Introduction to CFD
Prof. Arnab Roy
Department of Aerospace Engineering
Indian Institute of Technology - Kharagpur

Lecture - 31
Numerical Solution of Liner Wave Equation (Hyperbolic PDE) (continued)

We continue our discussion on linear wave equation in this lecture.

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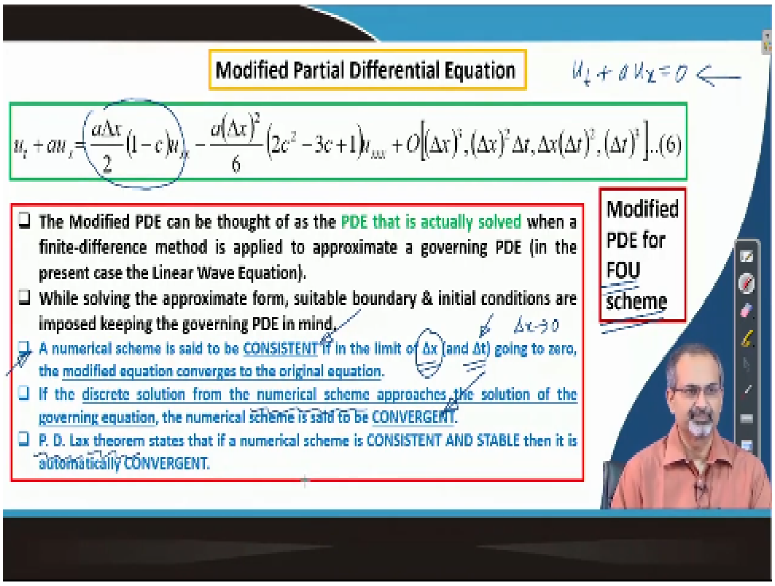

In the previous lecture, we had discussed about the modified partial differential equation. So, we are revisiting that equation here in the starting slide. So, we remember that we did it for the first order upwind scheme for discretizing the linear wave equation and we could identify that there are error terms in the truncation error and incidentally, the leading error term happens to be a second order derivative in space associated with a coefficient.

So, we had also discussed that the modified partial differential equation is effectively the partial differential equation which we end up solving when we approximate the governing partial differential equation using a numerical scheme. And though we enforce the initial and boundary conditions, which are appropriate for the exact equation, we end up using them also for the modified partial differential equation.

One of the important things that we would like to discuss at this point is that a numerical scheme would be considered to be a consistent numerical scheme. If in the limit delta x or delta t going to 0, we can limit the modified partial differential equation to the original partial differential equation. That means, if you look at equation 6 above, if you were to limit the delta x is that you see over here to 0, then you should be able to retrieve the exact equation.

And we can very easily check that that is what happens over here. Incidentally, you are not able to see terms which contain delta t because we have converted all the temporal derivatives to spatial derivatives or even the mixed derivatives where some part of it was a temporal derivative has also been converted entirely into a spatial derivative. So, you are finding terms with only delta x or delta x raised to certain power.

And as we limit delta x to 0, then obviously, you are able to retrieve the original partial differential equation. So, if that happens, then you have a consistent discretization of the original partial differential equation. So, in this case, as you can figure out that the FOU scheme, the first order upwind scheme is a consistent scheme.

Additionally, we have another property that if we are looking at a discrete solution of the original partial differential equation using the numerical scheme that you are proposing, the discrete solution coming from the numerical scheme should approach the solution of the governing partial differential equation itself as this limiting exercise is implemented. So, this property is of course, in terms of the solution itself.

So, consistent property was with respect to the modified partial differential equation limiting to the original partial differential equation, while the convergent property is linked with the solution itself of the discrete form coming from the numeral numerical solution of the original partial differential equation and whether that discrete solution is approaching the solution of the original exact partial differential equation.

If that is valid, if that is being satisfied, we have a convergent numerical scheme. Now, P.D. Lax was the scientist who proposed a theorem, which states that if a numerical scheme is both consistent and stable, then it is automatically convergent.

