Introduction to CFD
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Module - 1
Lecture – 4
Classification of PDEs

In this lecture, we are going to look at classification of partial differential equations, but before we discuss that we will have a quick look at the incompressible Navier Stokes equations, which we could not complete during the last lecture.

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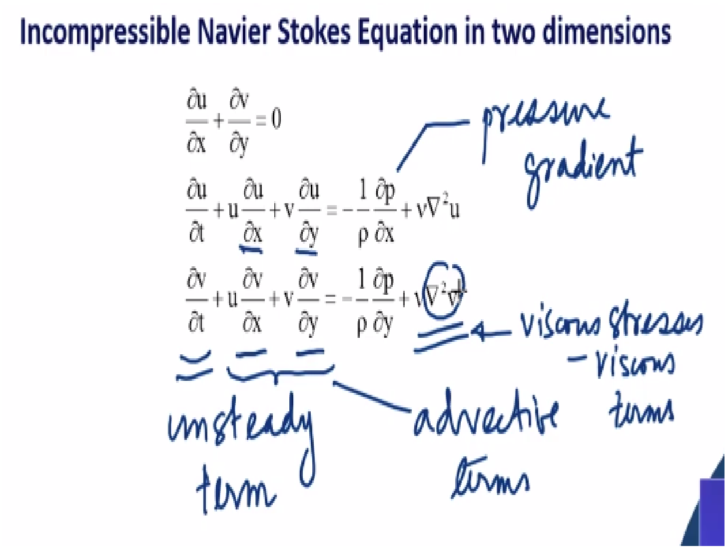

You can see the two-dimensional incompressible Navier Stokes equations in the screen. It is comprised of mass conservation and momentum conservation equations. So, the mass conservation equation is more often called as a continuity equation because it is two-dimensional Navier Stokes equation. So, you have 2 components of velocity respectively u and v and both of the velocity components are functions of x and y.

You can see partial derivatives of the u component of velocity in other equations where they could be derivatives with respect to x or y and similarly it is true for v. So, therefore, u and v are both functions of x and y. You can see the first term on the left hand side which is called as the unsteady term. That means any time dependence in the flow can be captured using such a term while the other two terms are more often called as the advective terms.

You have a pressure gradient term on the right hand side of the equation like you saw in Euler equation, but this is a term which is very typical of Navier Stokes equation, which is arising due to viscous stresses in the flow, and therefore we often call these terms as viscous terms. Note that they involve second order partial derivatives. You did not see second order partial derivatives in Euler equation for example.

Incompressible Navier Stokes equations assume that the flow behaves without any effect of compressibility which means that any variations in pressure in the field would be accompanied by variations in the flow field velocities, but it cannot affect the density. That means pressure and density are decoupled from each other in an incompressible situation. We will look at incompressible Navier Stokes equations in more detail in due course.

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- We need to classify PDEs into Elliptic, Parabolic and Hyperbolic type
- This classification helps in devising suitable numerical schemes for their approximation/ discretization
- We will see how to classify a single second order PDE
- We will also see how to classify a system of first order PDEs

We now look at how to classify a system of partial differential equations. When we classify of course we recall having looked at these terms elliptic, parabolic and hyperbolic and we need to know that when we are looking at a partial differential equation which kind of partial differential equation it is, and this classification is very important for us to be known ad hoc before we try to approximate the partial differential equation and try to solve it numerically.

Because the discretization that you apply would depend on the kind of partial differential equation you are handling. We would first look at a single partial differential equation how to classify that, and we will consider it to be a second order partial differential equation and then we will have a quick look at how to handle a system of partial differential equations and classify them.

But we would like to have such a system of partial differential equations as a first order system of partial differential equations. So, note the difference; in one case it is second order when it is a single partial differential equation, while in the other case where we handle a system, we are looking at a system of first order partial differential equations.

