Introduction to CFD
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Lecture - 46
Numerical Solution of One Dimensional Euler Equation for Shock Tube Problem (continued)

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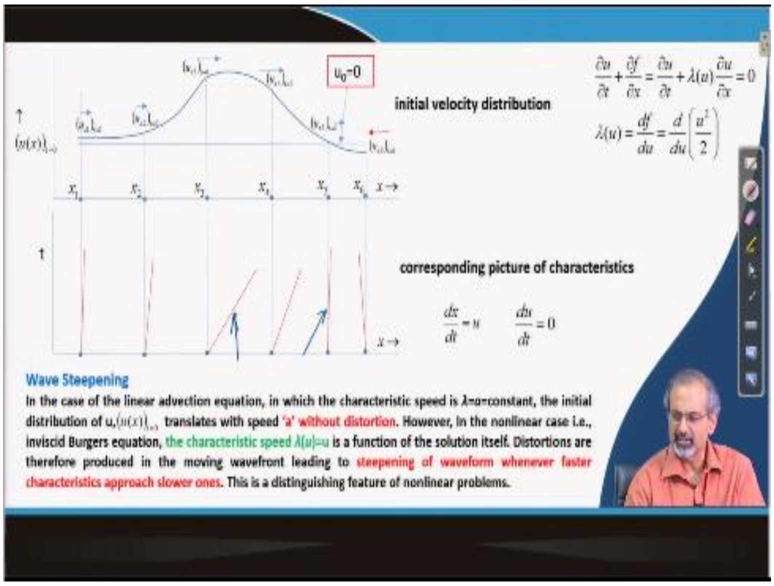

We continue our discussion on one dimensional Euler equations in this lecture. So, last time we had started discussing regarding this initial velocity distribution, and we had discussed as to how the initial characteristics would look like and remember that if this is the velocity distribution we have initially and we are solving the inviscid Burgers equation.

Then point to point these differences in velocities would have an influence on each other because this velocity distribution will not get translated at the same wave speed a like it happens in linear wave equation. But rather here they can interact with each other in inviscid Burgers equation and then there could be situations where they get close enough to each other generating compression actions.

When they move apart from each other, there could be expansion actions. Now, just to recollect very quickly that larger positive velocities are more slanted towards the right like we see over here. Smaller positive velocities are mainly tilted towards the right, while negative velocities are tilted towards the left; because this comes entirely from slope information. Because dx dt is equal to u.

So, depending on what the value and sin of u is accordingly dx dt gets defined and therefore that straight line gets defined. So, with that in mind remember that these lines whatever we have drawn, they are true ideally only a t = 0. Now, as time emerges, we have to figure out whether we can continue drawing these lines as straight lines which are independent of each other or they are going to get sufficiently close and start interacting with each other.

And whether characteristic lines can cross each other or not such issues have to come into picture, because you can clearly anticipate that these lines are going to distinctly get closer to each other as time progresses and then can they really intersect and cross over. Now, before answering all those issues, let us look at the concept of wave steepening what we have written at the bottom of the slide.

So, in the case of linear advection equation in which characteristic speed is constant, it is a. The initial distribution would translate with speed a without distortion. So, it will just get translated. However, in the non-linear case of inviscid Burgers equation, characteristic speed is u and it is a function of the solution itself. So, distortions are therefore, produced in the moving wave front leading to steepening of waveform whenever faster characteristics approach this lower ones and this is precisely what is happening here.

A foster characteristic may approach the slower ones here. That means they are going to get closer to each other and interact. So, this is a distinguishing feature of non-linear problems and this is essentially the mechanism with which non-linear waveforms are generated.

