Introduction to CFD
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Lecture - 47
Numerical Solution of One Dimensional Euler Equation for Shock Tube Problem (continued)

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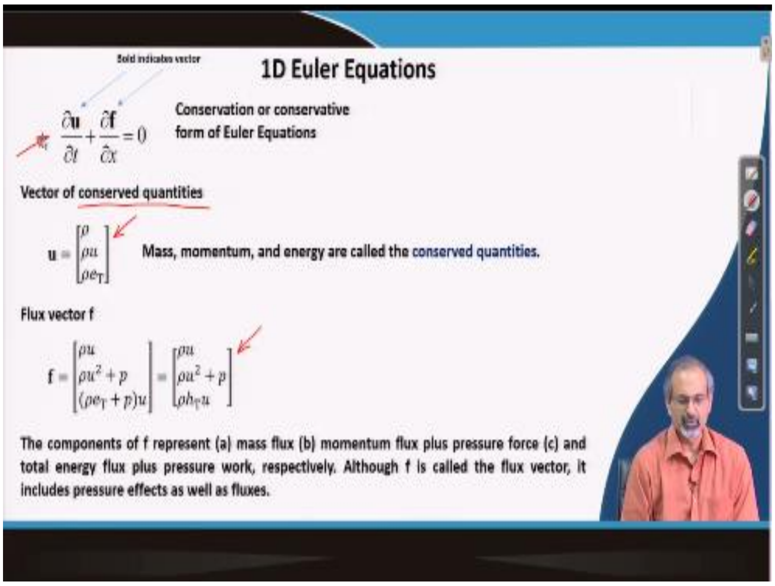

We will continue our discussion on one dimensional Euler equation in this lecture. So, in the last lecture, we before concluding we had talked about the one dimensional Euler equation in their vector form. And if you recall, we talked about 2 vectors; the vector u which is concerned with the conserved quantities conservation of mass momentum and energy and the flux vector f which comprised of the different fluxes, different flux quantities.

So, if you look at the components of the u vector, you find density rho as the first component then the product of density and the velocity here it is a one dimensional problem. So, there will be only one component of velocity namely u and then you have the product of density and internal energy in its total form and in the flux vector f, the first component is again density times velocity; the second component is density times velocity squared plus pressure.

And the third component is a summation of density times total internal energy plus pressure the whole multiplied by the u component of velocity. So, these are the different components, we are talking about vector conservation law. So, we are no longer having one equation to handle but multiple equations to handle here and we therefore, represent them in a more compact form in the vector form. So, this is the compact representation of the one dimensional Euler equation.

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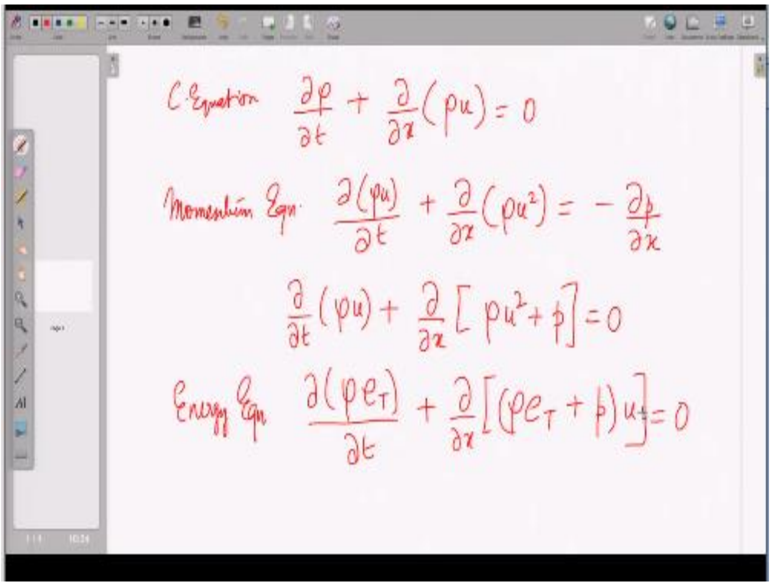

So, just to look at the familiar forms once again we if we look at the continuity equation. Then it is the time derivatives of density plus d dx or other del del x of rho u equal to 0. The momentum equation, so, as we mentioned earlier that we are going to prefer the conservative form of equations, because they are inherently more suitable to handle flow fields with discontinuities like shocks and contact discontinuities.

