Introduction to CFD
Prof. Arnab Roy
Department of Aerospace Engineering
Indian Institute of Technology - Kharagpur

Lecture - 45
Numerical Solution of One Dimensional Euler Equation for Shock Tube Problem

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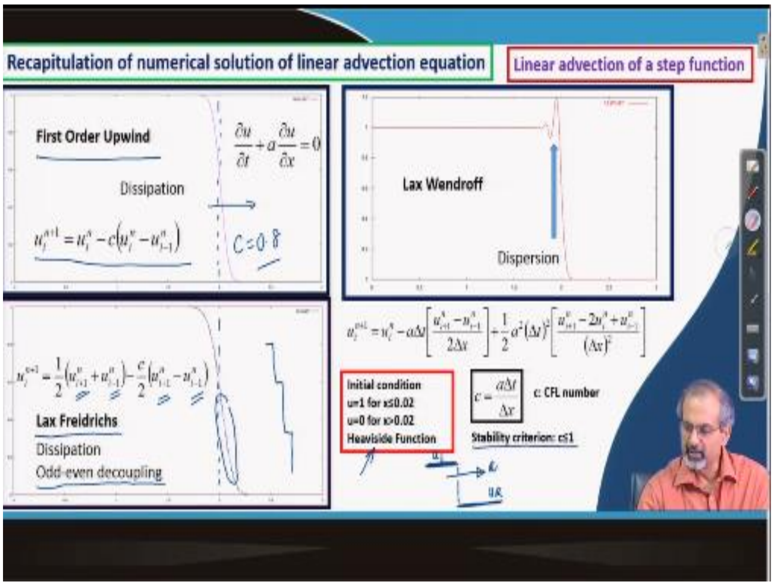

In this lecture, we will begin our discussion on one dimensional Euler equation and its application to the shock tube problem. So, we begin by doing some recapitulation about numerical solution of linear advection equation. And we gradually take it forward, we will discuss about inviscid Burgers equation in one dimension. Before we go into the core part of this module, where we discuss about one dimensional Euler equation and its application to shock tube problem.

So, we may recall that in the linear advection equation problem, we had dealt with a situation where first of all, we had only one transport equation. So, such situations are called as scalar situations, because you have only one variable to solve and it is a conservation equation. So, it is essentially called as a scalar conservation law. So, linear advection equation falls under the category of scalar conservation law.

So, we are talking about advection of what advection of the property u and the property u adverts with wave speed a. That is the physics behind linear advection equation. So, we can look at some typical examples for testing different numerical schemes, how well they can simulate the linear advection equation physics. So, linear advection equation accommodates sharp changes in functional values of u and its advection.

That means you can have a step function of this kind, which may be approximated by a Heaviside function. And if you say that this is the distribution of u. That means you have say a left state; we have a right state and there is a sudden jump in between so, it is better represented this way, we have a left state and a right state and there is a jump in between. So, such discontinuities are accommodated by linear advection equation.

And then if you say that this font will advect at a speed a, then linear advection equation will make this shape to move with the speed of a, say from left to the right as long as a is positive. So, we have tried to show how some of the numerical schemes behave when you have this advection of a sharp font like this through the linear advection equation. So, when you try doing it using a say, a first order upwind scheme then the scheme looks like this.

But of course, C is given by the CFL number or Quran number and we know that the stability criterion is C less than equal to 1. So, we choose a C which is bounded by that value and if we do that, then we will find the step adverts. So, the step actually lies somewhere here. So, the linear upwind scheme makes it to advect but there is dissipation layer as expected; because it is a first order accurate scheme.

But there are no oscillations; there are no overshoots and undershoots because there is heavy dissipation. Again, remember that if you had chosen C equal to 1, possibly first order upwind would have done a far better job than this. Incidentally, a C of 0.8 has been used in this case. So, if you look at another first order scheme that could be the Lax Friedrichs scheme and again you have this dissipation problem, additionally you have the odd even decoupling.

