Introduction to CFD
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Lecture - 34
Numerical Solution of One Dimensional Convection - Diffusion Equation (continued)

We continue our discussion on one dimensional convection diffusion equation.

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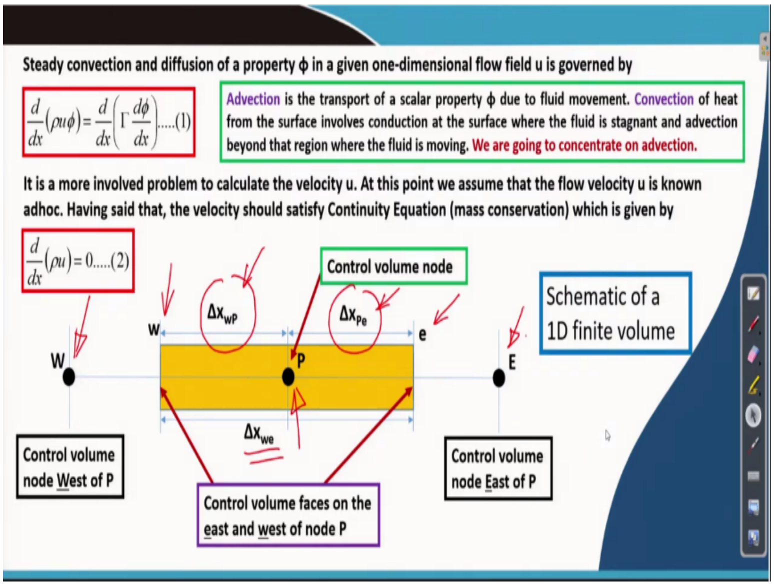

Let us look at the transport equation for property phi which involves the advection term on the left hand side and the diffusion term on the right hand side. Now, let us recall the definition advection is the transport of the scalar property phi due to fluid movement. And when we say convection, convection has a slightly broader meaning it in the context of heat would mean that there will be convection of heat from a surface involves conduction at the surface where the fluid is stagnant because of the boundary layer effect.

And there will be advection beyond that region which is stagnant or sticking to the surface where the fluid is moving. So, advection becomes active in the layers where fluid movement takes place. So, we are primarily going to concentrate on advection. At this point, we are not attempting to solve for the velocity field because that is a more involved problem, which we will see in the later part of the course.

At this point, we are assuming that the flow velocity u is known to us; it is provided to us in an ad hoc manner. And having said that the velocity should of course, always satisfy the mass conservation equation or the Continuity Equation which is given by the second equation. So, the first equation involves the advection diffusion of property phi and the second equation involves the mass conservation and therefore, involves the velocity field u and the density rho.

So, if we now look at a control volume surrounding the node P, then we can see its features over here. So, it has a east face it has a west face and the node P is usually defined at the center of the control volume and it has a certain width delta x small w e. Now, if we want to define other distances, let us say from node P to face e or node P to the face w etc., this is how the distances could be specified.

Again, there will be neighboring nodes like we have over here. Then node which is to the east would be generally marked as capital E to the west would be marked as capital W and so on. So, this is a typical schematic of a one dimensional finite volume. And you may recall that we decided to take the approach of finite volume to integrate the equations and proceed with the solution.

So, let us see how we can proceed with this definition of control volume and the governing equations, how they would work out in this framework.

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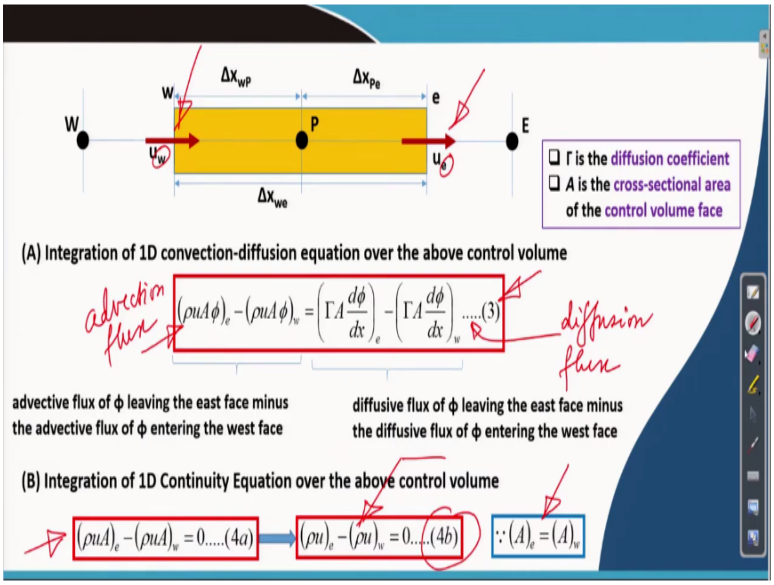