And Euler equations are capable of capturing such discontinuities. So, we would like to have them in the conservative form, where we handle fluxes and though there could be discontinuities in certain derivatives like say velocity derivative across such discontinuities, but the fluxes would not jump and coming to the energy equation. So, these are the familiar partial differential equations that we are aware of, to represent the conservation of mass momentum and energy.

So, when we look at the flux vector form, it is a very compact representation of the same. So, coming back to the flux vector f, we are talking about the different components in terms of mass flux, momentum flux, in addition to the pressure force, which is involved through the pressure gradient term in the momentum equation. And the third component is a total energy flux plus pressure work.

So, these comprise the different components of the flux vector. So, we may very often in vectorial form represent this vector u by its components, u 1, u 2, u 3 and the flux vector in terms of its 3 components f 1, f 2, and f 3. Step at this point would be to represent the different components of the flux vector in terms of the different components of the conserved quantities. So, let us try to look at that aspect gradually.

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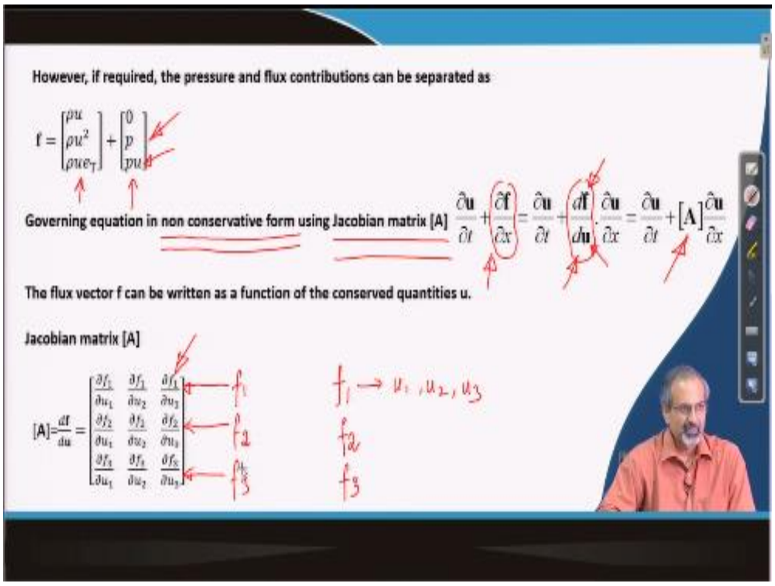

So, before we do that, we would like to mention that sometimes the flux vector is split up, because you have seen that the second and third components of the flux vector comprise of 2 terms, not a single term. And very often, we would prefer to segregate out the pressure contribution from the regular fluxes. So, in that case, you would write the flux in the form of summation of 2 separate vectors, one with only the influence of the velocity field, the other only with the influence of the pressure field.

Of course, here, as far as energy equation is concerned, there is also a velocity field involved, but it is primarily based on the pressure or the other part that we segregate this. And then, a very important part that we can look at, in terms of information about how the equation behaves actually comes from the non-conservative form, where we define the Jacobian matrix A.

So, in order to define the Jacobian matrix, we start from the conservative form of the equation, the left hand side of the conservative form of the equation, and then we split the spatial derivative. So, we split it into 2 portions, one is df du and then we multiply it with del u del x, and we call this as the Jacobian matrix. Now, remember that both f and u are vectors.

