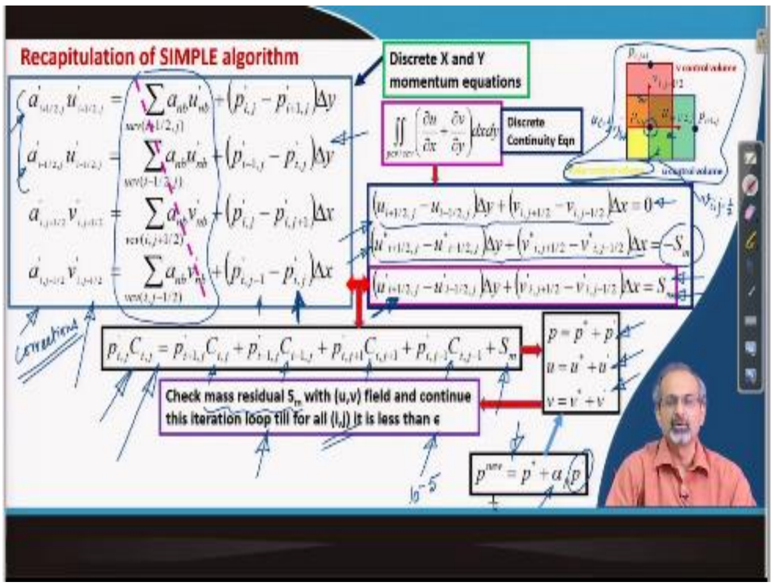
Introduction to CFD
Prof. Arnab Roy
Department of Aerospace Engineering
Indian Institute of Technology – Kharagpur

Lecture - 43
Numerical Solution of Two Dimensional Incompressible Navier Stokes Equations (continued)

(Refer Slide Time: 00:37)



In this lecture, we continue our discussion on two dimensional incompressible Navier Stokes equations. So, we would just do a quick recapitulation of the SIMPLE algorithm, which we discussed in the previous lecture. So, you recall that we wrote down the discrete form of the x and y momentum equations for the velocity components located on all the four faces of the scalar control volume.

So, going back to the grid, once before we look at the equations once more. So, this was the grid, we were talking about the staggered grid arrangement. So, the central cell is the scalar control volume, the center of that cell locates p i j. And we have u i + half j shifted to the right of that cell that means on the east face of the cell. And on the north face of the cell is v i j + half. So, this is how we had located the pressure and the two velocity components in the cell.

So, we are trying to come up with a discrete form of the continuity equation, and for that, we need to write expressions for the velocities on all the four faces of the scalar control volume. So, you will find that on the east face you will have u i + half j, on the west face you have u i - half j, on the north face, you have v i j + half and on the south face, you have v i j - half. So, this is how the four face velocities are represented.

Now, going back to this leftmost block, we have written the discrete form of the momentum equations for expressing these four velocity components. So we have two equations for u i + half j and u i - half j. These are coming from the discrete form of the x momentum equation, but then, we all recall that we have these coefficients defined or other the velocities defined in terms of the corrections. So, that is why we have them as dashed quantities.

So, we recall that we defined the correct values of pressure and velocity fields without any superscript. And they were expressed in terms of summation of the guest quantities, which were in the form of star quantities, and the corrections, which were in the form of dashed quantities. So, we wrote down the momentum equations both for the correct fields and the guest fields, and then we subtracted to come up with these equations expressing the velocities in the correction form.

And then we came up with the major approximation of the SIMPLE algorithm that is the velocity corrections involving in the neighbor cells where all neglected. So, we have struck them off. And so, what we were left with where corrections of pressure on the right hand side. Now, if you rearrange this equation, and then try to substitute it in the discrete form of the continuity equation, then you come up with pressure correction equation.

Before you do that, we have a quick look at how the discrete continuity equation looks like. So, the discrete continuity equation for the divergence free velocity field will yield a zero on the right hand side. So you can understand that these velocity components on the four faces of the cell are such that they exactly satisfied divergence v = 0 condition. That is why you have on the right hand side. However, you do not expect to do that with a guess or estimate.

And therefore, when you have an estimated velocity field, you are going to come up with a source term on the right hand side let us call it as – S m; m for mass, because we are handling mass conservation when we are talking about continuity equation. And once we do that, we have these equations. We can subtract the estimate equation from the correct field equation and then we come up with an equation in terms of the corrections.