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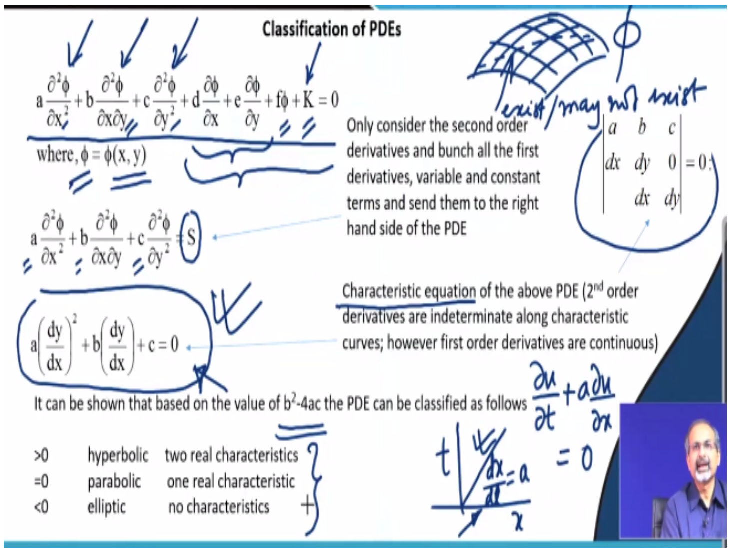

When it comes to classification of a single partial differential equation, you can write it down in a form like this, where you can see second order partial derivatives written of the dependent variable ϕ and ϕ is a function of two independent variables x and y. In this case, we are assuming that there are only two independent variables. In that case, you can have these three possible partial derivatives of second order. One is purely with respect to x, $\frac{\partial^2\phi}{\partial x^2}$ another purely with respect to y, $\frac{\partial^2\phi}{\partial y^2}$ while the other is a mixed derivative $\frac{\partial^2\phi}{\partial x\partial y}$.

Additionally, you can have lower order derivatives, which means first order derivatives in our case $\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}$ and the variable ϕ itself with a constant coefficient and then this is a purely numerical constant K or it could be purely a function of x and y that you have in a term like this. The whole equated to zero gives you the partial differential equation. Now, if you bunch all these terms comprising of first order derivatives, the variable itself and constants and bunch them together as a term S on the right hand side of the equation, then you are left with the higher order derivatives only on the left hand side of the equation along with their coefficients, which will make a difference. These are the coefficients which will finally decide the behavior of the partial differential equation whether it will be elliptic or parabolic or hyperbolic.

Now, how is it that a, b, c are going to decide that. For this, we need a characteristic equation. You may recall that when we were looking at the linear wave equation, we wrote the equation in this form and we plotted the behavior of this equation using a line like this in the x-t plane, where we said that this straight line has a slope dx/dt = a, this is a typical characteristic curve or line.

Now, when you are looking at a solution of a second order partial differential equation like we have shown over here, the solution will actually turn out to be a curve in the x-y plane. So, this is a space curve. Let us say it looks like this and then you will find that on this curve, which is essentially a solution of the partial differential equation, there are some specific lines which exist which are responsible for carrying information in a way similar to how this line carries information in a linear wave equation.

However, sometimes such lines exist while at other times they may not exist. So, they may or may not exist and our job is to find out that for a certain partial differential equation whether we have such lines existing on the solution $\phi(x, y)$. We have to remember that the characteristic equation of the above PDE would have to be obtained in order to extract information regarding the behavior of the PDE.

So, in order to obtain the characteristic equation of the PDE, we make use of two very important facts. One is that since we are handling a second order partial differential equation, the second order partial derivatives are indeterminate along these specific curves, which we are calling as the characteristic curves. However, the first order derivatives will remain continuous.

Based on these two facts, we would be able to obtain a condition for which characteristics either exist or they do not exist and that is the basis on which we have a certain determinant shown over here. This equation basically is an outcome of the determinant. What you saw as the dx/dt = a information for a linear wave equation is actually figuring in the form of dy/dx information in the x-y plane for a second order partial differential equation here.