If you watch carefully at this variation, then there are step like structures is in this front which is captured by Lax Friedrichs that is because of this odd even decoupling. That means you are updating the value at i grid point, but only i + 1 and i - 1 values are influencing the update at the grid point i. So, there is apparently a decoupling happening. So, that is the fallacy and that is why there is a kind of serrated structure of the front.

Additionally, there is dissipation because of the first order nature of the scheme. So, if you want to reduce the dissipation errors, then of course, you can go for higher order schemes. But then we know from our prior exposure that if you are not doing something special to take care of dispersion error, then there will be some oscillations overshoots and undershoots and that is what becomes evident when you try to use Lax Wendroff.

So, the scheme is given here and you can see that the front has been captured more sharply by the second order accurate scheme. There is less dissipation issues, but there is a significant dispersion issue. There is a wiggle. So, what we discussed earlier in the context of one dimensional advection diffusion equation if you are choosing mono-term schemes TVD schemes. You can probably mix and match the properties the favorable properties.

You can have higher order accuracy, but by damping out oscillations overshoots and undershoots. So, that can possibly be an answer to handling these oscillatory effects, whenever there is a sharp change to be captured using higher order schemes. So, this is a lesson that we need to keep in mind. Because when you are going to solve Euler equations, it will involve discontinuities like shocks or contact surfaces.

So, many of these schemes which are of higher order will show oscillatory behavior while the lower order schemes will show dissipated behavior. We just look at another example problem again linear advection but now with a sinusoidal wave packet and the green circled portions show the initial waveform.

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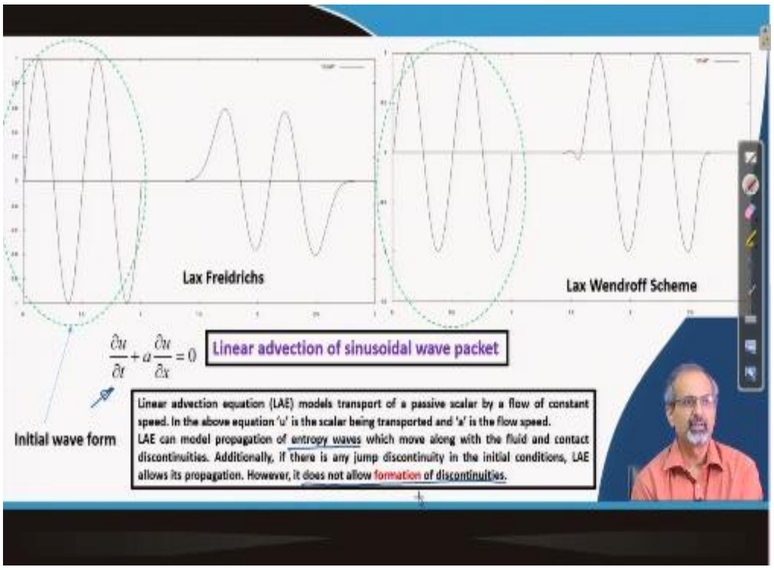

So, you introduce 2 wavelengths of a sinusoidal wave into the domain and ask linear advection equation to advect that waveform. So, we can see what Lax Friedrichs and Lax Wendroff does do that. So, Lax Friedrichs, because of its dissipative nature attenuates the amplitude. So, that is essentially due to the dissipate in nature, but there are no oscillations anywhere.

Whereas, Lax Wendroff does not dissipate the peak to peak is maintained, but there seems to be some oscillations here. Things can get worse, if you introduce still higher frequency waves and try to test these schemes. So, the dissipative and dispersive whereas, both can become more of aggravated. But, this is an instance where we can see that if there are cyclic variations, then how these numerical schemes responding when the transport equation happens to be linear advection equation.

A few important points at this point, which we need to know what linear advection equation. The basic definition is of course, that linear advection equation takes care of transport of a passive scalar by a flow of constant speed which we often refer as wave speed. And, if you are talking about this equation, then in the above equation u is the scalar which is being transported and a is the flow speed.