So, when you take a derivative of this form, which involves vectors, you would come up with matrices, because there are multiple components contributed from both ends, which lead to a matrix formation. That is why what we come up with is a matrix here, a matrix A which we would call as the Jacobian matrix. And a lot is interpreted about the nature of the equations from this Jacobian matrix.

We will see soon and that is why we take this non conservative form to come up with the Jacobean matrix and try to utilize it to extract more information about the equation. So, when we look at the Jacobean matrix, we see different components of this kind. So, remember that you are taking a derivative of f with respect to u. So, if you look at the first row of the matrix, you find the first component of the f vector figures here.

So, derivatives of that component with respect to the different components of u, they will form the elements of the first row of the Jacobian matrix. Of course, they are all partial derivatives. So, this makes it obvious that we would like to have the functional dependence of f 1 on u 1, u 2 and u 3, only then can you evaluate these derivatives.

If you go to the second row of the Jacobian matrix, you can similarly see that the partial derivatives of f 2 come in over there with respect to the 3 components of the u vector. And similarly, the third row involves f 3. So, these are the 3 rows involving partial derivatives of the different components of the conserved quantities. Alright. So, the next step, the next obvious rational step that we need to take is try to express f 1, f 2, f 3 all of them in terms of u 1, u 2, u 3.

Only then we can obtain the different elements of this Jacobian matrix. And then the normal course of action which you are aware of even from earlier modules, where we have discussed about partial differential equations that we try to extract Eigen values of such a matrix in order to understand the behavior of the system of equations. So, that comes later. So, let us try to do this activity of representing the different components of the f vector in terms of the components of the u vector.

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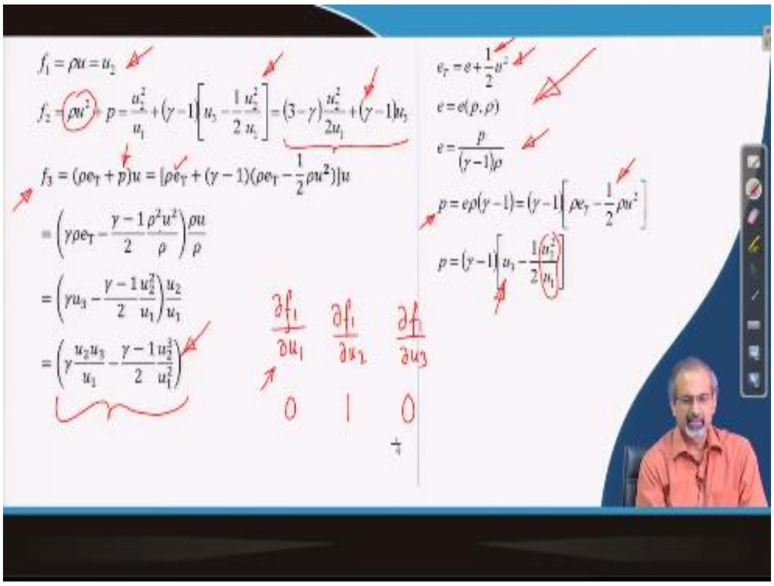

So, let us see how we go about doing it. So, if you just keep the 2 vectors in front of u, you can clearly compare that f 1 which is rho times u is nothing but u 2 directly. It does not have any dependency on u 1 or u 3. But it directly equals with u 2. How to find out f 2? f 2 is by definition rho u square plus p. Now, we can obtain rho u squared other easily because u 2 is rho u and u 1 is rho. So, if you take a ratio of that you will get rho u square.

So, it is u 2 square by u 1, but we do not know how to go about with pressure. So, for that, we do a simple calculation which we have shown here on the right side. So, we first define the total internal energy in terms of its static and the dynamic part. And then, we realize that we can actually express the static part in terms of say two thermodynamic variables like pressure and density.

So, this is a form of the equation of state itself, where e is equal to p by gamma minus 1 into rho. So, if you have this relation, then that helps us because, we can then go ahead and write an equation for pressure which is equal to e rho times gamma minus 1 keeping gamma minus 1 outside the bracket. We replace the e by e t and u square half u square inside the bracket and then multiplied with density.