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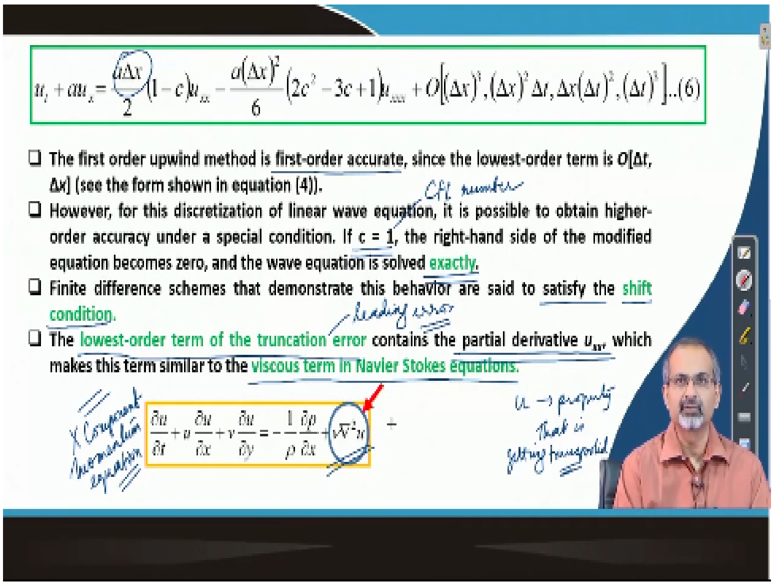

We see that the first order upwind scheme is a first order accurate scheme both in space and time. That is obvious from the leading error term that you have over here. And we also notice another very interesting property that if you set c = 1 that means the CFL number is set equal to 1, then all the terms in the truncation error seem to vanish. That is a very, very important property that means you were going to solve the linear wave equation exactly.

In that case, you will not be having these error terms at all. This is often called as satisfying the shift condition that means a certain situation where you are able to take the truncation error part completely off that means you are able to limit to the original partial differential equation itself. Now, the lowest order term that you have in the truncation error, which is often called as the leading error term.

If we look at that term that is of significance to us, because that is supposed to be the most significant error term, which will affect the solution that we see contains a partial derivative u xx, the second order derivative in u. Now, u is essentially the property that is getting transported. So, if you are talking about a second derivative of that property, that is similar to the viscous term in the Navier Stokes equation.

So, we have just written the x component momentum equation here of Navier Stokes equations for easy reference and we find that this is a viscous term that we see on the right hand side of the Navier Stokes equation. And the leading error term in the truncation error for the first order upwind scheme looks very analog to **(()) (07:00)**. Of course, you do not have a viscosity coefficient out there, but you have a certain coefficient, which we may call as a numerical viscosity coefficient, let us say.

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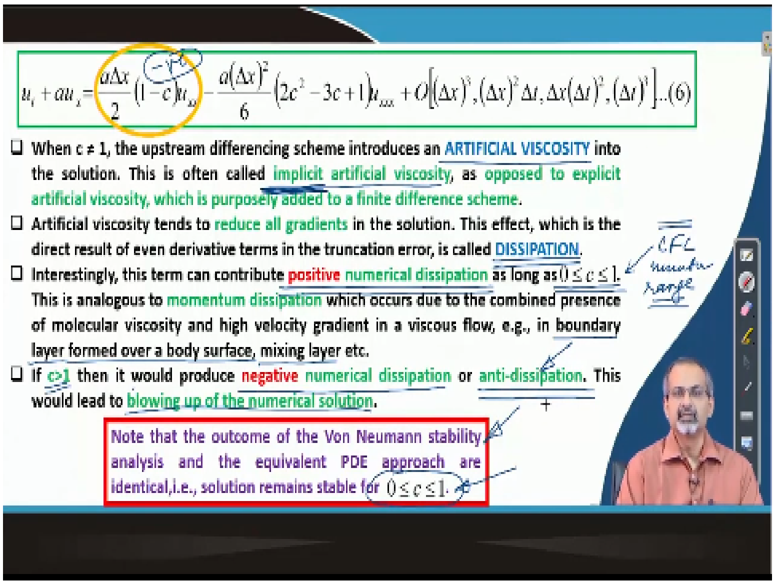

Now, whenever c is not equal to 1, what this term is going to do is. It is going to introduce a very important property which is called as artificial viscosity. That means, it is going to play the role similar to molecular viscosity, what you see in viscous simulations of the problem through Navier Stokes equations. So, this comes automatically by virtue of the numerical scheme that you were using.

