Introduction to CFD
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Module - 1
Lecture – 3
Governing Equations of Fluid Flow Continued

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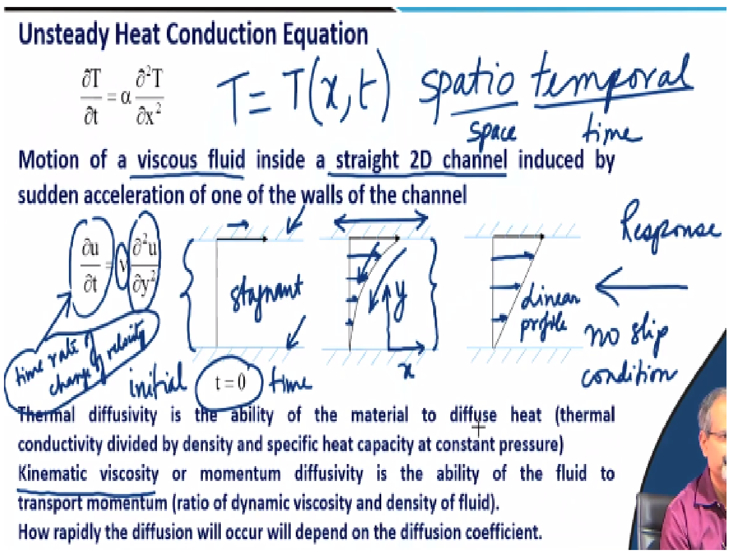

We continue our discussion on governing equations of fluid flow today and we start from where we left in the last lecture. If you recall we were talking about the unsteady heat conduction equation. In this equation we mentioned earlier that temperature, which is the dependent variable, now becomes a function of both space and time. So, the evolution will actually be spatio temporal. The term spatio stands for space which is quite obvious and temporal stands for time.

So, we have to look at the problem in both space and time. Incidentally, the governing equation for a fluid flow problem is also very similar to the unsteady heat conduction equation. That problem is stated here. We have a motion of a viscous fluid through a straight two-dimensional channel and at t = 0 (this is the initial time) we find that the liquid is stagnant. You have the two walls; one on top, another at the bottom and the region of confined fluid within the gap.

Now, once you have passed t = 0, the upper wall moves impulsively which is indicated by the black arrow towards the right and it is moving with some velocity which is indicated by that arrow in a vector sense. We want to find out that how will the flow which is confined in the gap respond to this movement. So, the response will not happen instantaneously.

That is the anticipation in this case, and that is why you have a term sitting in this equation, which is indicating the time rate of change of the velocity component u. So, what we assume is that the x direction points this way and the y along this. So, we are talking about a variation of u in the y direction here, that means along the gap direction as you move from the bottom to the top.

We assume that because this plate stretches infinitely in the negative and positive x directions, you will not see any changes in the velocity profile along the x direction at all. Whatever changes take place will be confined to the y direction. Therefore, you do not see the influence of any velocity derivatives along x direction coming up in this equation. Now, without going into the specifics of the solution, we try to intuitively figure out what is going to go on after the plate has moved.

This is a viscous flow. So, what happens is that the flow which is next to a wall sticks to the wall. This is what we call as the no slip condition on walls in viscous flows. So, if there is a movement of the upper wall, what it will do to the flow is that it will drag the flow along. As it drags the flow along there will be diffusion of momentum, which occurs due to the presence of kinematic viscosity coefficient, which stands ahead of the second derivative term.

The larger the value of the viscosity coefficient, the greater will be the momentum diffusion. What is diffusion? You see, that momentum is gradually percolating from the upper plate towards the lower part of the flow and more and more of the flow closer to the lower wall is gaining momentum which manifests through increase in velocity. So, this part basically indicates the velocities at different levels of the fluid and if you allow it enough time, you will finally see a velocity profile looking like this.