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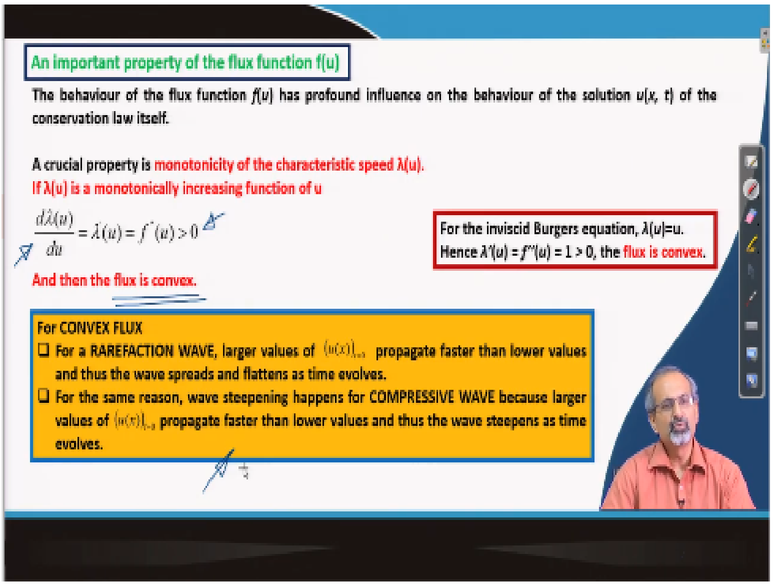

Another important property which we will have a quick look at which must be in short regarding the flux function f u. So, the behavior of the flux function f u it has an influence; a very strong influence on the behavior of the solution itself. The crucial properties monotonicity of the characteristic speak lambda and lambda u is a monotonically increasing function of u.

When if you take a derivative of lambda with respect to u, it should be greater than 0 then it is monotonically increasing. If that is the case, then the flux f is defined to be a convex flux. Now, what happens in the case of convex flux we will discuss later in this yellow highlighted portion later. Now, let us try to find out that do we really have a convex flux in the case of inviscid Burgers equation. So, here lambda is equal to u.

So, if you take a lambda dash, it will give you one which is greater than 0. That means the flux is convex. What happens for a convex flux? That is what is of interest to us. So, in a convex flux situation for a rarefaction wave. If you have larger values of the initial velocity, they propagate faster than lower values and thus the wave would spread and flatten as time evolves.

So, this would happen if the leading waves are moving faster than the trailing waves. So, the gap between the leading and the trailing waves keep increasing with time. The faster ones continue to move faster; slower ones continue to remain slow. This is what happens in a convex flux situation. And therefore, you have a fanning out of the flow or an expansion fan developing there.

Just the converse happens, if it is that the characteristics are moving closer to each other. So, for the same reason wave steepening would happen for a compressive wave situation; because larger values of u which are defined from initial condition would propagate faster than the lower values. And thus the wave steepens as time evolves. Now, this would happen if the leading waves are moving slower than the trailing waves.

So, the trailing waves would catch up with them leading waves leading to merging of waves compression effect and therefore, development of shocks. So, in the convex waves, flux situation, we have a slower wave continue to moving slower; a faster wave continue to move fast and therefore, these kind of interactions are possible.

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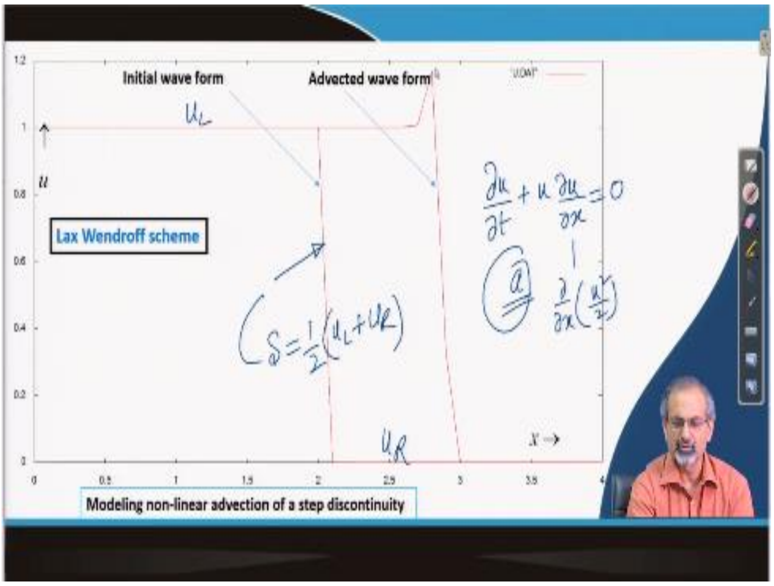