So, it is implicitly introduced; it is not that you are explicitly adding this kind of numerical viscosity or artificial viscosity, but it is coming implicitly through the kind of numerical scheme that we are using to discretize the linear wave equation. Now, what would this do this would introduce dissipation, it is going to reduce the gradients in the solution. This we have seen when we have talked about say elliptic partial differential equation.

Now, whenever positive numerical distribution is introduced for the range of values of C that we see over here. So, this is essentially the CFL number limit or the CFL number range that keeps first order upwind methods stable. Then we have positive numerical dissipation in action. And this would play a role similar to the kind of momentum dissipation we see in viscous flow problems.

And in viscous flow problems, how momentum dissipation takes places is simultaneous presence of the molecular viscosity coefficient and fairly high velocity gradients in some regions of the flow which can make the viscous terms very significant. Now, that is often seen in regions like boundary layers, which are found on solid surfaces, mixing layers and things like that.

Now, if you have a situation in the first order upwind equation where c is greater than 1, then that is going to create negative numerical dissipation because that would give rise to a negative term here or anti dissipation that means, a phenomena which is opposite to dissipation, and this would lead to blowing up of the numerical solution. The slightest of errors that build up in the solution would get amplified very severely and gradually the solution will go out of control.

So, this is something that one must avoid when one is proposing a numerical scheme to represent the linear wave equation in an approximate manner. Now, interestingly, this outcome that we have found here that keeping the artificial viscosity coefficient positive gives us the same outcome as the outcome that we saw through the Von Neumann stability analysis.

So, that both the approaches have led to the same outcome that means the c value that CFL number value should be bounded in this manner. So that we can keep the system stable. So, this is a very interesting outcome, which we are seeing through two apparently different ways of looking at the same problem.

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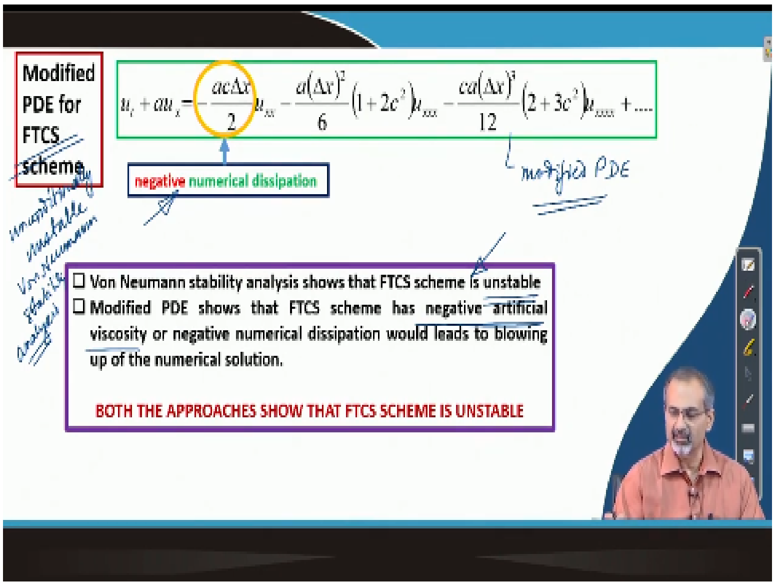

If we quickly recall about the forward time central space scheme for discretizing linear wave equation, we remember that it did not work because it was unconditionally unstable. This, we have shown through the Von Neumann stability analysis. So, though it is not an useful scheme, we would like to revisit it once more just to see that how does the modified partial differential equation form of the FTCS scheme look like?

Does it give us a clue similar to what Von Neumann stability analysis gave us. So, we find that there is negative numerical dissipation coming in the leading error term in the truncation error for the FTCS scheme. So, that is the recipe for trouble. So, like Von Neumann stability scheme has shown that it is unconditionally unstable.

Here we see through the modified partial differential equation approach that it has a negative artificial viscosity and that means anticipation and therefore, it is going to be unstable. So, both approaches are consistently showing the same outcome.