Now, this appears to be a linear profile and if we remember the solution of the last model equation that we discussed which was Laplace equation, we also had a linear solution in that case. So, is there any connect between the two? This is a very obvious question and it is interesting to note that yes indeed there is a connect. The point is that as momentum diffusion happens from the higher velocity regions to the lower velocity regions through the shearing of different layers of the fluid.

Shear which exists between layers of fluid tends to drag more and more of the low velocity fluid and speed them up. We move towards a steady state finally, and at steady state, there would be no more time rate of change of the velocity.

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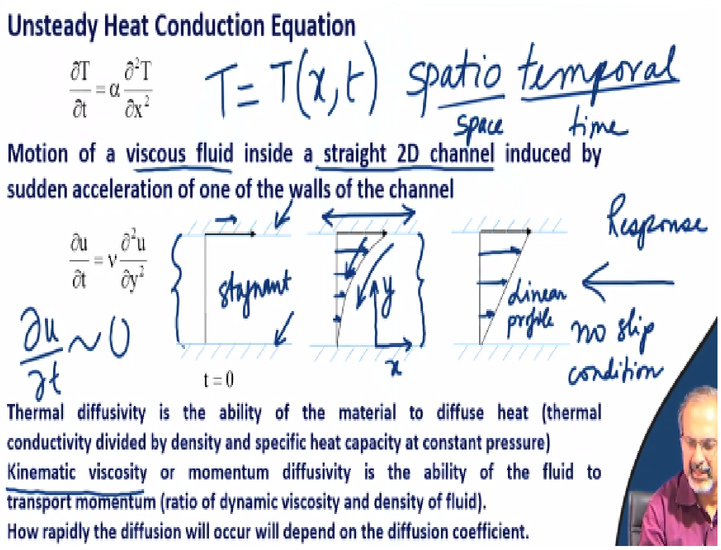

Which means that gradually we are moving towards a condition where we can more or less say that the $\frac{\partial u}{\partial t}$ term is approximately zero. If that is the situation, then you see that the equation essentially boils down to Laplace equation in one dimension.

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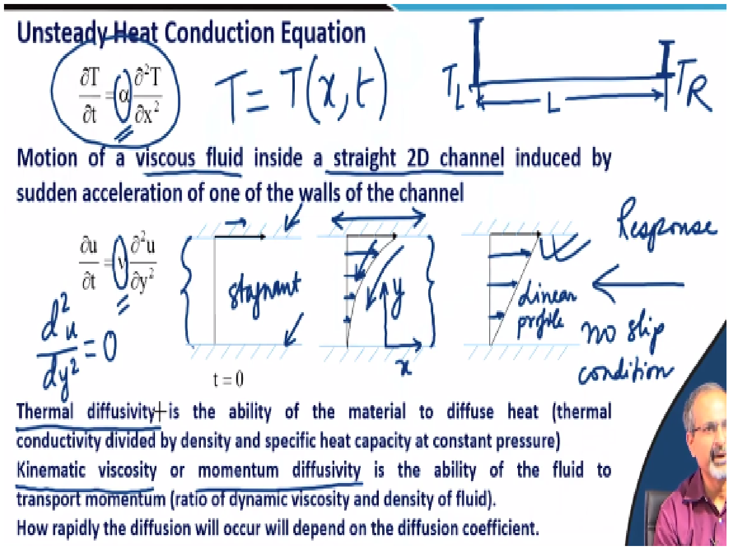

So, what will the equation look like? Under such situation, the equation will look like a form which you have already handled, $\frac{\partial^2 u}{\partial y^2} = \frac{d^2 u}{dy^2} = 0$. So, the last equation you solved was $\frac{d^2 \phi}{dx^2} = 0$ when you solved one-dimensional Laplace equation. That and this in principle is the same and that is why you see a linear variation over here. This is a very interesting aspect in problems when you try to track unsteady effects that means effects which vary with time.

So, this is a problem, which we see for the first time in our course, where both space and time seem to be important. As steady state is approached, the solution asymptotically approaches that of Laplace equation. I leave this as a homework exercise for you to figure out how the equation will work out when we are looking at heat conduction problem.