Let us try to look at a few numerical results. So, here we see non-linear advection of a step discontinuity. Now, we have also looked at step discontinuity being advected with linear advection equation. Remember that that was introduced as an initial condition there and it continued to move with the wave speed a whatever we defined. Now, incidentally in this case, where the equation is like this or this term can be written as the flux term.

You have no definition of a. So, at what speed will the front move that is a question in itself. Now, the equation decides on its own at what speed the front will move, it emerges as a part of the solution. This is something that we have to realize, because you are not defining at what speed that discontinuity will have to move and neither can you define it because the moment you define it, then it becomes linear advection.

Here, it is inherently non-linear advection. And the fact that you have a sharp discontinuity means as though there is a moving shock already introduced into the domain. Now, the point is that if the shock movement is governed by this equation, then the equation decides for itself that at what speed it will move. Incidentally, we will see slightly later that it moves with the speed of the discontinuity, which is an average of what you have in the left state and what you have in the right state.

And this condition essentially comes from the Rankine-Hugoniot conditions. So, you have a left state where u is equal to 1; a right state where you is equal to 0 as though a shock is moving into a stagnant region, such situations we will see even in the shock tube later, and then at what speed will this discontinuity move that is given by this relation; the shock speed relation.

And what we have done is we have tried to simulate this situation using the Lax Wendroff scheme. Now, because the Lax Wendroff has the dispersion issues; you of course, have wiggle formation over here and so on. But you can see that the wave front is moving as time elapses.

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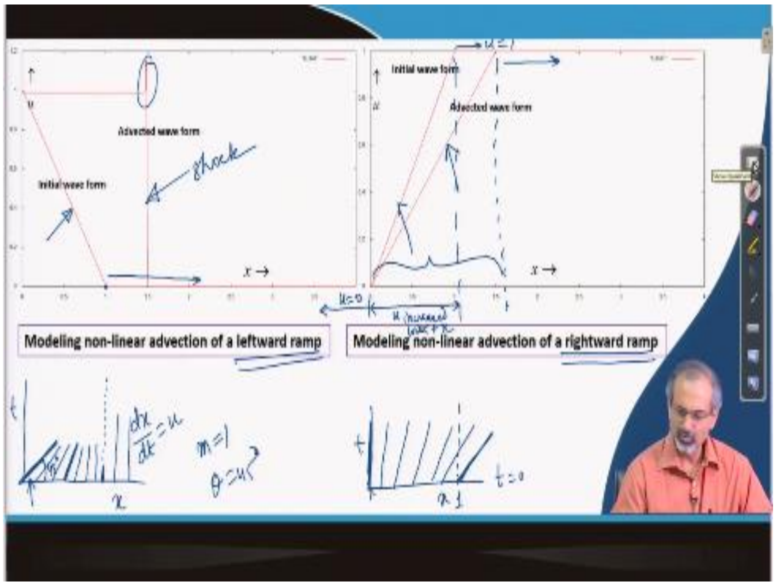

Let us look at a few more examples. So, here on the left you have an initial waveform which looks like this. Now, u is plotted along the y axis that means you have value of u = 1 at x = 0 and then it drops to 0 at x = 1 that means u uniformly reduces or linearly reduces to a value of

0 over a unit length. Now, if you allow this as an initial waveform, then allow the Burgers equation to solve this problem.

It leads to a sharp wave front like this again with the oscillation here because Lax Wendroff has been used. But if you ignore that part, you can see a sharp wave front. Why is it happening? As we said that if you look at this problem from the point of characteristics, then in the characteristic space, how can you define it at t = 0, you have vertical characteristics here.