So, we should not mistake u as velocity. It may not be velocity, most often it is a passive scalar and then linear advection equation can model propagation of so called entropy waves which move along with the fluid. We will discuss more about this when we discuss Euler equations. They can also advect contact discontinuities, we will look at contact discontinuities even when we discuss the shock tube problem.

Additionally, if there is any jump discontinuity in the initial conditions, linear advection equation allows its propagation. So, in the first slide, you remember, we actually talked about such a jump discontinuity and about its advection. So, that jump discontinuity was introduced into the domain as an initial condition itself. And we were looking at its linear advection. That means that jump would propagate through the domain on attenuated ideally and just get advected at the wave speed a.

So, such advection will be allowed by linear advection equation. However, linear advection equation will never allow formation of discontinuities from a smooth variation of property u.

Linear advection equation will never generate discontinuities. It will not allow formation of discontinuities until unless discontinuities introduced through initial conditions. So, this is a very, very important point that needs to be kept in mind.

That it will never allow formation of discontinuities. However, in the next transport equation that we look at, that is inviscid Burgers equation in one dimension. You will see that such discontinuities will actually form as time progresses, when you have non-linearity coming through the advection terms.

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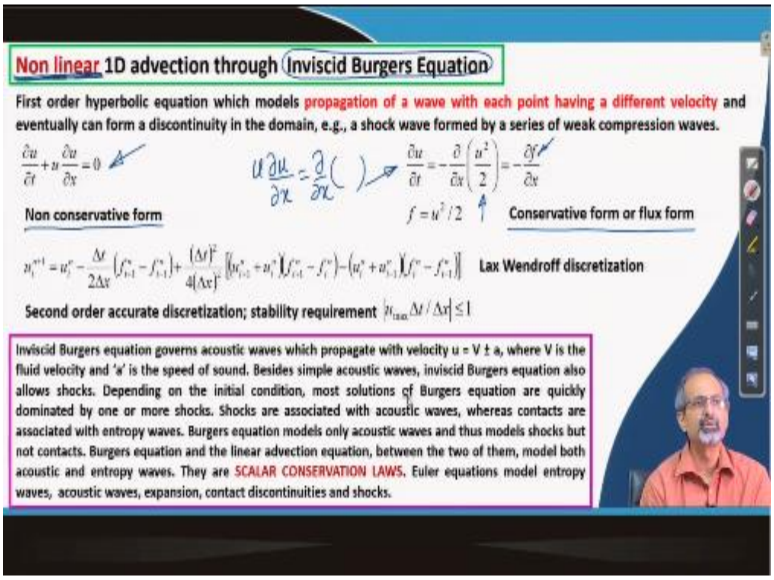

So, we come to non-linear one dimensional advection through inviscid Burgers equation. So, this is the equation, we are going to discuss here. We have already mentioned that this is a non-linear advection equation, we will see soon why. So, it is once again a first order hyperbolic equation and it models propagation of a wave with each point having a different velocity.

So, this is where it differs from linear advection equation, because, we remember that linear advection equation propagates the wave with identical velocity in each portion of the wave. That means each portion of the wave has the same velocity a. Right. While in this case, propagation of the wave will take place with each point having a different velocity. And in this equation, u means the velocity itself that is another change that you need to note.

So, in linear direction equation when we say u, we may mean even a scalar property, but in the case of one dimensional inviscid Burgers equation when we say u, we mean velocity and because propagation of the wave with each point having different velocity eventually can lead to formation of discontinuity in the domain. Inviscid Burgers equation can accommodate formation of shockwaves.

So, shockwave how does it get formed in this case, it forms by interaction of a series of weak compression waves, which finally lead to a strong compression front, which is the shock. And interactions are possible because different portions of the wave are propagating with different velocities that is accommodated in this equation. So, this is a major change between what we handled before and what we handled here.

So, the governing partial differential equation again is in scalar form. So, we call it again as a scalar conservation law and this is in non-conservative form and the corresponding conservative form or flux form can be represented like this. So, what we do is, we write the advective part on the right hand side and we introduced it entirely inside the derivative. So, how can you express u del u del x as del del x of something.