Try to do it yourself and try to understand how you might solve a problem where you have a one dimensional rod, where you have two different temperatures at two ends, let us say T left and T right and you have a large enough temperature T left defined over here, which I am indicating with the dark portion and a comparatively smaller temperature on the other side. Then how the temperature will vary along the length of this one-dimensional rod assuming that this property is constant all over?

Again in the fluid problem 2, it is important to ensure that these coefficients which influence momentum diffusion or thermal diffusion remain constant. If they do not remain so, then it is more difficult to tackle the equations. You can ideally do it, but the efforts involved will be more. In CFD, we can accommodate such inhomogeneity in properties in different kinds of problems, where in a flow domain or in a solid material there could be property variations.

Most often property variations can easily occur when you have temperature dependence for example.

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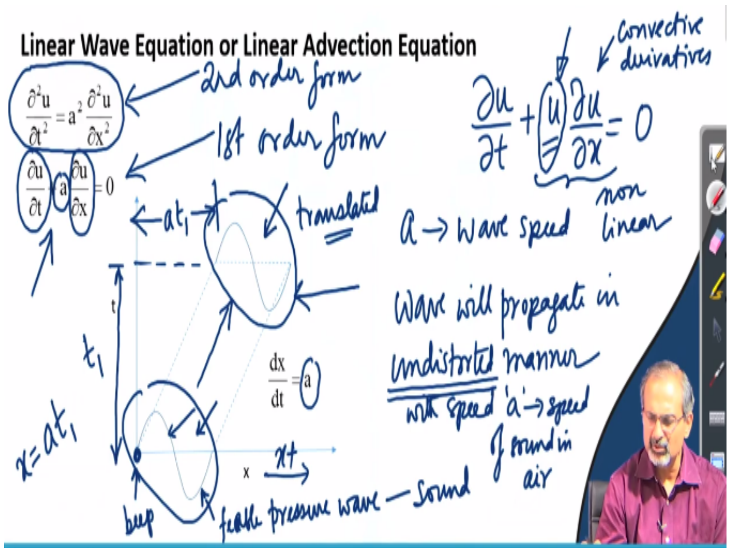

We look at another very important model equation, which we call as the linear wave equation or linear advection equation. The equation can be written in two forms, which we have shown over here.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

However, the plot that we have shown over here is catering to one of the forms, i.e., $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$.

$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is the second order form of linear wave equation, while $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$ is the first order form. Now, what are the terms we are having over here?

If I am looking at the first order form, then I have a time derivative term, I have a space derivative term and I have a coefficient 'a', though at this point I do not know the behavior of the coefficient. Very often, we call this coefficient as the wave speed. This is basically the speed with which disturbances can propagate through the domain of interest. What kind of disturbances are we talking about?

The disturbance could be that in this domain at x = 0 and time t = 0, you create a small beep using a certain device, which means it actually creates a very feeble pressure wave which we experience as sound. This is just an arbitrary representation of that small pressure wave which the sound creates. Now our common experience says that if we create a sound in air, that sound seems to propagate in all directions at the speed of sound in air (that is the rate at which it propagates in air; if it propagates through a different gas, the speed will change, even in air the speed will change if the thermodynamic state of air is changed, say its temperature is changed).

However, here we are just confining that movement to one dimension and not only that we are confining the movement only along the positive x direction. If that is the case, then the ripple will probably find its way at the speed ‘a’ along the positive x direction and because it is a linear wave, the most important fact that we need to keep in mind in this case is that the wave will propagate in undistorted manner with speed ‘a’, which in our case is the speed of sound in air.

That means, this equation is capable of handling the movement of sound waves how they propagate through air. Remember that we are talking about weak pressure waves here and therefore the model remains linear. When we handle stronger waves than what we have mentioned here, then the model equation may start becoming nonlinear. For example, a typical form of such a nonlinear equation may be looking like this.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

There is a very small but not at all trivial change which has come about now. The coefficient of $\frac{\partial u}{\partial x}$ is now u which is the dependent variable. So, the dependent variable gets multiplied with the derivative and we said that this is a recipe for non linearity. In fact, in computational fluid dynamics, a lot of effort goes into tackling these kinds of nonlinear terms. These are often called as convective derivatives.