So, for that we will go to the next slide. And in the next slide, what you see is that we have further mark that the velocities exist and the velocities could be defined at the interfaces like we have marked over here. So, u with a suffix e corresponding to that interface, small e and u with a suffix w corresponding to the interface w. And now, if we look at integrating the equation, then this is how it works out.

So, if you integrate the advection term it yields the term rho u capital A times phi at the east face minus the rho u A phi at the west face. So, this is essentially the advection flux which we have on the left hand side and what you have on the right hand side is essentially the diffusion flux or the diffusive flux. So, these are the terms we have on the two sides of the equation.

And then if we look at the integration of one dimensional continuity equation, this is how it works out. So, the product rho u A if you take a difference between the east and the west face that should equate to 0 which essentially means that the product rho times u will remain constant provided that the areas are identical. So, the kind of control volume that we are looking at, we do not have a variation of area as we move along the x direction.

And therefore, A suffix e = A suffix w and therefore, it boils down to this equation 4.4b where the **(()) (05:25)** term goes out of the equation.

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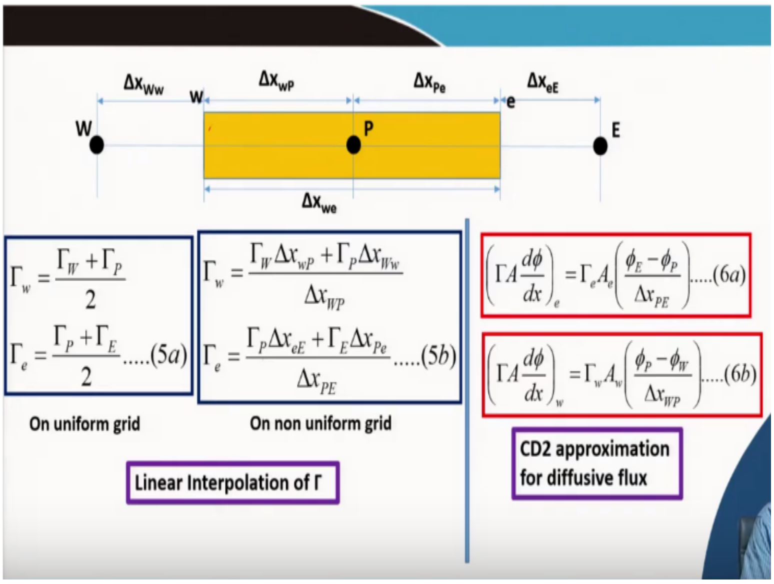

Now, let us look at the diffusive part. So, gamma is the diffusion coefficient and we can work out and arithmetic average to define gamma at the interfaces W and E based on the neighboring cell center values. So, that holds good for a uniform grid. However, if you have a non uniform grid, then you would have to have suitable weightages to multiply the cell center values and then again divided by the distance separating the cell centers, which we have to invoke for obtaining the interface values.

So, uniform and non uniform grid calculations would vary based on whether we have to have these weightages in place or the weightages very simple, it is just point 5. So, that is the calculation of gamma part. And then we have to finally approximate the diffusive flux for which the derivative comes into picture. So, you see equation 6a and 6b where the derivative calculations are shown.

And we can see that the CD2 approximation has been used for calculating the derivatives. So, that is again based on the neighboring nodes and the neighboring nodal values for calculating the derivatives at the respective interfaces.