So, that something will become the flux. You can very easily understand in this case it will be u square by 2. Because that generates u del u del x on the taking the derivative. So, that is the simple problem here. So, that u square by 2 becomes f, the flux in this case. So, in scalar conservation laws, you just generate one flux. In vector models, we have several conservation equations to handle like in one dimensional Euler equation.

You will see later that there are at least three conservation equations that you have to handle in one dimension. Then it becomes a victim model and then flux will not remain **(()) (15:54)** longer a single term but will become a vector. So, we call it a flux vector in that case. Generally, single fluxes like the one we see here for the scalar conservation law will be represented by an unbolted f.

While when it becomes a vector, we may often write it as a bold f. So, these are differences in the nomenclature also which we have to keep track of. Now, if you were to write down in inviscid Burgers equation in Lax Wendroff form of discretization, it would show up like this. So, you can take down this form and try to derive it on your own. It is also available in some of the references.

You can see that the fluxes are participating here very often in flows, where we handle discontinuities. We prefer the conservative form or the flux form over the non-conservative form. There are reasons behind it. Very often, the strong reason behind this is that we often find that in non-conservative forms, the derivatives that we handle can encounter difficulties when we cross shocks because they become enormously large, enormous large changes occur.

However, there are no such large changes or there are virtually no changes occurring in terms of the flux derivatives. So, there is a distinct convenience in handling the same equation in a different form the conservative form and there are jumps in the non-conservative form while there are no jumps in the conservative form or the flux form. So, you see the same thing happening in the Lax Wendroff formulation that we end up using the fluxes.

So, more of this will be discussed later, as we go on through this module. We also look at a few points below in the box, where we find that inviscid Burgers equation it governs acoustic waves, which propagate at velocities with respect to the flow velocity, you have an addition or subtraction of the sonic speed. And besides acoustic waves, they can also accommodate or allow shocks and Burgers equation can allow generation of shocks.

So, shocks may not be there or discontinuities may not be there through the initial conditions, but discontinuities is can get generated in course of the solution. So, depending on the initial conditions, most solutions of Burgers equation can be quickly dominated by multiple shocks and shocks are associated with acoustic waves whereas contacts are associated with entropy waves.

Burgers equation models only acoustic waves and thus models shocks, but not contacts. Burgers equation is a linear advection equation between the two of them model both acoustic and entropy waves. So, entropy waves we already remember in the previous slides we said that, entropy waves can be captured by linear advection equation. It can also handle contact discontinuities. Right. And Burgers can handle acoustic waves.

Now, both linear advection equation as well as Burgers equation fall under the category of scalar conservation laws because we are handling single transport equations or conservation laws. Euler equation on the other hand is a vector model. We have multiple conservation laws to take care of and Euler equations can handle entropy waves, acoustic waves, expansions, contact discontinuities as well as shocks.

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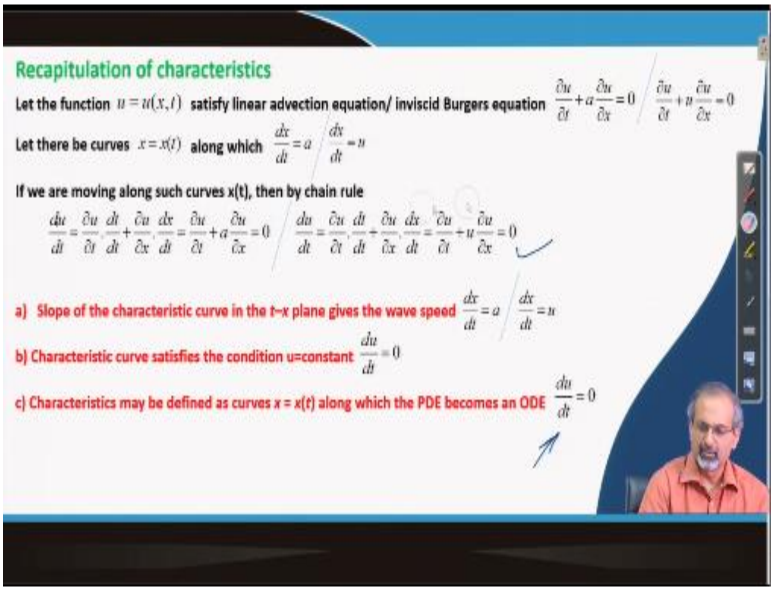