Remember that the wave equation has been represented as a first order equation here, we might as well represent it in a second order form. So, you can imagine that the solution of a second order form of the wave equation will have more terms in it to accommodate for the higher order. Essentially what we see is that in this case the solution looks like this that the ripple which was created at t = 0 in this form, gradually translating and just displaced as it is to a new location after a certain time.

If you quantify this time as an interval, let us say t1, then how far has the wave moved away? It has moved a distance given by x = at1, that is the distance it has moved. So, what it has done is it has just translated and what is important to remember is that it has not distorted. As though you just picked up the wave packet here and you dropped it here. Of course, it happens as a continuous space time evolution process.

But since you are looking at the pictures at two different discrete times, these are the two locations where you find the wave, but the interesting thing is that the wave shape remains the way it was in the beginning, and this is a typical feature of linear waves.

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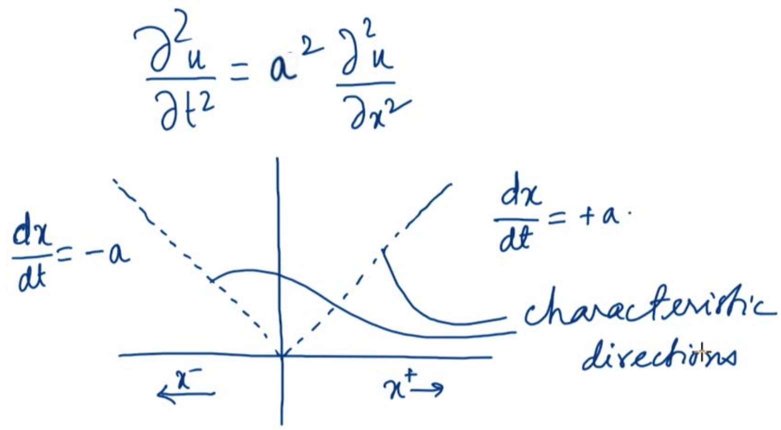

Now, the question to ask is that if it was the second order wave equation, then how would the solution look like? In that case, you would actually see that the initial ripple will travel in two directions, one along the x+ the other along the x- direction and at what rate will they travel? The one towards the right will be governed by this, the one towards the left will be governed by this. So one moves away at a speed of a+ which means that it is moving towards the positive x direction.

The other moves away towards x minus at a speed of a- or -a rather (magnitudes of both a+ and a- are equal to a). So, there is a symmetric emergence of the packet in the two directions. We will later see that these are very, very important directions, which we often call as characteristic directions.

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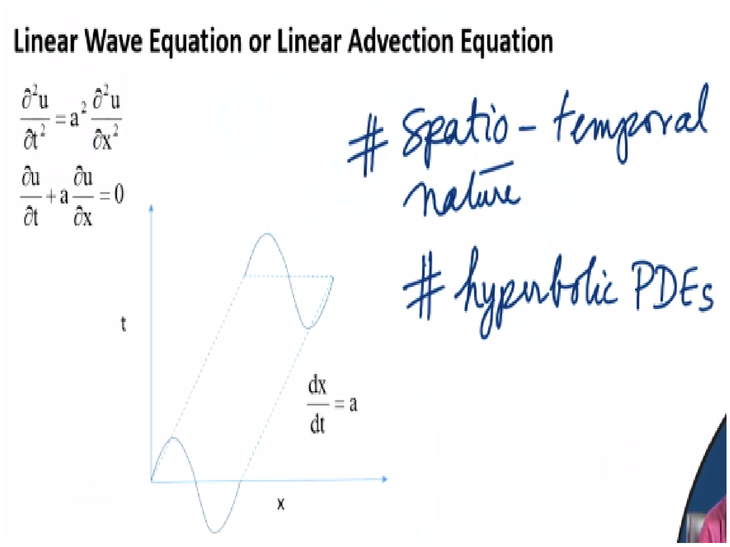

One more important fact about the wave equation is that you have noticed not only that it is linear in form, but it also has spatio-temporal nature. That means, you have to track its evolution both in space and time. Also keep in mind that the wave equation falls in the category of so called hyperbolic partial differential equations.