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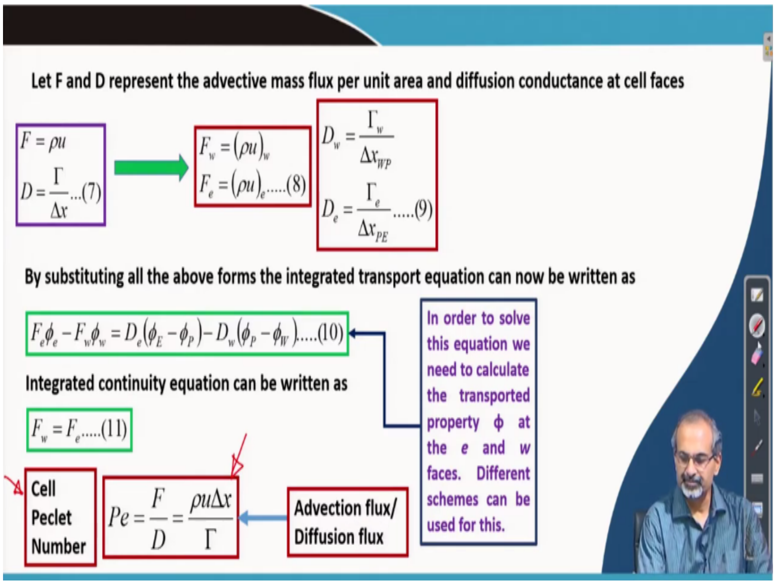

So, now, let us define the fluxes F and D. So, we will call F and D to represent the advective mass flux per unit area and the diffusion conductance at the cell faces. So, we can define F as the product of rho and u and D as the ratio of gamma and delta x. So, you can check easily that both of them will yield the same kind of dimensions. And then you define the advective fluxes at the interfaces e and w with F w F e.

And then similarly, the diffusive flux conductances at the same interfaces with the formulae 8 and 9. And then you finally substitute them in the integrated transport equation, which looks like equation 10. And then you realize that in order to solve this equation, we would certainly have to define the phi’s at the small e and small w locations. Because, on the right hand side of the equation, you find that the phi's are already available from the cell centers, where they are supposed to be updated from time to time.

But you do not have a ready definition available for phi small e and phi small w because they are the interface values. And in finite volume approach, we always have to think about strategies of calculating these interface values. And there could be different possibilities here, which we will explore in due course. Further, now, that we have defined F and D. Based on that we come up with a definition of a very important non dimensional number, which is called as the cell Peclet number.

So, cell Peclet number is defined as the ratio of small f rather capital F and capital D and that works out to be rho u delta x by gamma. Now, since we are using cell dimensions for calculating the D therefore, we call it as the cell Peclet number. It is nothing but a ratio of these fluxes the advection fluxes and the diffusion fluxes.

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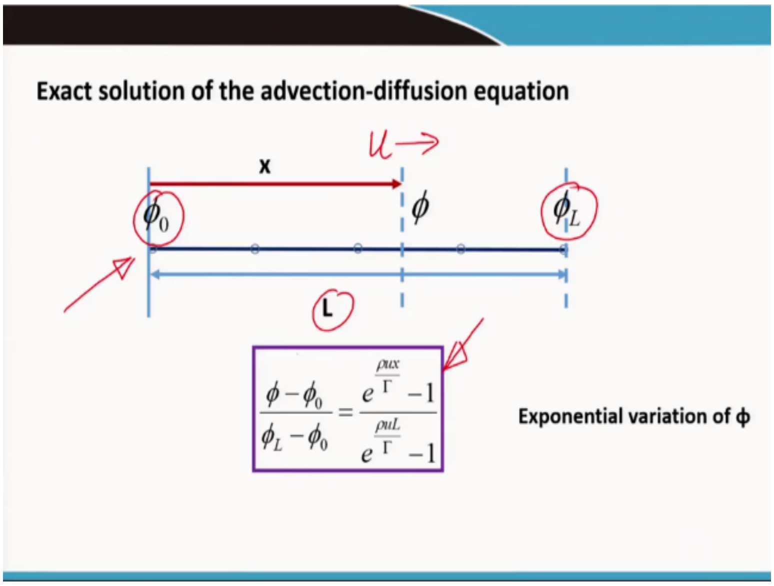

Now, if you look for an exact solution of the advection diffusion equation, then you would have to define a framework like this that you have a one dimensional domain of length capital L. And then you have a variation of the transport property phi within this domain which of course, would depend on the strength of the velocity, because, that would decide that how strong or how weak the advection is and rest of it is of course, diffusion of the property phi.