So, what we are essentially doing, we are substituting the equations that we generated on the left block into this equation. Right. Once you do that, you will find no velocity is left because the right hand side of the velocity equations on the leftmost block contain only pressures and therefore, you will you come up with this pressure correction equation. And each one of the pressure correction terms for each cell will have a coefficient.

You can do a slight exercise on this as a homework. So, that you can work out all these coefficients on your own. And then you will find that these individual coefficients that you see on the right hand side if you sum them up, they will exactly equal the coefficient on the left hand side. Right. Additionally, you will find the mass defect on or the source term S m which is coming from an incorrect velocity field in the estimated field.

So, that will be a source term in the pressure Poisson equation. So, if the source term did not exist, then you would actually have a Laplace equation for pressure, but because you have a source term. it is inhomogeneous. Therefore, it is a pressure Poisson equation. So, in the limit when you are able to send this residual gradually to zero, you will actually be able to go to a Laplace equation.

And as you do all this, what are we going to do? We are going to check for the p dash field that you generate by solving this pressure Poisson equation iteratively. Once you generate the p dash field from these equations. You can generate the u dash fields and v dash fields and then what do you check for. You essentially check for the new mass residual that that u dash and v dash field generates.

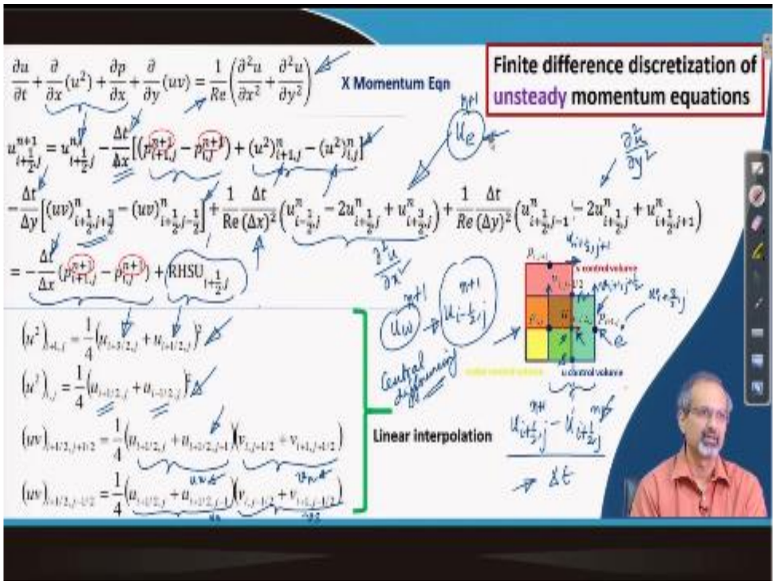
That means you go back to an equation of this form and see what happens to the right hand side. Is it reducing? If the algorithm is working perfectly, it should be reducing significantly. Alright. Where do you stop? You check this mass residual for all the cells i j in your domain and you find or you ensure that each one of them have reduced the mass residual below a small quantity epsilon.

That epsilon may be a very, very small quantity say 10 raise to power - 5 or - 4 suitably normalized and as you do all this, you may actually need to incorporate under relaxation. So, the pressure Poisson equation when you solve because you have neglected the neighbors cell contributions. You may find difficulty in converging. So, you make use of under relaxation strategy.

So, you do not use all of the p dash to be added to p star rather you use a coefficient alpha p; p for pressure as an under relaxation parameter and that alpha varies anywhere between 0 and 1 of course. If you set 0 value then of course, you do not add any portion of the p dash to p star. So, you cannot change the pressure field. So, of course, it has to be nonzero but it has to be below one depending on the convergence behavior of the scheme.

So, this is broadly what SIMPLE algorithm is all about and this process of pressure and velocity corrections will go on till you have essentially reached near divergence free field and you stop at a suitable point below which reducing the residuals become very costly.

(Refer Slide Time: 09:52)



So, we were discussing about finite volume implementation through the simple scheme. Right. Let us also discuss a little bit of finite difference implementation. And if you recall, we discussed about a steady set of momentum equations in the simple scheme. Now, we will also incorporate the time dependent term into the momentum equations. So, with a finite difference discretization of unsteady momentum equations.