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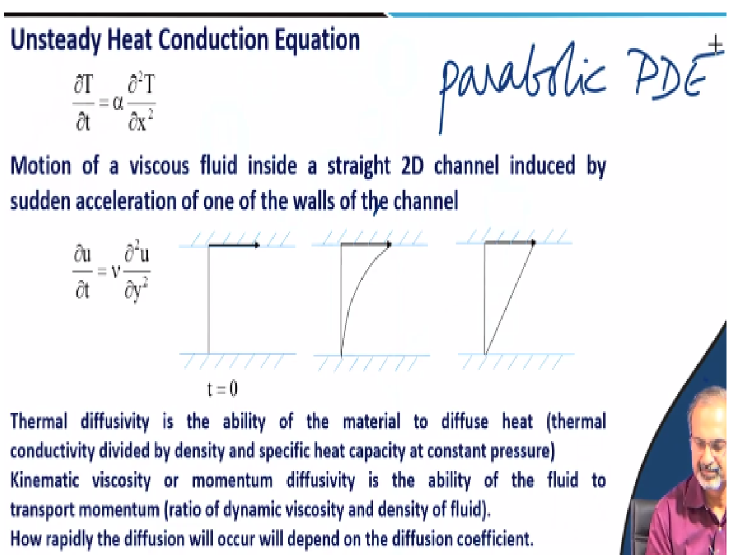

One thing that I forgot to mention was that when we were talking about the unsteady heat conduction equation, that family of PDE falls in the category of so called parabolic partial differential equation. We will come to these classifications of elliptic, parabolic and hyperbolic partial differential equations a little later in the course.

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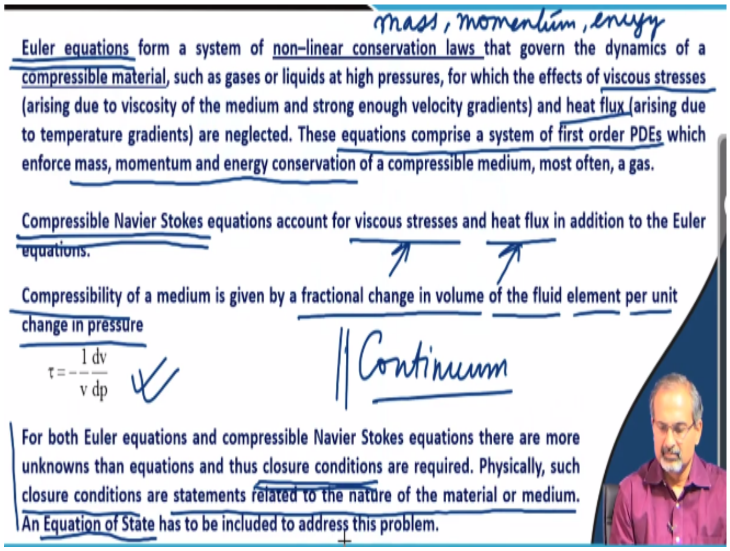

We may have more complicated situations to handle than can be handled with the previous three model equations. We try to recall that we looked at Laplace equation, we looked at transient heat conduction equation, we looked at linear wave equation or advection equation as model partial differential equations. Now comes the time where we may need to handle more complicated phenomena, which cannot be tackled alone by one partial differential equation.

That is what induces the need for invoking system of partial differential equations. You may have heard about Euler equations. Euler equations form a system of nonlinear conservation laws. I had talked earlier about conservation of certain flow properties. So, what conservation are we talking about. We may be talking about conservation of mass, momentum, energy and more.