So, they are kind of analogous. However, it is a little more involved to obtain the conditions under which you can have real or imaginary characteristics here. So, what you do is you essentially look at the discriminant of this quadratic equation in dy/dx and then based on $b^2 - 4ac$, you try to work out the conditions for the existence of real or imaginary characteristics.

So, when you have the discriminant greater than 0, this leads to 2 roots of the equation, that is typically the case for hyperbolic partial differential equations. So hyperbolic partial differential equations have two real characteristics. When the discriminant becomes 0, then you are left with only one real characteristic because there is essentially one root. When it becomes less than 0, then it ends up giving you complex conjugate roots and that is the situation for elliptic partial differential equations.

So, this is essentially the basis on which we try to classify a second order partial differential equation where there are two independent variables. The question is that what would happen if you have more than two independent variables? Is there a way that we can have for classifying such an equation? To figure that out, we will have to do a little more work, but before we do that, we work out a small problem based on what we learned just now.

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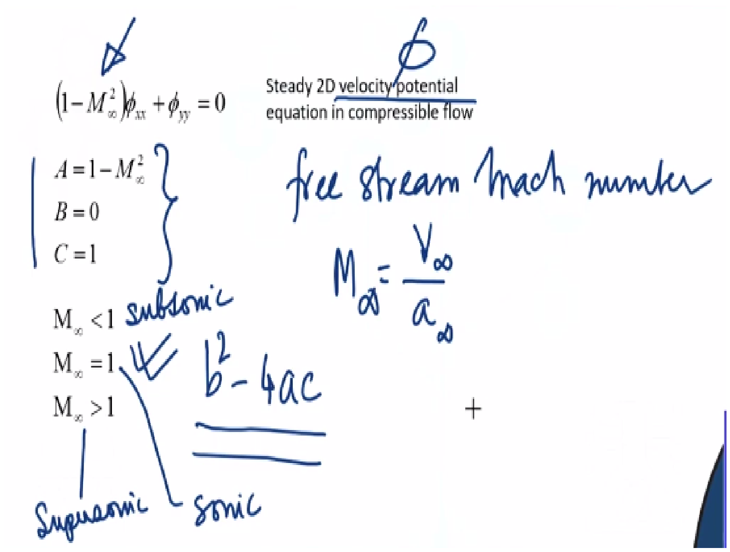

Here is an equation which is used to solve steady two-dimensional velocity potential equation in compressible flow. So, ϕ is the velocity potential that we are talking about and this equation certainly has dependence on the free stream Mach number, which we are calling as M_∞. If you take a ratio between local velocity and the sonic speed you get the local Mach number.

And here you actually take a ratio between the free stream velocity and the free stream sonic speed in order to get the free stream Mach number. Now, based on what we learned earlier, we can figure out that the coefficients of the partial differential equation, which we call a small a, b c in this case would be looking like this. Only thing is that here we have written them as capital A, B, Cs.

Now, based on this, you can of course work out what $B^2 - 4AC$ is and then if you do this exercise, you can get the behavior of the partial differential equation for a range of Mach numbers. You can do this as a small homework problem and see what happens when you have a Mach number which is less than 1, when it is exactly equal to 1, and when it is more than 1. So, as you know that $M_\infty < 1$, this would be the situation of a subsonic flow, $M_\infty = 1$, this would cater to sonic flow, while $M_\infty > 1$, this would cater to supersonic flow.