And then based on the boundary values that you have at the two ends of the domain, phi nought say at x = 0 and phi L at x = L. You will need to define a variation of phi based on exact solution of the advection diffusion equation and incidentally the solution comes out to be like this. Now, you could of course, attempt to solve it yourself as a homework problem for which we will just discuss about one or two small steps to be noted down which could help you in solving this problem at your spare time.

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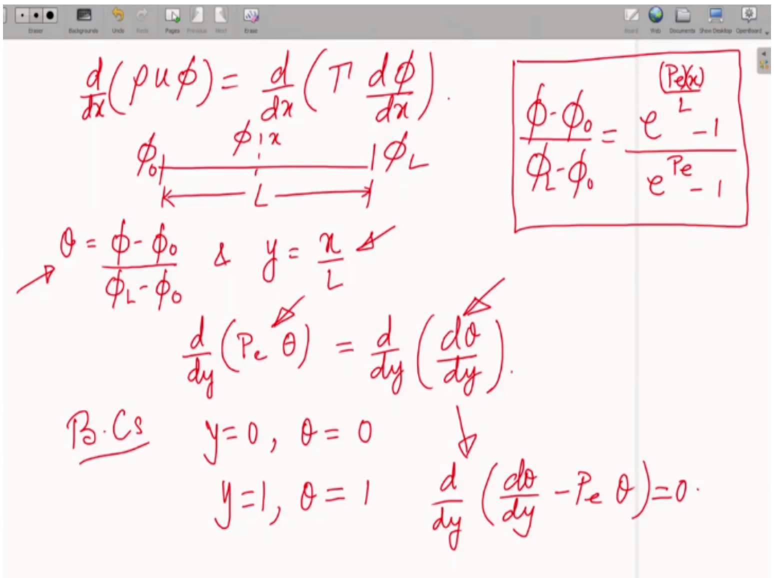

So, when you try attempting to solve this problem. Let us write down the transport equation once more, yeah. So, this is the equation we are trying to solve. And remember that we are solving it over a domain of length L and we have the boundary values phi nought and phi L. And at any arbitrary location x, we are trying to find what the value of phi is. So, for that we can define a non dimensional parameter theta let us say.

Taking clue from the solution that this kind of a term figures in the solution and we try to non dimensionalize the length by define dividing the length scale x by the total length of the domain capital L. And if you use these parameters, you can show that the governing equation can be reduced to a much simpler form which looks like this. So, we have d dy which is essentially the non dimensional length **(()) (12:41)** applied on the product of Peclet number the cell Peclet number and theta and that is equal to d dy of d theta dy.

So, of course, you can understand that this is essentially the transformed advection term and this is the transform diffusion term of the transport equation. And we of course, have to define boundary conditions based on which we are trying to solve this differential equation. So, at y = 0, you can easily show that theta = 0 because at y = 0, phi will be equal to phi nought and therefore, theta becomes 0 and then at y = 1 theta becomes = 1.

And then essentially this equation can be further written in a more compact manner like this. And then you need to integrate it; integrated and apply the boundary conditions. And then you should be able to come up with the final solution which can be written as in a more compact manner than what we showed in the earlier slide by introducing the Peclet number there.

So, you have exponential terms the one in the numerator would look like this, e to the power of P e times x by L - 1 and what you have in the denominator is e to the power of Peclet number - 1. So, this is how it figures. So, this is the exact solution of advection diffusion equation in one dimension.

