generalized in the form of upwind scheme and we have used first order upwind and therefore, we could abbreviate it as say FOU scheme first order accurate upwind scheme.

So, F for first; O for order and U for upwind. So, that gives you a kind of an abbreviation for the first order accurate upwind scheme. And here, the first order accuracy is with respect to the spatial derivative. Could there be further interpretation of what we have now introduced as upwind scheme. So, we will try to explore another way of looking at the whole problem.

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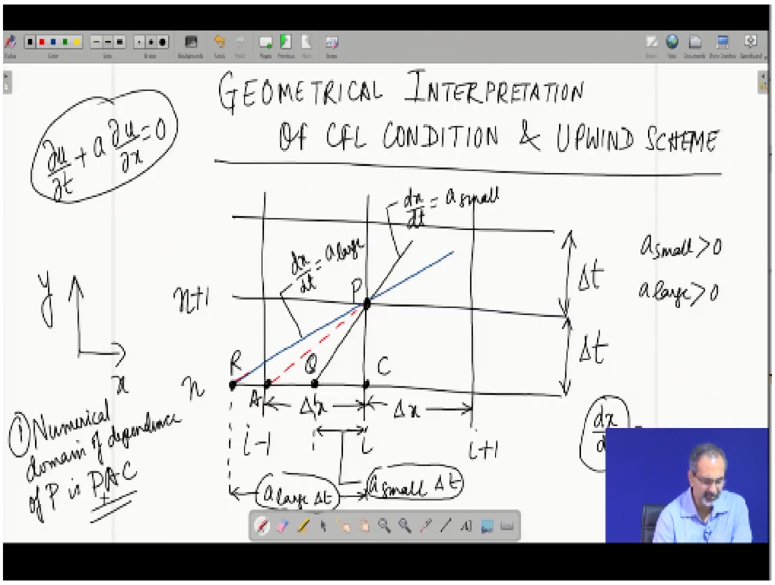

So, we will call it as a Geometrical representation and **(()) (022:57)** use this approach to explain the significance of the CFL condition further. So, it is a geometrical approach of course, so, geometrical interpretation of CFL condition and the utility of an upwind scheme.

So, let us see how we do it. For that we will make a small sketch to explain the time and space discretization in the form of grid lines.

So, let us say along the x direction, we will have the grid points; the spatial grid points i, i + 1, i - 1 and along the y direction will have the temporal levels; the time steps. So, of course, the time intervals remain constant. As you go from 1 time step to the other. It is true even for the spatial width. So, we have drawn 3 lines here. We will also put a few points in order to discuss some important concepts here in diagram.

So, P is this point, we have point C here, let us mark point Q, A and say, R. Let us see what we have on the diagram. The straight lines the black and the blue straight lines, these are essentially characteristic lines. We had discussed briefly about characteristics earlier also. So, these lines have constant slope dx dt, and the slope essentially gives information about the wave speed which they represent.

So, of course, both the characteristic lines here are governed by the linear wave equation. Now, we are having two possibilities; one where the wave speed is on a lower side which we are calling as a suffix small and another possibility where the wave speed is on the larger side and we are calling it as a suffix large. And the interesting thing that you can understand from these 2 straight lines is that the dx dt equal to a small has a steeper slope in this plot, the x t plot here compared to the dx dt equal to a large.

Now, why is it so? If you think carefully that for a given time interval the a small wave should be traversing a lesser distance than the a large wave. So, if it is looking steeper here that basically means, for a given period of time delta t the displacement of the wave would be on the lower side. And that is what you see in the displacement a small into delta t; it is a smaller distance compared to the displacement a large into delta t.

Remember that these displacements are monitored over a time interval of delta t. So, if you were to follow these characteristic lines, you would undergo these respective displacement. Now, you have another line, which is a dotted line and indicated with red color in between. Now, what is the significance of that line, that line is not a characteristic line, it is just connecting 2 points in the spatial temporal grid that you have defined.

So, it is connecting the spatial grid i - 1 at nth time level with the spatial grid, i at the n + 1 at level that is all that it is doing. Now, having defined the geometrical setting of the problem, let us try to understand what is the significance of these lines or these regions that we have defined here. So, we also need to take care of the points that we have defined, let us say P, R, A, Q, C these all these points are important for us to understand what is going on here.

So, we will define 2 domains, which are very important for us. Let us say to begin with a numerical domain of dependence of point P and we will define that as PAC.