And then you see conveniently that the first term is nothing but u 3 and the second term just like what you did over here can be easily represented in terms of u 2 square by u 1. So, pressure now is expressed in terms of u 1, u 2 and u 3. You have the functional relationship. So, that is plugged in over here. And then you can actually come up with the final expression. Sorry. So, we were looking at the second equation. We will go back to this.

So, it is plugged in over here and then we club the terms together. So, you now have the functional dependence of f 2 on u 1, u 2 and u 3. So, having said that, we go to the f 3 part now, and then using very similar approach we try to proceed. So, rho e t plus p the whole into u. So, we have already obtained an expression for pressure in terms of u 1, u 2, u 3. So, that functional dependence has been worked out already. It is convenient for us.

And then we all already know that rho e t is nothing but u 3. Right. So, this is known to you and so, is your pressure. And therefore, if you just put in those steps together, you will find that the function f 3 can now be expressed using this equation in terms of u 1, u 2 and u 3. So, we try to work this out and we have achieved this goal. We have found out the relationships and now, in order to fill up the different components of the matrix A.

We have to actually work out the derivatives. Right. So, in order to work out the derivatives, you have to work out terms like say del f 1 del u 1, del f 1 del u 2, del f 1 del u 3 and so on. So, in order to do that, you take the expression for f 1 for example, and if you take a derivative like this, then because it is only dependent on u too this will become 0. This will be 1. This will again be 0. So, that feels the first row of the A matrix.

If you go to the second row, then you have to work with this expression and then take the partial derivative with respect to u 1 first then u 2 then u 3. So, if you take with respect to u 1, then you will find the different components and you can do the simple calculations.

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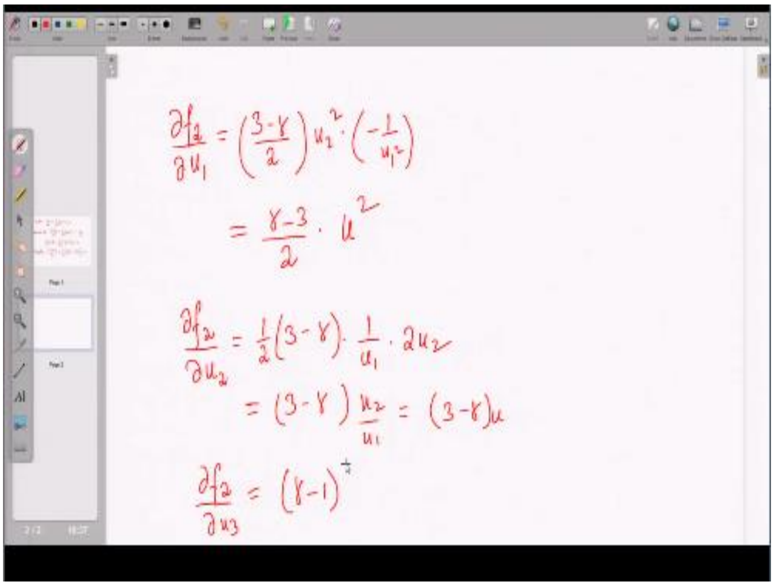

So, you can say, del f 2 del u 1. So, that will be 3 minus gamma by 2 and then it will be u 2 square minus 1 by u 1 square. And once you collect them together, so, it is we put it as gamma minus 3 by 2. And then if you gather those terms together, it turns out to be u square because u 2 by u 1. If you take a ratio, it will give u and then it is squared. So, that gives you del f 2 del u 1 will give u.

So, that is what it will give you. Then going to del f 2 del u 3 that will give you gamma minus 1.