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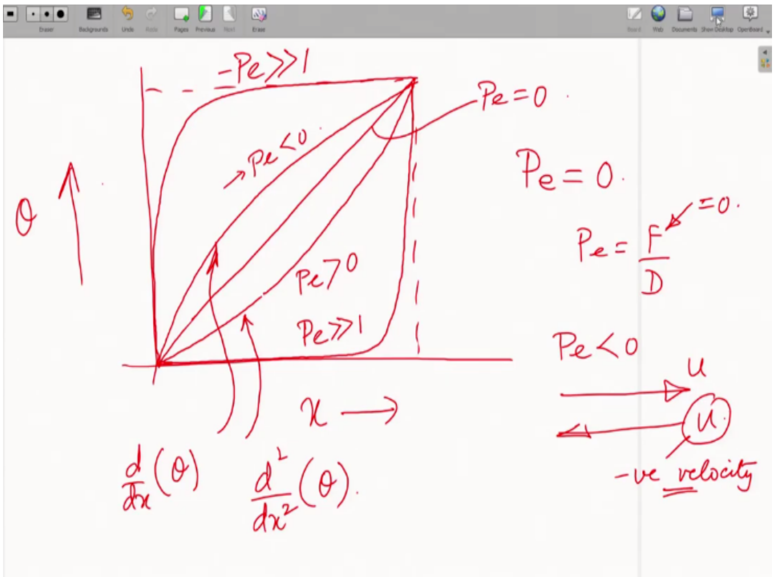

And then taking clue from this definition, one can derive very interesting results. Let us say if you are plotting a theta x curve or x theta then you can imagine that for Peclet number = 0, you would have zero advection effect. Right. By definition, as you can recall, this is equal to F by D, and therefore Peclet number = 0 when this goes to 0. That means you have no advection it is all diffusion.

And then if you look at the governing or rather their solution, exact solution, you will find that zero Peclet number will give you a linear variation. So, it will be a straight line. While if you vary the Peclet number from there, let us try to draw the trends, you will find that Peclet number > 0 would have a trend like this, Peclet number < 0 would have a trend like this.

So, if we imagine that Peclet number < 0 would mean that instead of going from left to right, the flow is actually moving from right to left, which will give you a negative velocity here; based on our reference direction. So, if we have a negative velocity based on that we can of course, define a negative Peclet number. And then it would show a trend like what we have drawn here.

And then if your positive Peclet number becomes very large than in the limit. For very large positive Peclet numbers, the trend would look like this. While for very large negative Peclet numbers, the trend would look like this. So, how can we ascertain these trends for that of course, we have to do a small exercise. For example, if you try to do a d dx of theta, then that would give you slope information.

And then further, if you take a second derivative of theta, it would actually define which way the curve would bend whether it could be a concave or convex kind of a curve. So, that can be explored by you further as a homework exercise. So, these give very interesting trends about how advection diffusion equation solutions are expected to behave.

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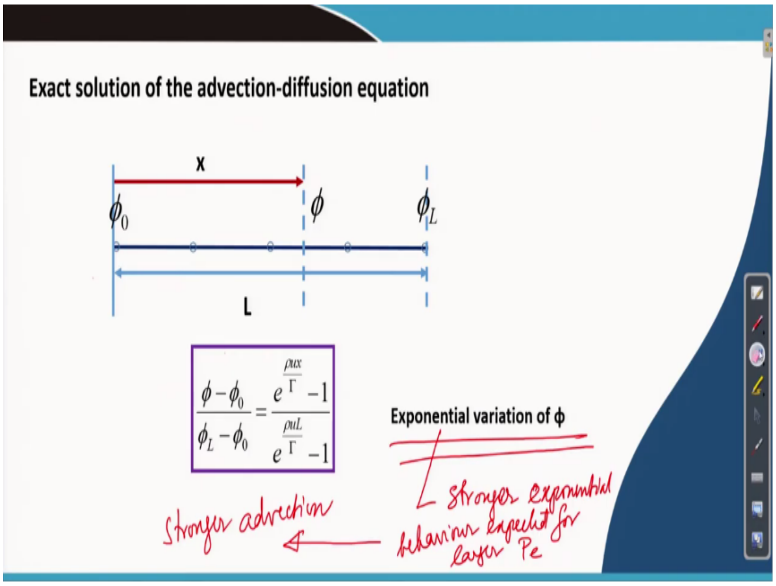

The most important take home from this exercise is that we have an exponential variation of phi which needs to be kept in mind. So, when we try to use different kinds of numerical schemes to approximate advection diffusion equation, we always need to keep in mind that the solution will end up showing some exponential behavior. And when does that behavior show stronger exponential nature? When the Peclet number becomes larger and larger.