And you have sloping characteristics like this here, and in between the slope increases in the y direction. Why do you have it this way? Because dx dt is equal to u along each characteristic. Right. So, with passage of time this characteristic does not see any change, because it is stagnant. They are all characteristics beyond that also are vertical, because it is stagnant here.

But, the one that you have over here at x = 0 is moving with u = 1. So, you will actually have a 45 degree inclination here because dx dt is equal to 1. So, that gives you m = 1 and therefore, the slope is the theta is 45 degrees. Right. What will be the intermediate characteristic slopes? They will vary gradually and the slope will start approximating towards this vertical line as you approach the end up.

That is the distribution of the characteristics at the beginning at t = 0. So, you clearly see that the characteristics are converging and will show soon that they cannot intersect with each other, but they can merge into a shockwave and that is what you see as a shock front over here. So, that is how the whole thing evolves, because you are allowing these characteristics to non-linearly interact.

The packet will not get translated as it is with a constant speed without the distortion like it happened in linear advection equation. They will interact and therefore, there will be non-linear waves generated. So, this is the basic lesson we are taking trying to take from inviscid Burgers equation which will be very helpful when we try to handle Euler equation because Euler equation also has these behaviors inherent.

On the contrary, if you have a distribute velocity distribution of this kind, you find that there are no discontinuities generated rather there is a smooth variation which continues to move towards the right. Now, if you look carefully at this initial waveform on the right picture, you have, we are calling this as a rightward ramp, while the previous problem we said it is a leftward ramp.

So, in a rightward ramp what is happening is that you have zero velocities here and you have $u = 1$ here onwards towards the right. That means the velocity in this intermediate range has changed from 0 to 1. Right. So, u has increased with positive change in x in this region. So, what does that mean? In the characteristic space, if you look at it as the initial distribution, so, here the line is vertical because dx dt is equal to 0.

Here the line is again at 45 like we saw in the previous problem at $x = 0$, it was 45; here at $x = 1$, it is 45 and what will happen to the intermediate lines? They will start behaving like this. This is a picture at $t = 0$ essentially. Now, the interesting thing is at no point these characteristics will come close to each other because they are getting further and further apart as time grows.

And therefore, there is no possibility of compression wave generated out of this. Rather there would be an expansion fan generated out of this and that is why you find widening of the field as time progresses and so, now, at a later time you are finding the fan has expanded up to this extent. It has become wider, which is shown in the bracketed region. So, this is how the things are working.

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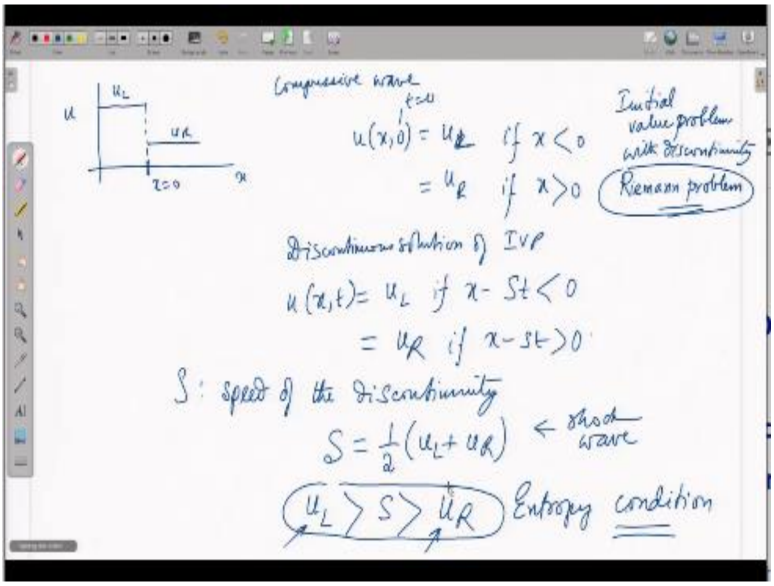

So, we just try to add some more detail here. So, if you look at a compressive wave like this as a sharp discontinuity. This is the u distribution and let us call this as x = 0 . So, what we have done is that u at x 0. So, this is t = 0 essentially is equal to u R if x is less than 0. Sorry. It should be u l and this is u R if x greater than 0 and the discontinuous solution for this initial value problem.