At this point, it may be worthwhile to have a quick recapitulation of the concept of characteristics. We have discussed this before also, but once again in the context of linear advection equation and inviscid Burgers equation, let us have a relook at the concept of characteristics. So, we begin by saying that let there be a function u of x and t which satisfy linear advection equation as one possibility and inviscid Burgers equation as another possibility.

So, we are handling both the scenarios at the same time here under the common umbrella of characteristics, and we are trying to figure out how the concept fits in the characteristic concept fits in. So, we will very often use this kind of slashes just to make sure that we are looking at both the equations simultaneously. So, when we look at the linear advection, it is this and in the case of inviscid Burgers equation, it is this.

And remember that the function on one hand can satisfy linear advection equation. So, you can have one function u which can do that, again you can redefine that function, so, that it can satisfy inviscid Burgers equation two separate situations, but looking at them we are looking at them simultaneously. Now, let the curves x of t along which for linear advection equation dx dt is equal to a for the Burgers equation dx dt is equal to u. Alright.

So, these curves of course, must be existing in the x t or t x plane. Right. And along these curves we are defining the slope we are actually generating the slope information dx dt is the slope information. In one case it is equal to a; in another case it is u. So, remember that a is a constant while u can be changing because by definition u is a function of x and t. Right. Okay.

Now, if we are moving along such a curve x t, then for the linear advection equation, you can write an expression for the du dt. So, by chain rule, it will be showing up as del u del t plus a del u del x and because u satisfies linear advection equation. It must be equal to 0. So, this is the u which satisfies the linear advection equation. So, in that case du dt will be equal to 0. So, when is du dt = 0?

When you are following the curves x t and along those curves what is happening dx dt remains equal to a. So, the connections need to be remembered. And because dx dt is equal to a along those curves, we were able to replace this by a here. You must have noticed that. Right. Okay. Now, that is the situation for the linear advection equation. Now, if I now take a u which satisfies inviscid Burgers equation, and then we follow the same curve x equal to x t.

But now the curve satisfies this condition that along that curve dx dt is equal to u. It is no longer a but it is u. So, I can write du dt. Now, as I move along that curve x equal to del u del t plus u del u del x and which now again by definition should be equal to 0 but because this u satisfies the inviscid Burgers equation. So, what do I have as a chain of events? I begin by saying that I have a u which satisfies inviscid Burgers equation.

Then I choose a set of curves x t, such that dx dt is equal to u and then I am moving along those curves x t and I am trying to compute du dt. And I find that that du dt will be equal to 0; because it satisfies the governing equation as I move along those curves. So, it happens equally well both for linear advection equation as well as Burgers equation as long as I have chosen correct functions u do represent those two distinct cases.

Now, there are outcomes which are common, which we have written in red at the bottom. So, if we go through those points, we can sum up what we found. So, we find that slope of these characteristic curves which we are calling as these x t in the t x plane gives the wave speed dx dt equal to a that is a wave speed in linear advection equation case. It is dx dt equal to u in the Burgers equation. That is the first thing.

The second thing is characteristics curve satisfies the condition u equal to constant. That means as long as I am moving along x t. I am satisfying that. You find it happening for both linear advection equation as well as Burgers equation. Right. And then the characteristics may be defined as these curves x t along with the PDE now has become an ODE.

Earlier, we were handling these PDEs while we are now handling an ODE. So, these are things that we have to keep in mind when we look at the problem from the angle of characteristics. So, let us try to see how we can make use of these concepts.