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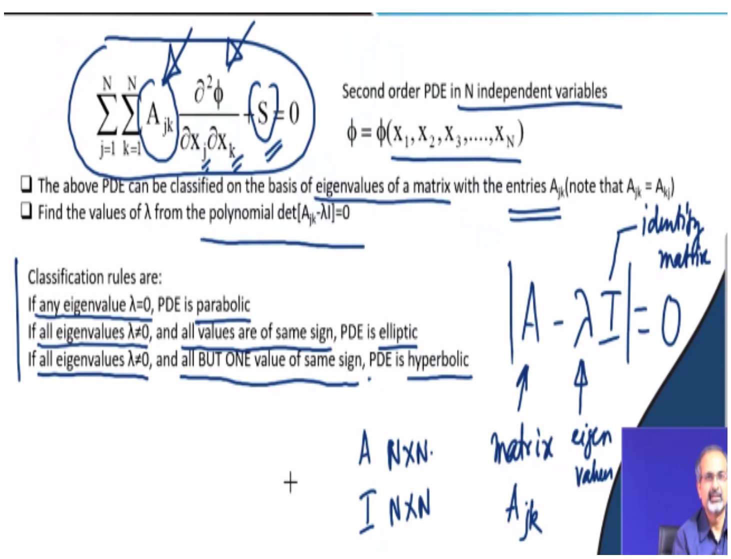

We next look at a situation where you have a second order partial differential equation where there could be more than two independent variables. There could be a fairly large number of independent variables x_1, x_2, x_3 up to x_n, which means you have n independent variables. In this case, the equation can be written in a manner where partial derivatives would be written in the form of indices, where the j and k indices run from one to n.

Then you have these coefficients associated with the partial derivatives and a possible source term like the one we had in the previous form of the partial differential equation, where many terms were lumped together into one term, which did not involve second order derivatives. So, remember that all the second order derivatives have to be expressed in the summation form, while the other terms account for the S term.

Now, to classify this kind of a partial differential equation, we have to obtain a matrix and then that matrix would essentially comprise of these coefficients, which we have associated with the different second order partial derivatives and using the matrix, we can obtain a characteristic polynomial which looks like this where **A** is the matrix which is comprised of all the terms which come from the coefficients.

So, the matrix which comes from A_{jk} terms and then λ will give you the eigenvalues and **I** is an identity matrix. So, if **A** is a NxN matrix, then obviously **I** also has to be a NxN matrix. After you work out this characteristic equation based on the determinant, you will be essentially having a polynomial whose roots have to be obtained. Roots of that equation will be the eigenvalues.

So, you will have a fairly large number of eigenvalues if N is large. Now, you look at how the eigenvalues are like. So, if any of the eigenvalues happens to be 0, then the partial differential equation will be characterized as a parabolic partial differential equation. If all eigenvalues are nonzero, and all values are of the same sign, then you have an elliptic partial differential equation.

Again, if all eigenvalues are nonzero and all but one value is of the same sign, then the partial differential equation is a hyperbolic kind. So a good exercise to do would be to go back to the previous example that we saw in the previous slide and try to put it into this framework and solve for the nature of the partial differential equation and see whether your solutions match by the two approaches. You could of course take even other equations and try them out.

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Two principal categories of physical behaviour:

- **Equilibrium problems**
- **Marching problems**

Equilibrium problems

- The problems in the first category are steady state type. Laplace's equation, which describes irrotational flow of an incompressible fluid and steady state conductive heat transfer are examples. These and many other steady state problems are governed by **ELLIPTIC PDE**.
- An important feature of elliptic problems is that a disturbance in the interior of the solution affects and alters the solution everywhere else.
- Disturbance signals travel in all directions through the interior solution. Consequently, the solutions to physical problems described by elliptic equations are always smooth even if the boundary conditions are discontinuous. To ensure that information propagates in all directions, the numerical techniques for elliptic problems must allow events at each point to be influenced by all its neighbours.

The physical behavior of the different types of partial differential equations:

We know that partial differential equations primarily cater to these two types of behaviors, the equilibrium type of problems and marching type of problems. So, what do we do in equilibrium type of problems? Let us see. The main thing is we use them for steady state kind of problem scenarios. We have had a look at Laplace equation previously, which models irrotational flow of an incompressible fluid.