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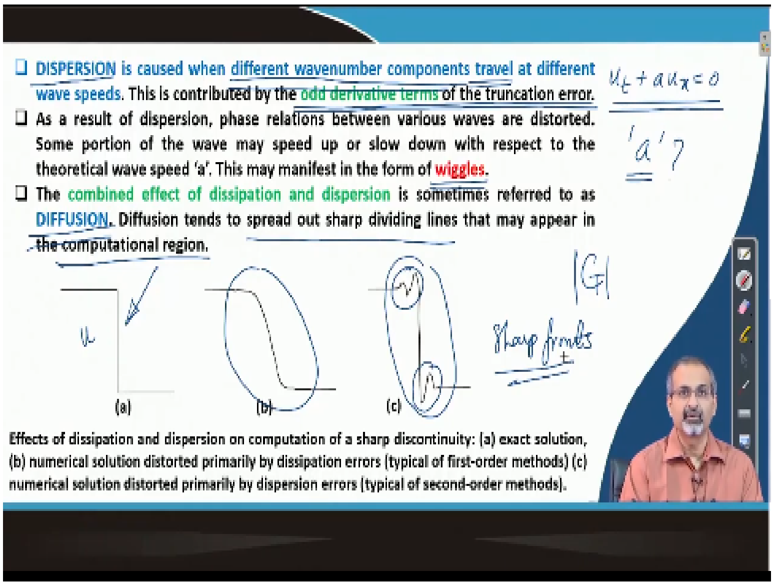

Another very important property of numerical schemes is what we call as dispersion. Will show some examples more concrete examples to explain how this person works, but, for now, let us first try to understand it from very basic perspective. We had done this in our previous lecture as well, but we are revisiting it once more to make sure that the concepts get through very clearly.

So, dispersion happens when different wave number components of the waveform that we are talking about travel at different speeds that means, not all the wave number components travel at the same speed. Ideally, if we were exactly representing this equation, then there was no reason for worry, because every wave number component would then have traveled at the same wave speed that is a.

So, does it really travel at a? That is a question to ask and the answer if the answer is no, that means, different wave number components are essentially traveling at different speeds. None of them are all or most of them are not complying with a and if that is the case, then we would have an error introduced into the solution which we call us this dispersion error. Usually, when we look at the truncation error part, we would find them in the form of odd derivative terms.

So, once we have found the modified partial differential equation in the modified partial differential equation, we look at the leading error term and if that happens to be the odd, happens to be an odd derivative term, then that would be responsible for this dispersion of the wave. So, earlier we were looking at dissipation, we remember that the first order upwind scheme shows positive dissipation.

So, dissipation is certainly connected with the modulus of G because that will be responsible for gradually decreasing the amplitude of the wave. So, that is its dissipation while if different components of the wave in terms of the wave number contributions get segregated out because they are traversing at different speeds then that leads to dispersal. Now, what is the typical signature of this person, you will often see wiggles formed.

Wiggles formed around what around the initial waveform. You would see wiggles formed either ahead of it or behind it. So, wiggles would be found in and around the original waveform as it translates. That is a typical signature of dispersion. Now, if we have a combined effect of both dissipation as well as dispersion working almost with comparable strength, then we may end up having a diffusive error.

That means, we are no longer talking about a segregation in terms of dissipation or dispersion, but in general a term diffusion which spreads out sharp dividing lines in the computational domain if they exist at all. So, here we are talking about a sharp feature, a sudden change in a property, say u which is getting transported. So, if you are trying to numerically capture this feature, then a diffusive scheme could capture it gradually like this.

As you timestep the problem, the sharp front will get weakened. A predominantly dispersive scheme will try to capture it this way. That means, it does not lead to decrements of the sharpness or even decrements of the amplitude, but it generates wiggles. Wiggles can even form simultaneously ahead and beyond of the front. And usually these errors get very much amplified when you have sharp fronts to be captured.

So, sharp fronts could be very challenging situation for numerical schemes. Typically, first order accurate methods like FOU would show predominantly dissipation error, while second order methods would primarily show dispersion error.