Let us see how the calculations can be accomplished. So, on top you have the x momentum equation in unsteady form and then it is of course, non-dimensionalized that is why you have Reynolds number coming up on the right hand side and then comes the issue of discretizing this equation. So, we have to remember that we have to discretize this equation in the control volume. So, if you follow the first order Euler scheme then the time derivative may be discretized in this manner.

And in that case, what you leave behind on the left hand side is the n + 1th time step velocity that you would actually like to generate. So, that you can move to the next time step calculations and this term is transposed to the right hand side. So, that is what you find over here. And because you have generated a delta t term in the denominator, if you multiply the whole equation by delta t, then delta t goes over to all these terms on the right hand side.

You can see that. Right. And if you notice carefully that one advective term and the pressure or derivatives partial derivatives of x. So, they are the ones which are put into this square bracket because they have a common delta x in the denominator on discretization and we have used central differencing if you notice carefully. It is centrally difference because pressure of course, on a staggered grid.

If you centrally difference, it will become p i + 1 - p i. That is what you have for pressure and then for velocity, this is the del u square del x term discretized. So, u square comes from this point because that is the east point. This is the west point of the control volume. So, at the east point, it is u square at i + 1 and then on the east point, it is u square at i. j of course remains the same because you are moving on the same grid line along the x direction.

And how do you calculate u square at i + 1 or at i? They calculated this way. So, you can make out they are linearly interpolated. So, for example, at this point in order to linearly interpolate, you define the neighboring u's. So, this is u i + 3 by 2 j and this is u i + half j. Those are the two u's which have been averaged. In order to give you, the u at the point i + 1 and then if you multiply them with each other, then it generates a one fourth of this bracket a term whole square.

So, that is how we are generating the u square values. We are also doing it at i j. So, when it comes to i j, you would actually have to use the values of u at i + half and i - half in order to generate the value at i through linear interpolation. Right. So, this is equivalent to using the central differencing. Now, coming to the derivatives del del y u v. So, that is the y direction so, you have to involve the north and the south points of the u control volume.

So, at the north point remember that you have to average the product u v. So, that is what we call us i + half j + half. That is the north point. Right. So, at i + half j + half, let us see how we average. So, the average value of u at that point will be u i + half j + u i + half j + 1 because you have to take this velocity here and the one on top which we are calling as u i + half j + 1. You have to average them to come to the value at u n.

This is essentially u n. How do we get v n? v n could be an average of v i j + half + v i + 1 j + half. So, v i j + half lies here. v i + 1 j + half lies here. So, you average them and you get v n. So, this is how you have to look at two different directions, when you pick up the nearest available velocity components to average them to come to the average value at a cell face. So, this is very similar to what we have done even in finite volume.

So, you can see the close correspondence between finite volume and finite difference when you have a regular mesh uniformly spaced mesh. They actually shrink to the same form. We can show that. Now, similarly, we can calculate the product u v at i + half j - 1. So, it is the south point and then this is the value of u at the point S, this is the value of v at the point S. Similar strategies are used like on the north point to calculate these valves.

So, that way you can calculate the values within this square bracket. So, that completes the calculations of the convective derivatives. Coming to the viscous terms, the second order derivatives, therefore, for second order central differencing, you will involve three points centered around i + half as far as u is concerned. Right. So, for the x direction you move between i - half to i + 3 by 2 to pick up three points in order to do the central difference.

So, that gives you the del to u del x squared discretization and of course, in the denominator you must have noticed it is going to become del x square because it is a second order derivative and then this comes as del 2 u del y square and now, we have to move along the y direction. So, now it moves between j - 1 to j + 1. So, the central point is j; the one on top is j + 1; one at the bottom is j - 1 and for all of them i remains fixed at i + half.

So, you are essentially moving along this line. So, pick up to pick up the necessary values. So, again, you see that derivatives along x or y direction would of course, make you move along their respective directions to pick up the velocities and fit them into your discretization scheme. So, this way you can finally, clump all the terms together and then finally, express it in this form where only the pressure terms are separated.

And then the rest of all the velocity related terms are packed into this so called source term, we just give a nomenclature right hand side of the u control volume. So, RHSU i + half j. So, that is the point around which the u control volume is centered right. So, this way you can of course, also generate the u velocity calculations for the west face. Sorry, not this west face, I meant to say that, you can go to the west face of the scalar control volume.

And you can likewise generate an equation for u i - half j n + 1. Like, we did even in the simple scheme that you generate a discretized form of the momentum equation for all the four faces of the scalar control volume. So, the u's are on the east and the west face and the v's are on the north and the south face. So, this equation is essentially the u on the east faced and equation for that for the n + 1th time step.