You saw that it actually gives you a steady state solution of the velocity potential of the flow field. You also saw the steady state heat conduction equation. So, these kinds of steady state problems are often called equilibrium problems and they are modeled using the elliptic partial differential equations.

A very important feature of elliptic problems is that if you have a disturbance anywhere in the interior of the domain where you are solving the equation, then that disturbance affects and alters the solution everywhere else in the domain reaching farther out towards the boundaries and the disturbance signal propagates in all directions through the domain and they essentially propagate at infinite speed.

So, there is no delay for propagation of the disturbance to all regions of the domain where you are solving the problem. So, these are very important characteristics of elliptic partial differential equations which are used for modeling equilibrium problems. There are another class of problems which we often call as marching problems and we will see that parabolic partial differential and hyperbolic partial differential equations fall in that category.

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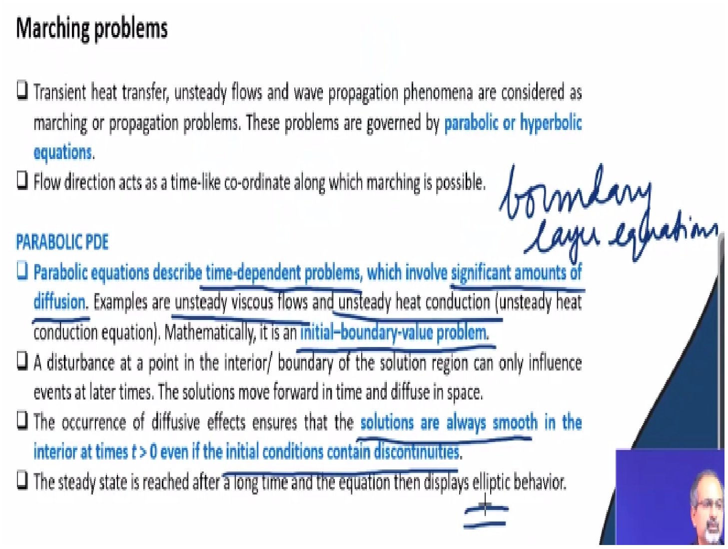

In marching problems, we can be solving transient heat transfer or we may be solving unsteady flows or wave propagation phenomena, where we can see the solution changing with time. These problems are governed by parabolic or hyperbolic equations and in these problems sometimes a particular flow direction can also act as a time-like coordinate along which we can march the solution. So, this is what we see in say boundary layer equations.

In parabolic partial differential equations, we look at time dependent problems and those problems involve significant amounts of diffusion. These problems are used for solving unsteady viscous flows or unsteady heat conduction and these are mathematically stated as initial boundary value problems because the solution depends both on initial condition as well as boundary condition and when you look at the solution, you will find that a disturbance at any point in the domain can influence events only at a later point of time.

They cannot influence backward in time, but they can influence events forward in time and they would diffuse the solution in space. So, these are very typical characteristics of parabolic kind of problems. Solutions always remain smooth in the interior and even if initial conditions contain discontinuities, they will never create discontinuities in the solution within the domain. The steady state may be reached after a very long time, and then the behavior of the equation maybe gradually approaching an elliptic one.

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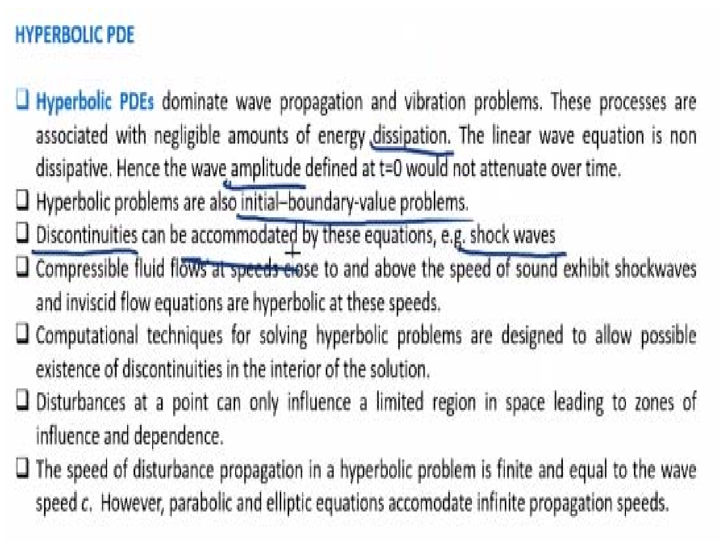