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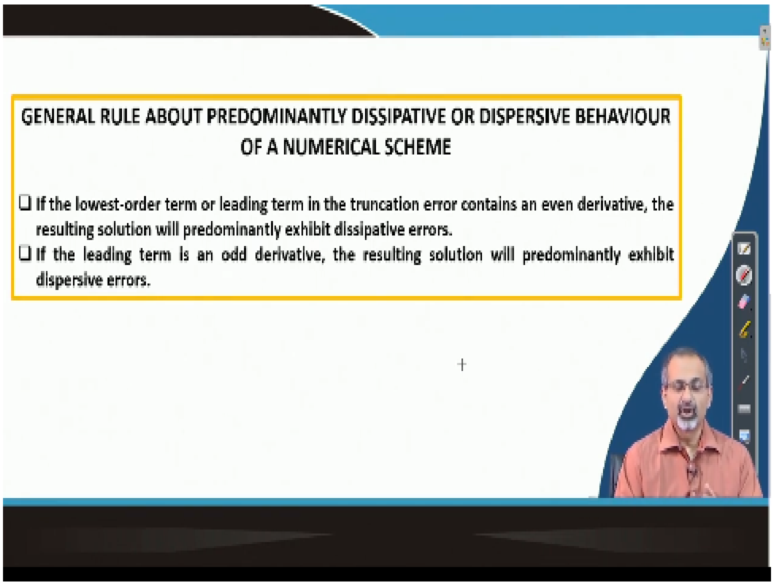

So, we are just trying to sharp on the general rule. So, if the lowest order term or leading term in the truncation error contains even derivatives, then we see predominantly dissipative errors in the solution while if the leading error contains or derivative, then we predominantly see dispersive error. To discuss 1or 2 more aspects which will give us more insight into dissipation and dispersion error.

Also, we would revisit the first order upwind scheme once more to show that we can represent it in a manner where it is shown as a combination of central differencing portion as well as an artificial viscosity portion. That means, we can in general show upwind schemes to be coming out of a central differencing summed with an artificial dissipation term with a tunable coefficient.

So, upwind schemes are all generally developed this way. So, let us try to see the example for the first order upwind scheme. And after this we will also look a little bit more on dissipation and dispersion.

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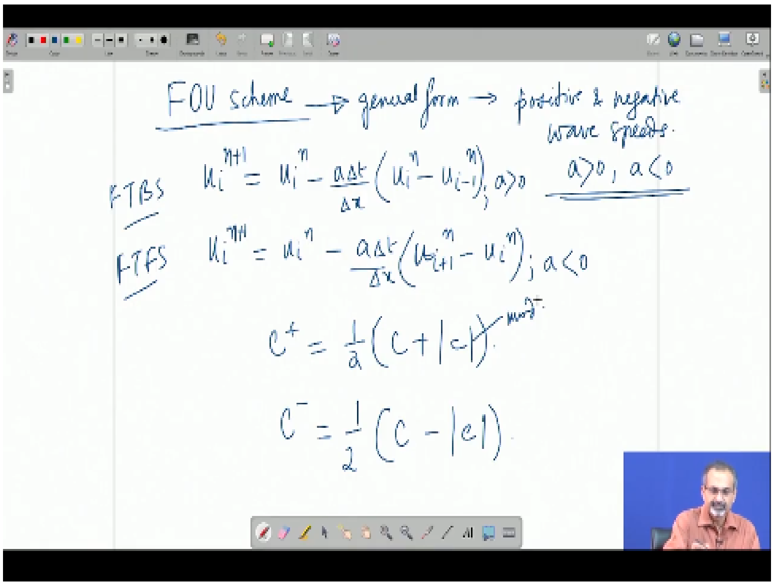

So, we are revisiting the first order upwind scheme. And now that we have understood the concept of modified partial differential equation as well as the artificial viscosity. We will see how these concepts come in, in framing the first order upwind scheme and now, as we frame it, we frame it in a more general sense, a more general form, where we account for both positive and negative wave speeds which means that a can be greater than 0 or a can be less than 0.

So, if you have any of such situations, can you have a more generalized representation of the first order upwind scheme. So, let us try to achieve that first. So, we write down the discretization for the FOU scheme first where a is greater than 0. So, as you remember that this would be the FTBS scheme, the backward differencing in space and when a is. So, this is for $a > 0$ and when it is for $a < 0$ then you essentially have the FTFS scheme.