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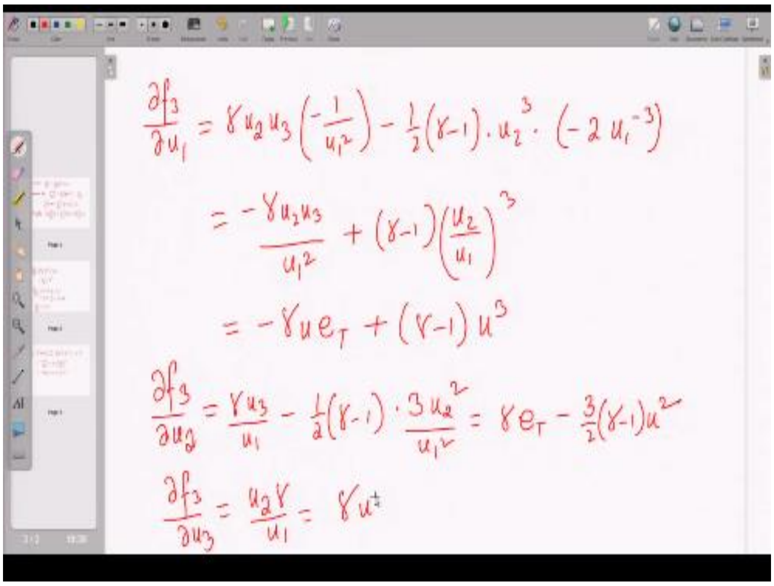

Now, we can just complete it by doing the del f 3 del u 1 which is the third row of the Jacobian matrix. So, you see it is gamma u 2 u 3 into minus of 1 by u square minus half gamma minus 1, u 2 cube into minus 2 u 1 to the power of minus 3. So, let us collect the terms, this is, what the expression will be. So, we have filled all the different components of the Jacobian matrix in this manner.

There are of course, alternative ways also to express these terms where we bring in say enthalpy. So, we have actually done this step already. We have just shown an example once again of one of those partial derivatives.

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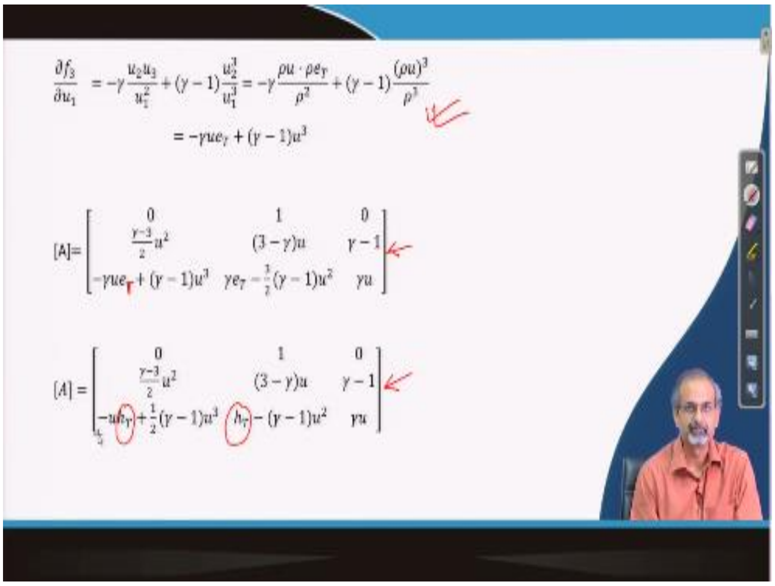

As I was telling you earlier that there are alternative forms of this equation. This should be T. And the one that we have obtained is essentially this. But there are alternative forms where you can have the total enthalpy coming in. So, in your spare time, you can try exploring the alternative forms. As we can see that up to the second row of the Jacobian matrix, there is no difference. It is only in the energy equation that you see the difference.

So, that is the third row of the matrix, which is essentially the third component of f, where the difference is.