In hyperbolic partial differential equations, which could be used for wave propagation problems and vibration problems, one very important characteristic is that there is negligible amount of dissipation and therefore what happens is that if you have a certain disturbance with a certain waveform and amplitude, then there is no attenuation of the amplitude as time progresses, but you would find that the wave form would be propagating through the domain spatially without any attenuation or diffusion.

Hyperbolic problems are also initial boundary value problems and they can accommodate discontinuities. In compressible flows, we can have discontinuities like shockwaves and such discontinuities can be accommodated in hyperbolic problems. A very important feature of hyperbolic problems is that the disturbances propagate at finite speed, whereas in parabolic and elliptic equations, the disturbance propagation speeds are infinite.

Therefore, in hyperbolic equations, it depends how quickly the wave would be able to reach out into the domain to carry along with it information coming from a certain region of the domain to influence certain other.

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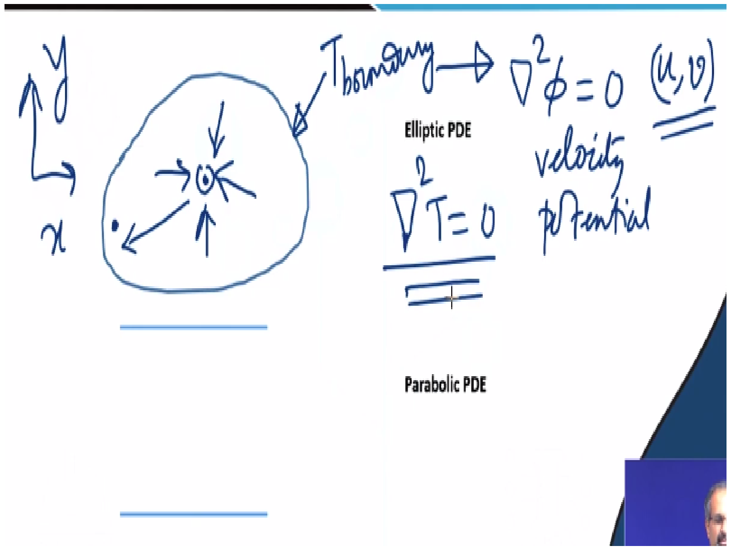

Here we have a curve drawn in two-dimensional space and we are trying to understand how an elliptic partial differential equation would behave when we are solving it in a domain like this. So, in this domain if you look at a certain point you will find that information is propagating into that point from all directions and also information from this point can affect other points in its vicinity.

That means, different disturbances can propagate and reach out to every corner of the domain and that too at infinite speed. For example, when you are solving $\nabla^2\phi = 0$ where ϕ stands for velocity potential, then you will see that with the given boundary conditions, it will give you a steady state solution. This means that from the solution of ϕ you will get a unique u and v field and it takes no time for the flow to reach that steady state u and v field.

How does it happen? In this model of the flow, the disturbances can propagate out into the domain at infinite speed and therefore the flow would reach equilibrium in no time, and therefore you get a steady state u, v distribution instantaneously. If you look at it from heat conduction perspective, then in reality it may take some time for the temperature redistribution, especially if you are starting from a temperature distribution along the boundaries of the domain, which are different from inner part of the domain.

However, the way we look at this problem here is that we are only concerned with the steady state solution of the temperature and then as long as $\nabla^2 T = 0$ is concerned, you just look at the equilibrium state. We will discuss about the parabolic partial differential equation in the next class.