So, these are initial values that you have imposed. So, this is an initial value problem with discontinuity and if you impose such discontinuities we often say this is a Riemann problem. So, the discontinuous solution of this initial value problem is given by u x t equal u L if x - S t is less than 0; is u R if x - S t is greater than 0. What is this S? S is the speed of the discontinuity which we were referring as shock speed a few slides back.

And it is defined as u L plus u R by 2. As I mentioned earlier, it should be possible for you to show it through Rankine-Hugoniot equations and this discontinuous solution of course, is a shockwave and you need to satisfy an entropy condition across the shockwave that u L is greater than S is greater than u R. So, this is called as the entropy condition. Solution must not violate this entropy condition. If it does, then the solution will become unstable.

And as you can understand these are the characteristic speeds in the respective domains and how do the characteristics interact? That is the interesting part of the story which we need to figure out.

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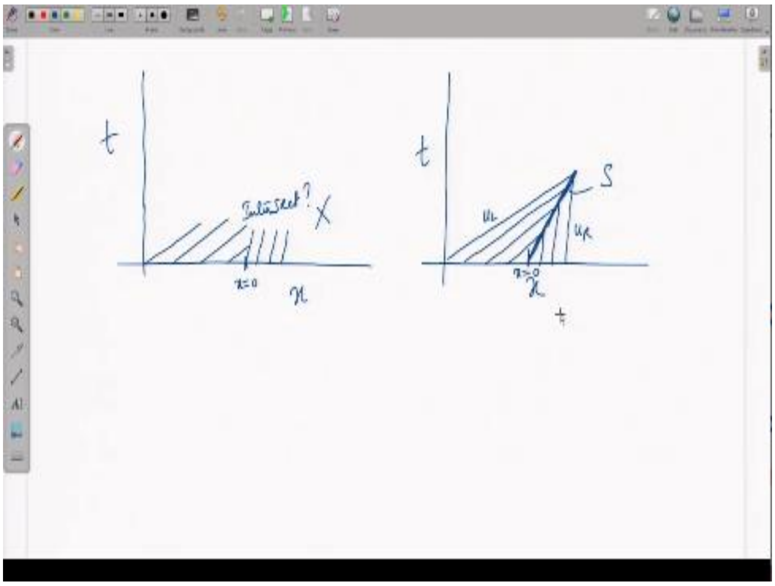

So, at t = 0 if you try to look at the problem, so, this is x = 0. So, the characteristics on the right are moving slow. So, they were looking like this. While the characteristics on the left where moving fast, so, they were looking like this. And now, not sure whether they will intersect or not. Actually, they cannot because then there is entropy violation. We are not going into details of this but from a thermodynamic perspective, if we analyze the problem.

And we can show that the characteristics would finally engage in this form that right from t = 0. A front will develop which we are calling as the discontinuity and this will move with speed S and the characteristics on two sides of the front as if get lost. In the front as the meet the front they cannot cross the front. So, the front as the acts like a black hole for all the characteristics which meets the front.

So, we now figure out that these are the characteristics coming from u L; these are the characteristics which are coming from u R and merge at the discontinuity and the discontinuity moves with speed S. So, this is how a compressive front works. What happens in an expensive case?