So, FTBS and FTFS schemes here. Let us introduce some new nomenclature say C+ will represent as this and C- as this. So, this is mod.

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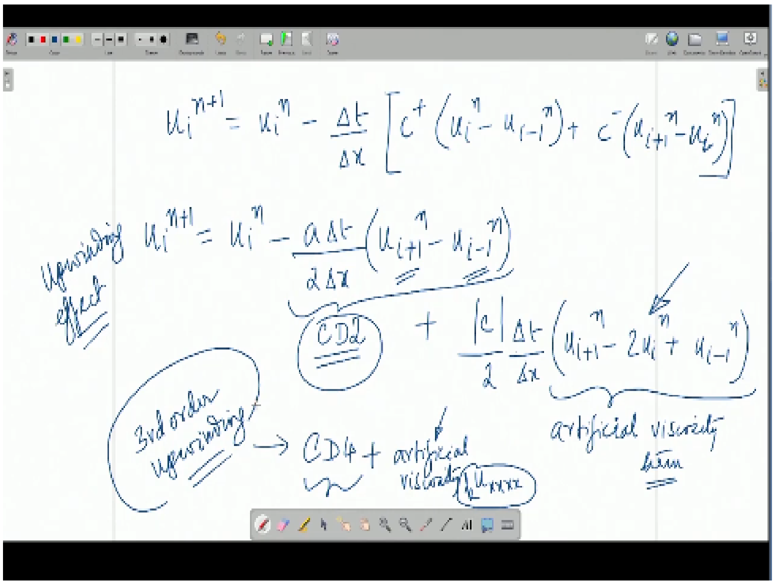

As we do this we can represent the first order upwind scheme in a more general form where you can cater to both positive as well as negative wave speeds. So, this is how one write it and if you substitute the values of C+ and C- and do a little bit of arrangement, you will be in a position to show it this way. So, this is the final form as you can understand that this contribution comes from a central differencing.

So, it is a second order central differencing which is involved because we have i + 1 and i - 1 terms. While this contribution is like an artificial viscosity term. So, you remember that artificial viscosity or the first order upwind scheme was showing up as a second order derivatives; the leading term of the truncation error. So, the second order derivative will be proportional to this bracketed term.

That means, this is connected with the artificial viscosity content, which is added to a centrally different portion which would give you an equivalent upwinding effect and this is a general rule. So, it works for more higher order accurate upwinding as well. So, let us say, if you are looking for a third order upwind scheme in order to improve your accuracy of upwinding in the formal accuracy.

Since, then you would go for CD4 and add artificial viscosity to it which would involve the fourth derivative of the dependent variable. So, let us say it will be u x x x x with a certain coefficient, which you can control. Let us call that coefficient as a k. So, that would be the kind of expression you would use to represent artificial viscosity to be added to a CD4 in order to give you a third order upwinding.

So, this is another aspect of how centrally difference scheme can be converted into an upwind scheme.

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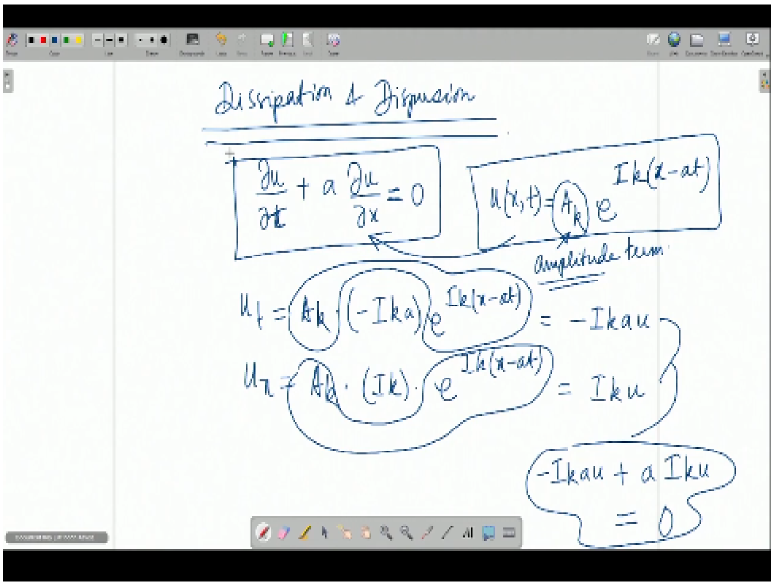