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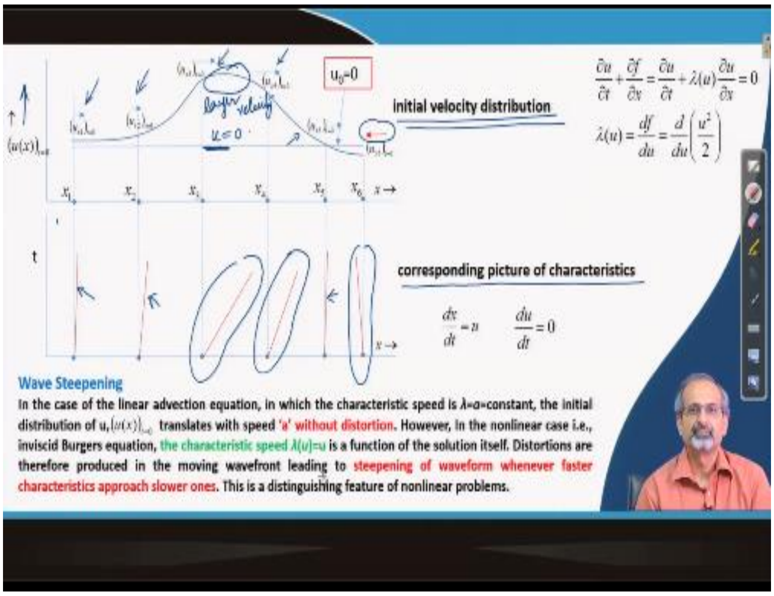

We will go to a discussion which is very pertinent for the Burgers equation now. So, in the beginning when we start the discussion, we will look at the equation once more. So, it is del u del t plus del f del x. So, That is in the flux form which should be equal to 0 and it can also be written as del u del t plus lambda u del u del x equal to 0 where lambda u is df du. What we have done essentially is we have rewritten this derivative as this by chain rule.

Where the flux is purely a function of u and therefore, we can have an ordinary derivative form of f with respect to u multiplied with a partial derivative del u del x. Right. And we are calling this df du as the lambda u. Now, what is this lambda u? This lambda u is often called as the characteristic speed. So, what is it in the inviscid Burgers equation? It is u itself. So, for Inviscid Burgers equation lambda u is equal to u.

What is it in the case of linear advection equation? For that you have to find out first how can we represent inviscid linear advection equation. So, linear advection equation is written like this. So, in flux form I can write it as del del x of a u equal to 0. Right. So, that is essentially the flux for linear advection equation. Correct. So, if you take a derivative of that flux with respect to u, you get lambda. So, what do you get back? It is a.

So, for linear advection equation, df du is d du of a u, which is a itself. So, that is the characteristic speed for linear advection equation. So, we begin by saying that let us imagine that there is an initial velocity distribution in the field and that initial distribution is indicated as u which is a function of x at t = 0. And you can see that at different spatial locations x1, x2, x3, x4, and so on.

At six different locations, we have defined six different velocities as initial conditions. So, if this is the line on which u is equal to 0 that means from x1 to x5, we all have positive velocities, where x5 is of course, a very, very small positive velocity nearly 0. And only at the point x6, do we have a negative velocity? So, this is the initial distribution of the velocities.

So, this is the picture in the x u plane. Right. What is the corresponding picture of these characteristics with the initial velocity distribution for that we need to go to the x t plane. Right. So, remember that if I have a very slowly moving portion of the wave, then the characteristics will be nearly vertical. If I have a very fast moving portion of the wave, they will snap closer to the horizontal because they are going to cover more space within the same time. That is the idea.

So, here, these are larger velocity portions and these are positive velocities. So, in the larger velocity portions, it is slanting further towards the horizontal. While in the remaining portions, they are almost close to vertical where the u’s are small by larger portions are slanting closer to the horizontal and there is only one point at which the velocity is pointing towards the other direction and therefore, it is slanting towards the opposite direction. So, we will discuss more on this in the next lecture. Thank you.