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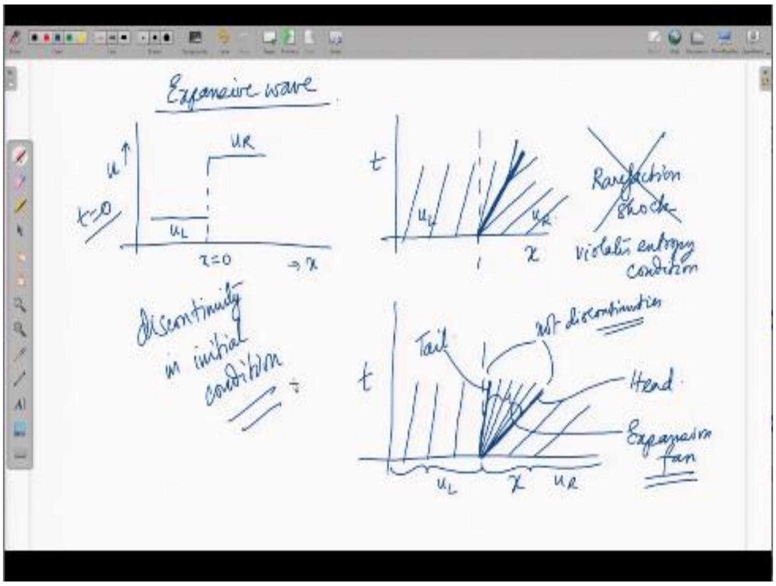

And in an expensive wave, let us try to first define the problem. So, you have a left state; you have a right state. Let us say this is x = 0 and this is essentially that u x at t = 0. This is the initial condition. Now, one can say that there could be a possibility like this where you have a so called rarefaction shock, just like we had a compression shock. we have a rarefaction shock from which the characteristics as they emerge as we move in time.

So, this is say that x t space. So, as you can understand these are the slower waves, these are the faster waves and beyond t = 0 as the rarefaction shock is emerging. But, as we said earlier that this condition where the shock absorbs all the characteristics that has to be kept in mind, but here we are seeing emergence of characteristics from the shock which again violates entropy condition.

And therefore, this is not a possibility which nature accommodates. Rather what nature accommodates looks something like this. You have the faster waves; you have the slower waves. And then to have an expansion fan generated in between. So, this is the extent of the slower waves; extend of the faster waves. And then in between there is a region which is defined by the limiting characteristics minded that these are not discontinuities.

These are the limiting characteristics and within those limiting characteristics you have an expansion fan, we call this as the tail of the wave. We call this as the head of the wave. So, what you have in between is an expansion fan. So, this is how an expansive wave would behave with again a discontinuity in initial condition. So, this is a centered wave. You could also have non-centered waves where you could have a region over which the velocity change occurs, like we have shown one of those cases in the numerical simulation.

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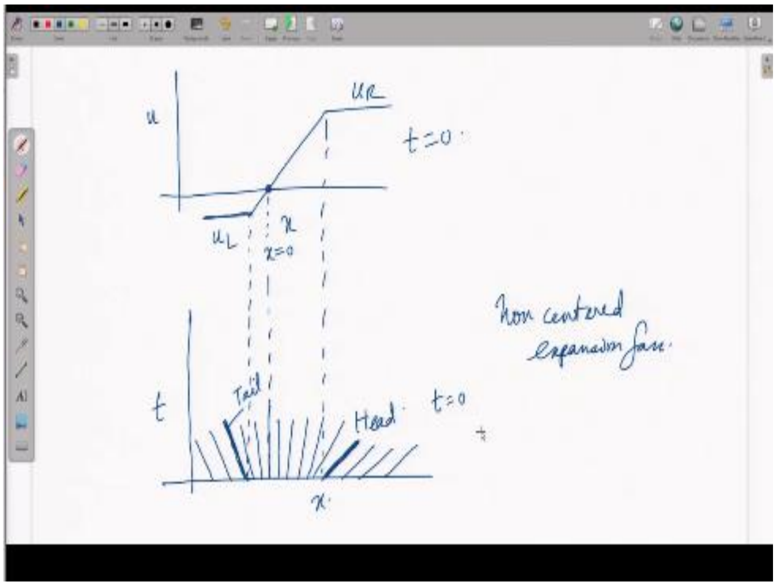

I will make a quick plot for you to explain that situation. So, you may actually have a situation like this as an initial condition. You have a u R here. You have a u L and a linear variation in between. So, let us say it crosses over, over here at x = 0 and this is what you have as an initial condition. Then you can show that in the S t space. This is where the characteristic will be vertical.