You similarly need an equation for u w for the n + 1th at time step and that is going to come from an equation for u i - half j n + 1. So, it has to be written in a way similar to what you have done for u e at n + 1. So, this way we discretize these equations. And we will also have a quick look at how we do it for them y momentum equation. So, that we can generate the discrete equation for v for both the north and the south face.

One thing that you have, you must have noted is that in this equation, we have circled the pressure term superscripts we have written them as n + 1. That means these pressures are actually going to work good for the next time step. That means in the next time step, whatever velocities are generated, they would be divergence free that means n + 1th at time step velocities that we are generating are divergence free under the circumstance that they have a correct pressure field also accompanying them.

And that correct pressure field is essentially the p n + 1 pressure field. That is what the discretized pressure Poisson equation is generating for us. That is why we are assign a superscript n + 1 for it. Like we have used linear interpolation, of course, we can use all the different schemes that we have discussed in the context of 1D advection diffusion equation earlier even over here.

We can also use more accurate schemes, the compact schemes that we have discussed earlier. Though it is more tedious to implement them over here, but they can be very much done to augment your accuracy.

Going through the y momentum equation, again similar straight strategies work over here. So, you have to find out how to average the velocities at different points for working out your convective derivatives. So, that is how we do the v squared and u v calculations. So, as you can understand the v squared calculations would come for the north and the south cell faces because, you can see the j + 1 point and j points.

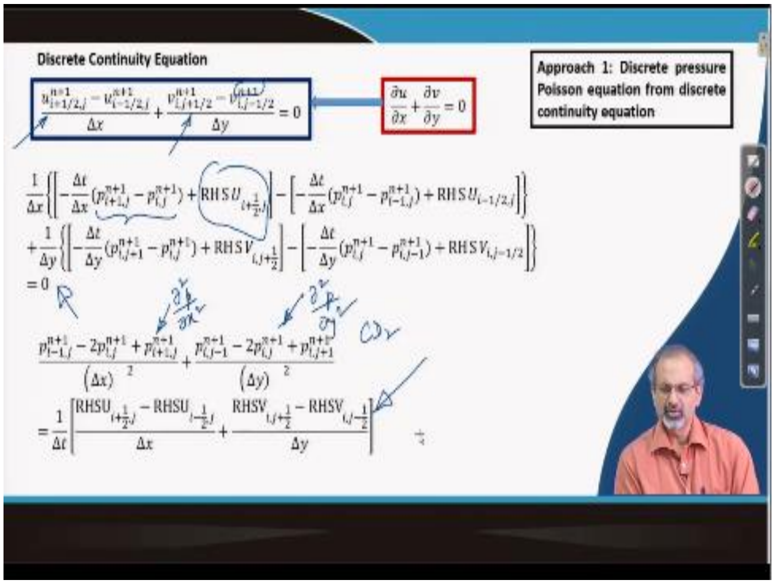
So, those are the north and the south faces and you have to go along the north direction to find v's from where you can average. So, the north face has j + 3 by 2 and the south and j + half while the south face has j + half and j - half to find u's from v values from where it can average. So, that way the v squares is average at the north and the south points. Similarly, u v this product has to be averaged at the east and the west faces.

Why is it? Because it involves an x derivative. So, depends on the derivative the direction along which you are taking the derivative. That is why you have to find ways of working out that product u v at the east and west faces which gives you the x orientation. So, u v at i + half j + half point that means the east point comes from the neighboring u's. So, the neighboring u's are at u i + half j that is this and u i + have half j + 1 that is this.

And then v comes from v i j + half and v i + 1 j + half. So, v i j + half is here and v i + 1 j + half is here. So, that way you can get these values of u and v at the east face. Similarly, you can find the values of u and v at the west face and so, this was v at n, this was v at S and so on. So, all this makes your advective term derivative calculations completed and then the CD2 scheme applied for your viscous terms.

Again you have to sweep along x direction in order to work out del 2 v del x square sweep along y direction to find del 2 v del y square and so on. And at the end, the pressure term along with the source term assigned for the particular cell that you are looking at. So, right hand side of v control volume centered around i j + half. That is the nomenclature.