So, that means advection becomes stronger and stronger. So, stronger exponential behavior expected for larger self acclaimed number, which means stronger advection. So, these are very interesting trends which we are getting from the exact solution. So, we will try to keep these trends in mind when we go in for testing the capability of different numerical schemes in solving advection diffusion equation in an effective manner.

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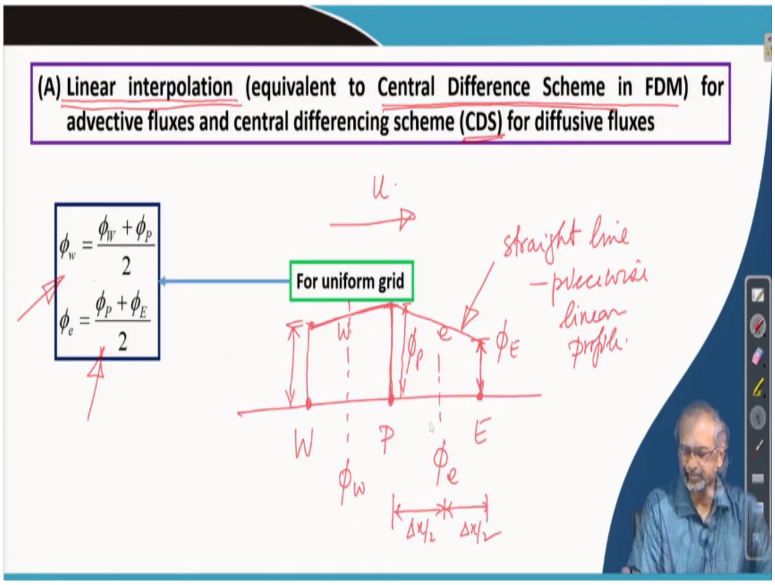

So, we were talking some time back as to how we should define the phi **(()) (19:18)** at the east and west interfaces because, as we said that these are not known to us ad hoc. So, there could be different techniques by which we define these different schemes by which we define these values of the interfaces. So, we are testing the first such technique here, which is the technique of linear interpolation. That means, let us try to make a simple sketch.

So here is our node P node E. You are not going to draw the control volume with all its features every time because we already have that picture in mind from our previous slides. Here, what we need to understand is that between P and E, we have the node e between W and P we have the node w and that essentially defines the control volume surrounding the node P. And the question is how to calculate these files.

So, if we try to represent the value of phi at the node P by this length, let us say. And then again we define the value of phi E by this length. Then if you look at the definition of phi E here we are just averaging phi P and phi E. So, what are we doing? We are essentially drawing a straight line connecting these two values. And when we have equal distances then obviously, it is just an arithmetic average.

But, as we mentioned earlier, you do not essentially need to have equal distances and this is one of the biggest flexibilities of finite volume technique that you can have unequal distances at your convenience because very often in fluid flow problems, we prefer on equal distances, when we have to capture strong gradients. So, we put a very fine grid in order to capture large gradients, and then we tend to put a coarser grid where the gradients are decaying and becoming weak.

So, we do not have large number of nodes in such regions because that saves us computing effort. So, if that is the case even then you can very easily try to define phi e for example, by having unequal lengths over here separating the P from small e and small e to E capital E. So, that can be a simple exercise for you. So, we find that whether it is equal or unequal distances, we are essentially connecting a straight line.

And that essentially means piecewise linear profile. Similarly, if phi w is indicated by this height, for example, we would again join a straight line here and define **(()) (22:30)** phi small w this way. So, this is the strategy of linear interpolation. Now, as we do that remember that we are handling advection flux and we are not paying attention to the direction of the flow direction, direction of the flow, sorry.

So, we are paying equal weightage to nodes irrespective of flow direction that is something that we are committing over here. Would it affect the calculations in any way? That would be answered later. But we need to keep this in mind. And we are going to go ahead with this definition and when we use this kind of a definition, correspondingly when we use finite difference method, we would have called it as a central differencing scheme.

So, very often linear interpolation of CDS essentially means the same. Again remember that all the different techniques that we are going to discuss about we are only going to touch about touch upon the advection terms for applying different techniques, while for the deficient terms, we are not going to touch the technique it is going to be uniformly CD2. Because we have learned earlier that diffusion does not have preferential direction. Right.