And then in this region the characteristics will be slanted with the slope defined by u R. And then this is a region where they will be sloping with u L constant slopes and the region in between will have a variation to match these two limits. That means the tail on one hand and the head on the other. So, this is how the intermediate characteristics will behave. Of course, remember that this is a t = 0 but then it will expand fan.

And they will not come close to each other. So, it will fan out. So, this is a non-centered expansion fan. So, these are very important concepts which one need to keep in mind when we discussed about numerical calculations and their implications and we will get a much deeper insight to analyze the results if we are aware of the physics which is working in the background.

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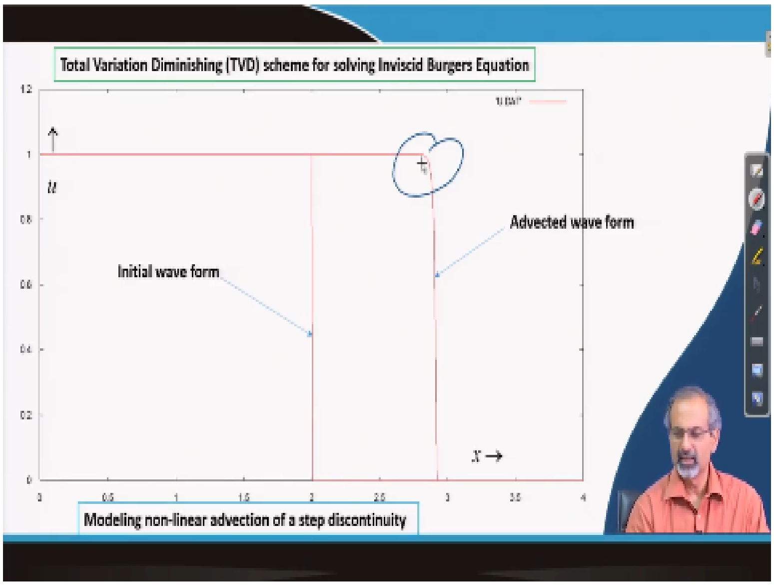

And the physics is distinctly different between the linear and the non-linear advection case. That is what we are trying to establish through our results. We have discussed about TVD schemes in our last lecture on one dimensional advection diffusion equations. So, there we said that schemes which maintain monotonicity do not allow occurrence of fresh overshoots and undershoots in the solution.

So, that comes in very handy when you try to handle discontinuities in initial conditions and that too when you are solving non-linear equations like inviscid Burgers equation. So, here you find that unlike what you saw for Lax Wendroff, there are no longer wiggles or oscillations found at the corner when you are advocating the sharp discontinuity. So, that is by virtue of the TVD nature of the solution.

We are not going into the details of the algorithm, but nevertheless the solution proves that the TVD scheme is very useful in this instance.

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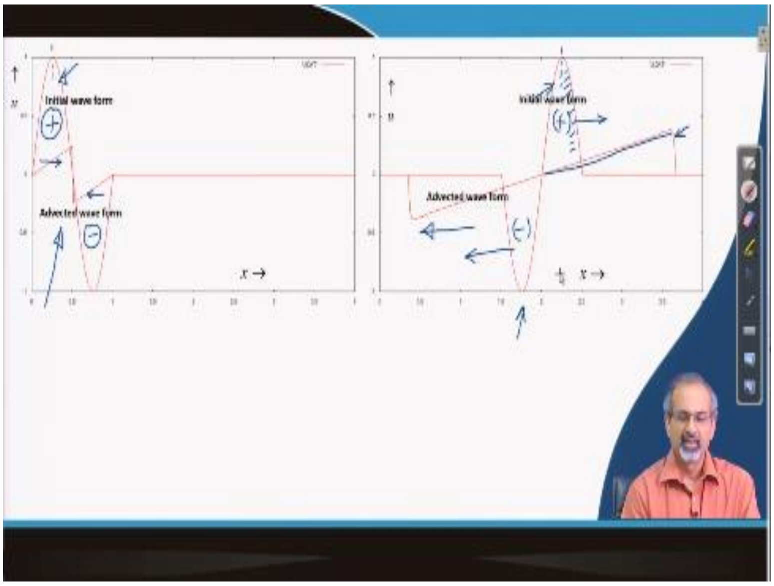