We do a little more discussion on dissipation and dispersion. Let us go back to our governing partial differential equation which is this and let us assume a solution of this form for u as a function of space and time. Let us say we assign a certain amplitude term and we purposely introduced a suffix k that means, amplitude which is a function of wave number and we include the usual exponential term which we have discussed in previous lectures as well.

Especially, where we had tried to quantify dissipation and dispersion error of numerical schemes and compare them with the exact solution. So, in that case, we had not included this amplitude term, but to give more general flavor we would include an amplitude term as well over here. And then we know that if we substitute it into this governing equation, this would be a solution for the governing equation.

Let us do it very quickly. So, if we do a time derivative, then what do you get? You get the amplitude term here and then you will get a - I k a into e to the power of I k x - at. So, this would be the time derivative. And then if you do the space derivative, you will get A k times I k into e to the power of I k x – at. Now, obviously, the A k into the exponential term that is nothing but u itself.

So, it is - Ik a u for u t and it is I k u in case of u x. So, now, if you substitute these 2 expressions into the governing partial differential equation, you get – I k a u + a into I k u and which of course is equal to 0 and therefore, that proves that this form satisfies the exact partial differential equation.

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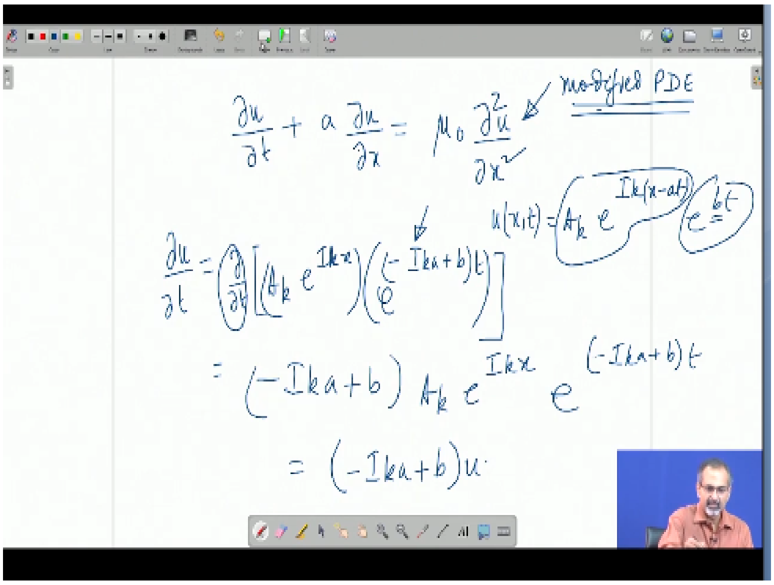

So, we would like to see further that instead of the linear wave equation, if we include a dissipation term on the right hand side of the equation, then what kind of a trial solution might work. This, we are interested because now we have seen modified partial differential equation emerging from the different numerical schemes that we are trying to use.

So, if this is the form of the modified partial differential equation accounting only for the leading error term, that then what kind of trial solution might work in order to solve it. So, let us try one. So, we just include another exponential term here, in addition to the functional form that we had already assumed for the wave equation alone, and then the challenge would be to solve for the unknown parameter b.

So, how do we go about it? So, we first take the time derivative, and we find that that would be equal to let us try to segregate the space and time out, because that will help us calculate the time derivative more easily, this is the beauty of partial differentiation. So, this operator will essentially operate only on this part. So, what what are we coming up with it is giving us – I k a + b into the spatial part along with the amplitude and then the temporal part.

So, as we can understand this is nothing but – I k a + b into u. So, that is del u del t. And then we have to work out the del u del x. So, we actually have to work out a del u del x and that is very easy to calculate. It is i k into u. That is more convenient to calculate. Right. So, now, if you substitute these 2 forms into the partial differential equation that we are looking at, which is essentially linear wave equation with the second order dissipation term on the right hand side, then what do we get.