So, that is the basis on which we will continue to use central differencing for the diffusion terms.

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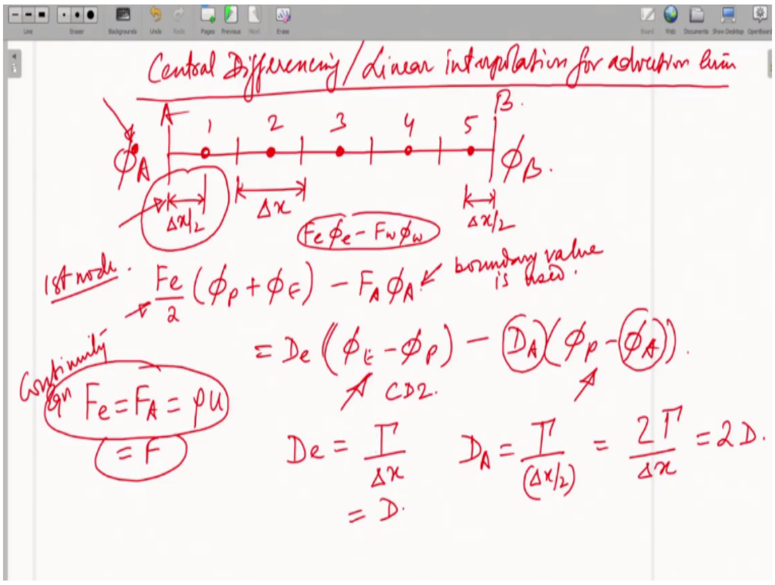

Now, let us try to work out central differencing or linear interpolation for advection term. So, we will make a small diagram like this. Let us say, it is a length A B with boundary values of phi given by phi A and phi B and let us put few nodes inside. So, these are the control volumes and at the center of the control volumes we have the nodes. So, incidentally we have marked four nodes five nodes rather. And let us use a uniform grid for the purpose.

So, if we have this then we can go ahead and start fighting the discrete form of the transport equation, node by node. So, what we will do in the beginning is write down the discrete form of the equations. Let us say, for the first node. Let us try to do that. So, for the first node, let us see how we write down the equation. We write this for the advection term. Remember that the left hand side of the equation, what does it have? It has a difference between F e phi e - F w phi w.

This is what the left hand side of the discrete equation has. So, this is F e phi e, phi e being phi P + phi E by two - F w phi w. Now, how do you do a phi w based on the linear interpolation. For that you should have had a node somewhere here which you do not have because that lies beyond the boundary. So, what do you do you truncate the domain and you try to calculate using a different strategy and incorporating the boundary value itself.

So, you define the convective flux F as say F A and or rather the term F as F A and then the advective flux then becomes F A times phi A where the boundary value is used. What happens on the right hand side of the equation? The right hand side of the equation looks like this. Let us try to put the right hand side values. So, here you will notice that this is a regular calculation based on the CD2 scheme which we have already talked about.

While again over here the boundary value has to come in and then D again has to be defined in a different way. Let us see how we do these definitions. First thing is that F e = F A = rho u. This has to be convert station to station. It cannot change. Right. That is why the continuity equation. So, you may as well define it as say capital F if you wish. What about D e? D e would be defined as gamma by delta x.

And what would be D A? D A would be gamma by delta x by 2. Remember that here the distance that you are using is delta x by 2 not delta x. So, the D at the boundary A becomes different. It becomes 2 gamma by delta x. So, if I call this as capital D then this becomes 2 times capital D, twice the effect. Right. So, this is how I would write down the discrete form of the equation at the first node. How would I do it for other nodes?