Again, we are showing another one or two interesting examples using the one dimensional Burgers equation where we have an initial waveform of a sinusoidal nature like this and with the advection, it finally develops discontinuities. So, as you can understand this part of the wave has positive velocity; this part of the wave has negative velocity and the larger velocities are somewhere in the middle, which tend to make the characteristics move towards the right.

And then there are characteristics moving towards the left from the negative velocity region. And they try to, you know kind of collide with each other. Thereby leading to a sharp discontinuity in between, which essentially remain stagnant. It cannot move anyway. Because it is being barraged by characteristics from both ends. While here, you have a situation where you have positive velocity here; negative velocity here.

And positive velocity moves towards the right; this towards the left, but then the maximum is somewhere in the middle. So, those characteristics would catch up with the slower ones towards the right and therefore leading to a sharp discontinuity. While the ones which are left behind, move slower, therefore, there is a kind of expansive nature here. So, there is a discontinuity. There is an expansive nature, hand in hand.

Same thing happens to the negative velocity region where the feature moves towards the left again with a shock and an expansion. So these are very interesting test cases where with the concept of characteristics now, hopefully, you should be in a better position to analyze the situation.

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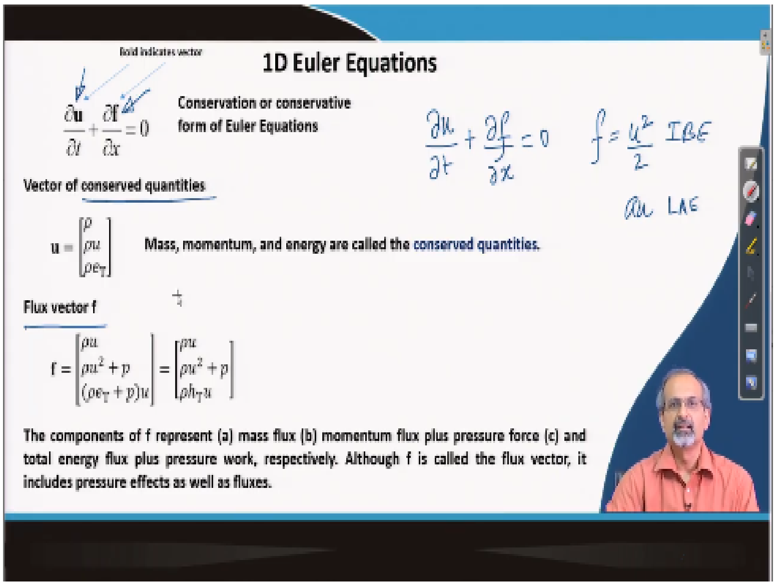

And then we just spend a minute to have an appearance of the one day Euler equations before we close this lecture. So I am just going to harp on only one point that is, in the earlier instances, we were all looking at equations where we were writing them in unbolted form like this. And you remember that you had a flux term u squared equal to 2 for inviscid Burgers equation or equal to a u for linear advection equation.

While now we have come to problem domain where we have multiple conservation equations. So these are no longer scalar conservation laws. So, we have vectors involved. So, both u and f are vectors, u is called the vector of conserved quantities, while the f vector continues to be called as a flux vector. So, we will discuss more on this in the next lecture. Thank you.