(Refer Slide Time: 25:18)



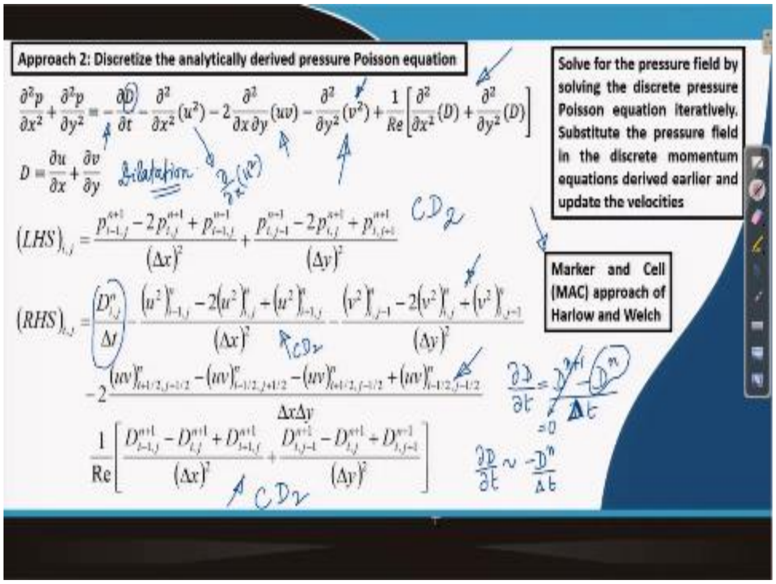
Now, once you are done with these calculations, we have shown how to calculate u i + half and v j + half already in a similar manner you can do u i - half and v j - half. And once you are done you substitute into the discrete continuity equation. And of course, we are substituting the expressions for the n + 1th time step which we have shown in the previous two slides how to do them. So, as you recall that all of those expressions had a portion coming from pressure.

And another portion which had the source term this is easy to do because it simplifies your nomenclature otherwise the equation will look enormous and clumsy. And when you do a computer program anyway you do not need to make it clumsy, because those calculations can be stored into arrays. Once you are done with that, the discrete form of the pressure Poisson equation looks like this.

So, as you can understand the pressure terms can be clubbed together and the source terms can be clubbed together. So, the pressure terms fall on the left hand side of the equation; the source term lie on the right hand side of the equation and you can clearly identify that this is nothing but del 2 p del x square. This is del 2 p del y square in a CD2 form and on all the source terms go over to the right hand side.

So, this is one possible approach where discrete pressure Poisson equation is generated from the discrete continuity equation and then you can solve this discrete pressure Poisson equation iteratively in order to generate the proper pressure and then you can take the calculations to the next time step.

(Refer Slide Time: 27:17)



Another approach could be that we use the discrete analytically derived pressure Poisson equations. So, earlier lectures have shown how to obtain the pressure Poisson equation analytical by taking derivatives of the moment of the equation suitably summing them up and then we come to the pressure Poisson equation. So, we have actually picked up that form and we are trying to demonstrate that even from that form, you can possibly solve pressure.

So, in that form, if you recall, we had purposely retained the Dilatation term and if you retain that, let us see what happens. We earlier said that this term maybe discretized like this. So, we may be having some Dilatation at the nth time step. It is not very small. So, the aim is always to reach a even lower level of Dilatation at the next time step. So, the idea is that we would try to limit the next time step Dilatation to zero if possible.

So, if that is achieved, then we are in an ideal situation. So, we would like to discretize this del v del t term as minus D n by del t. This is what is usually done. So, that is what you can actually see over here and because it carries a minus sign ahead of it. So, it becomes D i j n by delta t over here. And then, if you look at the left hand side of this equation, then we have actually used CD2 to discretize the pressure terms, the second order pressure terms.

And the remaining terms on the right hand side are all terms looking like convective derivatives or the viscous terms but with an additional order of differentiation associated with them, because that is how we could actually come to the pressure Poisson equation from the momentum equations. That is why the additional derivatives are come up because normally a term as an advective derivative.