Euler equation does look at these conservations of a compressible material which could very often be gases, sometimes even liquids at enormously high pressures and under such pressures liquids also may behave like compressible materials and then we neglect two effects, one that we neglect the viscous effects the other we neglect heat transfer effects. Euler equations form a system of first order partial differential equations.

And as we said earlier, they conserve mass, momentum and energy. One more level of complication increases when we handle a compressible medium using the Navier Stokes equation. Good reason for doing that could be that we are interested in capturing viscous effects or we are interested in capturing heat transfer effects or both most often both and then these are the additional effects that Navier Stokes equation can handle over and above what Euler can.

For both these systems, there is another assumption that the medium behaves as a continuum. That means, the molecules are packed so tightly that no matter how small a domain you will look at, you can see a continuous medium that is essentially the continuum hypothesis and both Euler equations and Navier Stokes equations operate under this hypothesis. Of course, there may be situations where you may be violating this condition.

Then you have to look at other equations which can tackle that. In the meantime, you may be recalling about compressibility of a medium and then what we talk about is that if we have a fractional change in volume of the fluid element per unit change in pressure, we have compressibility. So if volume change is happening due to change in pressure, then we have compressibility. There are more formal definitions, but if you can understand this bit, it should be okay for you at this point of time.

Remember that just Euler equations or Navier Stokes equations are not necessarily sufficient to solve for the dependent variables that you are interested in. So, we may need to invoke more equations which we call as closure conditions. That means you invoke more equations into the system so that you can finally manage to have a match between the number of variables that you are solving and the number of equations that you have, but of course these additional conditions are not arbitrary.

In this case, for example, the closure condition will state the nature of the material or medium, which often happens through a so called equation of state.

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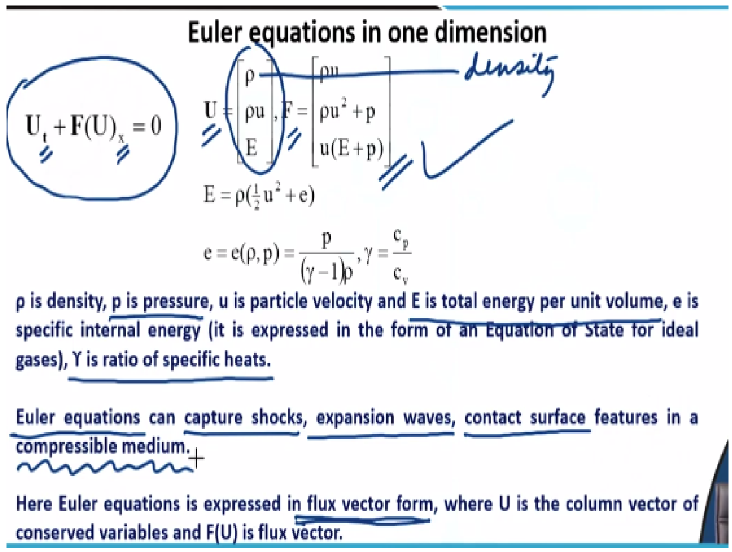

We will close this lecture by looking at the form of Euler equations in one dimension. This is one possible way that you can represent the equation. There are numerous other possible ways that you can do it. This is the so called flux vector form, where you may note that you have a column vector here comprising of quantities, which are actually being conserved. So, ρ stands for density. Of course u stands for velocity, E stands for the total energy per unit volume.

Here 'p' stands for pressure and γ stands for the ratio of specific heats. Now, how is Euler equation represented here? It is apparently represented in a very simple form here. So, you have a time derivative operating on a column vector which is given by **U**. You have a space derivative operating on another vector, a column vector which is given by **F** and those two terms when summed together and equated to 0 gives you the full system of equations.

So, when we have systems of equations to be represented, this is a very convenient way of representing them and this is called as the flux vector form. This form is very preferred in CFD literature. We need to know what Euler equations can do for us. They can capture shocks, they can capture expansion waves, they can capture contact surface features. These are all very often found in compressible flows. We will talk more about these features in future lectures. Thank you.