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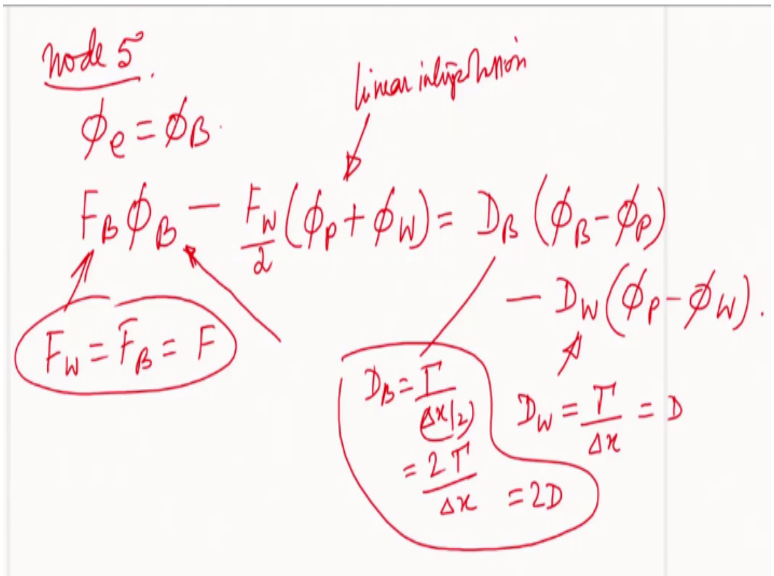

Because try to go to the node 5. So, remember that phi e = phi B for the node 5 because that is how we bring in the boundary value while phi w will continue to be estimated based on the linear interpolation. Right. So, once we take note of that we can write down the advective flux like this F B phi B at the east boundary - F w by 2 into phi p + phi w which remains as the regular linear interpolation.

And then on the right hand side, what do we have? We have D B times phi B - phi p - D w times phi p - phi w. Again, the kind of changes we made for the boundary for the node 1 would work similarly over here. Let us see how we do it. So, you recall that F w will be equal to F B will be **(()) (29:56).** Right. let us call it F because that is again coming from continuity equation.

We are incorporating the value of phi B here without going in for any linear interpolation, because we have reached the boundary. And then as far as the diffusion part is comes up concerned, D w is a regular definition. So, D w will be given by gamma by delta x, while D B will be defined as gamma by delta x by 2, which makes it 2 gamma by delta x. So, things are somewhat similar to what we see in node 1.

So, again, if I call this as D, this will become 2D. Right. So, this is how we are able to figure out the equations for node 5.

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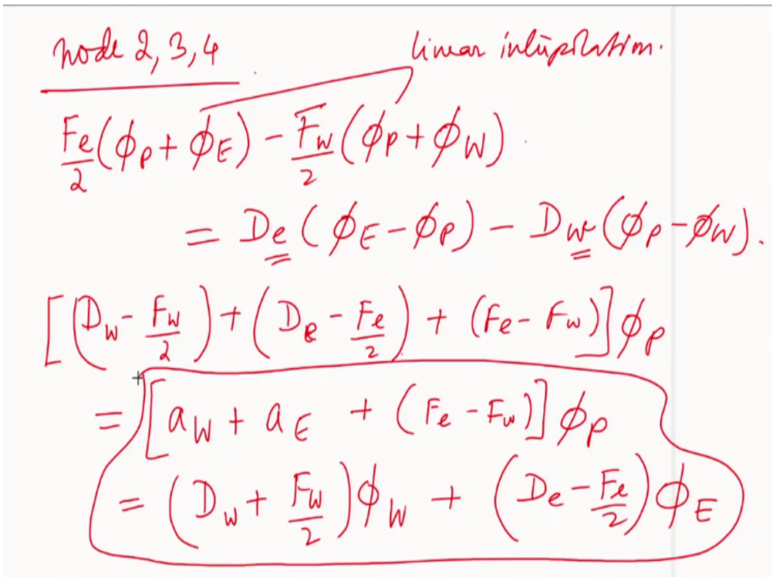

And now, we will try to look at node 2, 3, 4, for which the scheme will remain identical. So, that is rather a much simpler job, because we have already learned to do the more difficult part at the boundaries. So, this is how we do it for linear interpolation. And linear interpolation works equally well on both sides of the control volume because it is a internal or inner control volume, it is not sharing a boundary.

So, linear interpolation for both terms. Right. So, that is how it works. And then even on the diffusion part, you will see that you can use the regular CD2 without any difficulties. Right. So, remember that these are small e's and small w's. And finally, what do we have if we rearrange the equation it would look somewhat like this. So, this is how the left hand side would look and the right hand side will look like.

So, this is how the equation looks like finally for the internal nodes. So, we will continue this discussion in the next lecture. Thank you.