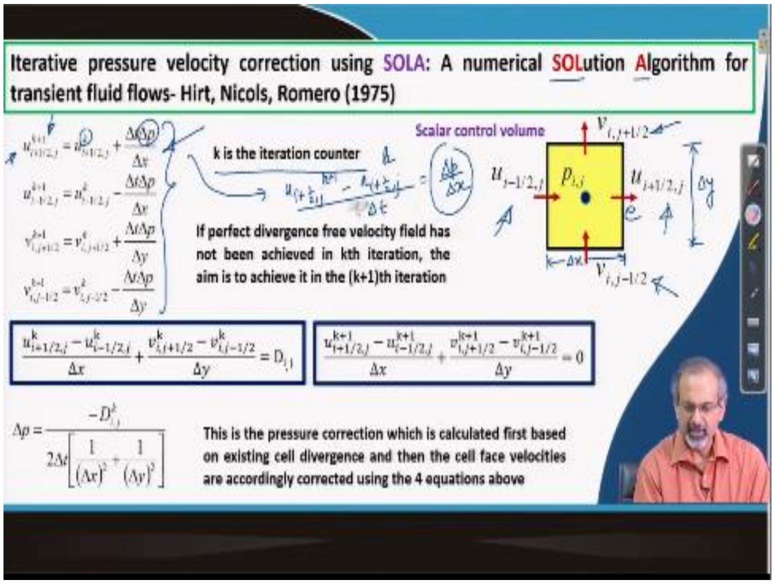
It would look like this saying the x momentum equation, but here it is a second order derivative. That second order derivative has come, because you had differentiated the x momentum equation with respect to x in order to generate the pressure Poisson equation. That is why an additional order of derivative has come up in these source terms. So, therefore, now, this will have to be discretized as a second derivative say using a CD2 skin.

Similarly, it happens for the v squared term which you have here and then you have a mix derivative over here. So, you have del 2 del x del y. So, it is a mixed derivative second order. So, you can show that it can be discretized like this. We have not explicitly dealt with mixed derivatives in class, but you can do it as a small homework problem. Now, that we have done enormous amount of exercises on finding finite differences.

This mixed derivative can be worked out in spare time and then finally comes the second derivative applied on the Dilatation. So, again CD2 discretization applied here. So, this is how we discretize and then if you solve this equation again, here is a way you can work out the pressure at the n + 1th time step.

So, the approaches that we discussed about in the last few slides essentially form the crux of an approach which was first proposed by Harlow and Welch, a couple of decades back, which became very famous, it is known as the Marker and Cell approach. We are not getting into the details of the approach, but we have learned the ingredients of the approach by and large in the last few slides.

(Refer Slide Time: 32:13)



And this is one of the most popular approaches in solving incompressible Navier Stokes equations. We are also proposing another very elegant iterative pressure velocity correction using a scheme which is called as the SOLA scheme. So, the full form of this scheme is a numerical solution algorithm for transient fluid flows, which was proposed by a few scientists headed by Heart.

And if you just pick up a few of those letters from the complete name of the algorithm, you actually come up with that abbreviation SOLA. So, once again we are looking at the scalar control volume over here and you have the four velocity components defined on the four faces here and you are going to see how this algorithm works in terms of refining these velocities. So, there is an iteration counter which we are talking of this is called as k.

So, that comes in the superscript. And if a perfect divergence free velocity field has not been achieved indicates iteration. Then the aim of course, is to achieve it in the k + 1th iteration. And this iteration process goes on and on till we have reached a sufficiently small divergence in all the cells like we discussed even in the context of SIMPLE scheme. So, in the SOLA algorithm, this is the proposed methodology by which the velocities at the faces are defined at each iteration level.

So, if you look at the u component at the i + half j point that means the east point then the k + 1th iteration value will be the summation of the kth value plus a correction which is delta t that means the time step into some pressure correction delta p by the cell width. So, the cell width and height are of course, delta x and delta y. We are talking about a pressure correction delta p though we are not sure what that value is going to be, but we are proposing a form.

Now, interestingly if you rearrange this form, it gives you u i + half j k + 1 – u i + half j k by delta t is equal to delta v by delta x. That means as though the temporal change of u will depend on the ratio delta p by delta x, which is of course, a concept connected with what we see happening in the momentum equation in a highly truncated form though, because we are not talking about advection terms.

We are not talking about these customs. but we are just talking about a connection between the time rate of change of u with pressure gradient linked with pressure gradient. However, here we cannot actually say time rate of change because iterations are involved. So, it is only through an iteration process that such a convergence one can achieve, but always linking it with the pressure gradient in the respective direction.

So, that is the basis. Now, similarly, u at i - half j would be calculated based on the kth value minus the delta t delta p by delta x term. So, this is how the u’s at the east and west faces would be updated. We will continue with this discussion in our next lecture. Thank you.