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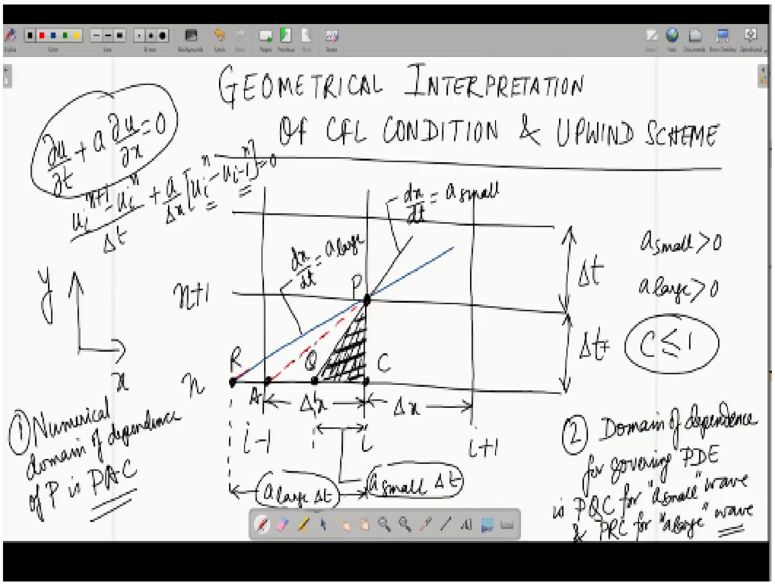

And we will define another domain, the domain of dependence for governing partial differential equation which is the linear wave equation. And that happens to be PQC for the wave with a small velocity, so we will call it as a small wave and the domain of dependence for governing equation is going to be PRC for the wave, which has a wave speed of a large. Now, what do we mean?

We mean that when we are looking at the point P and we are looking at the wave which is moving with wave speed a small then the domain of dependence as far as the exact wave equation is concerned the domain of dependence for the governing partial differential equation is PQC for the wave with a small velocity. That means, this is the region which that wave will encompass and this is the region which will influence the point P as far as the a small wave is concerned.

So, the point P is dependent on that region PQC for the information propagation as far as the a small wave is concerned. This region will be different, when we look at the wave with a large speed. So, in that case, what will be the domain of dependence for point P. It will become larger. It will become PRC because that is the region that the wave will sweep through from the time n to n + 1.

And therefore, as we reach point P will have to draw information from that entire region that that wave characteristic sweeps during that 1 interval of time from n to n + 1. Then we will be able to account for all the information that is coming from that characteristic. So, we have to understand that there are different domains of dependence if the wave speed is changing. It may be smaller or larger depending on whether the wave speed is small or the wave speed is large.

Now, when it comes to the numerical scheme, you have a certain grid distribution that you have created for yourself with delta x and delta t is defined along the respective directions. That is defining the so called numerical domain of dependence of P automatically. And in that case, because you are using a first order upwind scheme, remember that how have you discretized the equation, we just look back at the terms that you have used assuming that a is positive.

You have used this kind of a distribution to solve the problem. So, what is your spatial distribution comprised of. It is I and i - 1. So, you are drawing information only from i and i - 1 at the nth time level, in order to construct the value of u at the point i at the n + 1 at level. That defines the numerical domain of dependence of the point P. That is basically highlighted by that region bounded by PA that dotted red line and the other arms of the triangle.

So, it is PAC. That is the region from where the numerical scheme is deriving its information. So, that is the numerical domain of dependence of point P for the given discretization scheme. So, what we now understand is that the hatched region PQC remains bounded by the numerical domain of dependence and therefore, the numerical scheme will be able to capture all the physics that comes from that that hatched region.

So, the bounding characteristic line could be having a slope exactly equal to the red dotted line up to that the numerical scheme will be able to capture. if it goes beyond that then we have a situation where you have a large speed of the wave. And what you find is part of the information from that characteristic will remain on captured by the numerical scheme.

Because there is a region AR indicated in the diagram from where information cannot reach the point P, it is just impossible because of the kind of discretization we are using. Now, this might create numerical instabilities and also because information is not properly propagating to the point P from those regions. There could be accruing of errors, which can become unbounded as the calculations progress over time and then finally, make the calculations completely unstable.

So, we can show of course, that what is the bounding condition that that hatched or rather the dashed red line is showing that would give you exactly the condition that C is less than equal to 1 of course, I have not mentioned the lower bound, but C has to remain bounded in the sense that it cannot exceed 1 and the red dotted line will actually enforce that condition. So, this is another way of showing how an upwind discretization ends up conforming with the CFL condition.

And it is a geometrical interpretation of the problem from that perspective. We will discuss more about this in the next lecture. Thank you.