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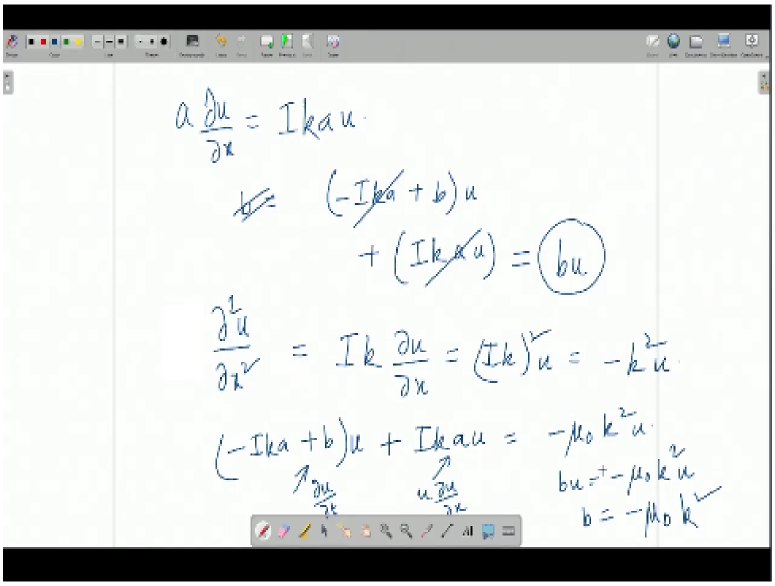

We substitute and we should be able to get, we will try to write it down this way. So, del u del t is - I k a + b into u and a del u del x is I k a into u. And if you check, that gives us, so, we are left with so this and this term and calculate will go away and we are left with b times u. So, that is what we have from the left hand side. Now we have to work out the right hand side.

So, for which we actually have to find out the second order derivative. So if you work out the second order derivative, you will find that this works out to be I k del u del x. So, that is I k square u, that is - k square u. And therefore, now, what we have essentially is we have worked out all the terms that we need. So, we had we had – I k a + b into u on the left hand side coming from the temporal derivative, then I k a into u coming from the spatial derivative.

This is del u del t. This is u del u del x. And then on the right hand side you now have - mu not k square u. So, as we showed that we will be left with b times u on the left hand side and -

mu square k square u on the right hand side which gives you the value of b which is - mu not k square. So, finally, we have solved for the value of b.

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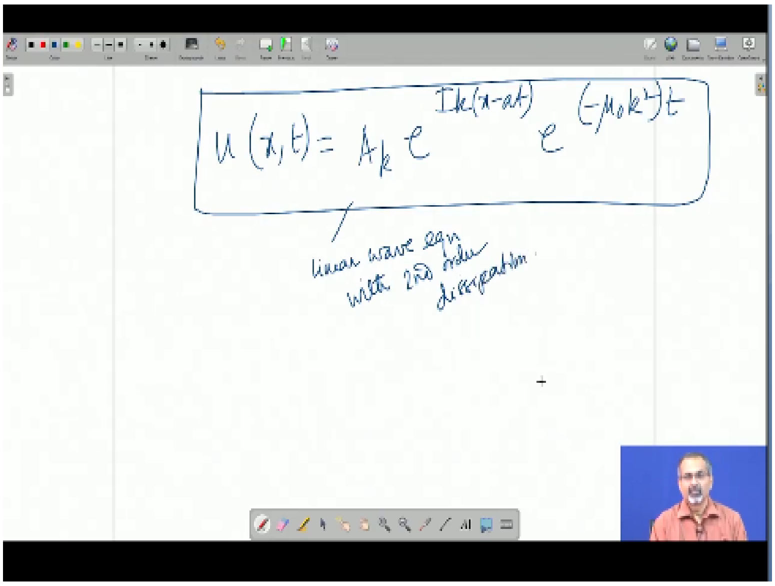

And therefore, the solution u x t will then be A k e to the power of I k x - at into e to the power of so, we now substitute the b that we have worked out. So, this is the final form for the linear wave equation with second order dissipation. So, we will discuss more about this equation in the next lecture. Thank you.