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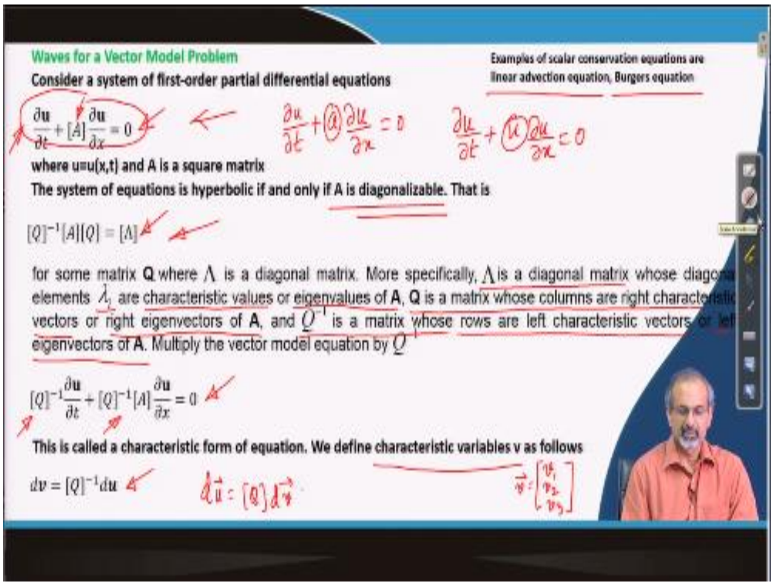

Now, in the previous lectures, you will recall that we have, we looked at the scalar conservation equations related to the wave kinds of problems. So, we looked at some of the results specifically for linear advection equation and inviscid Burgers equation. So, they were all scalar conservation laws. Now, once we get into one dimensional Euler equation, even with one dimension, we have a vector model.

Because we have continuity equation, one component of momentum equation and energy equation, which are a combined system of conservation laws, which have to be handled. Therefore, we come up a vector model problem. Now, when we talk about waves for wave kind of solutions for vector model problems, we represent the system in this form, where as you can see that u is now a vector. The coefficient matrix is no longer a single parameter.

If you recall that in linear advection equation or wave equation, this was a so it was just a constant or in inviscid Burgers question, it was u the velocity components of, but now, we have a matrix here, because we have a vector model in place. And we now know it is the Jacobean matrix and essentially is a square matrix. We have 3 conservation equations here. So, A is a 3 by 3 matrix.

Now, this system of equations is going to be hyperbolic in nature that means its wave kind of solutions only if this matrix A is diagonalizable. So, we come up with a new word diagonalizable. And we try to understand what that means. So, we write down an equation over here involving a matrix A, and we bring in another matrix Q, which is essentially derived from the matrix A itself, but it is not only Q that is involved in this equation, but it is also its inverse.

So, the left hand side of this equation, we can see it is a product of 3 matrices Q inverse A and Q. And remember that this matrix Q or Q inverse, they are derived from the matrix A itself. Now, this product that you have on the left hand side of this matrix equation will ultimately give you a diagonalized or diagonal matrix on the right hand side. That means the right hand side matrix will only have nonzero entries in its diagonals.

All the non diagonal elements will be zero. So, if this equation is drivable, only then will this system of equations be hyperbolic. So, if this equation is possible to drive, then we say that the matrix A is diagonalizable. So, let us look at a little more details on the matrix Q or Q inverse. So, as we already mentioned that the right hand side diagonal matrix has only nonzero diagonal entries.

And these diagonal elements are lambda i's, or characteristic values or Eigen values of A. So, sometime back, we were talking about extracting Eigen values of the matrix A because that tells us more about the behavior of equations. So, we now find that these Eigen values of A are available directly in the diagonal matrix capital lambda. And all these diagonal elements can be represented by the small lambdas in general, represented as small lambda i.

Now, coming to the matrix Q and Q inverse, so, Q is a matrix whose columns are right characteristic vectors or right Eigen vectors of the matrix A and Q inverse is a matrix whose rows are left characteristic vectors are left Eigen vectors of matrix A. So, this is how Q and Q inverse are defined. So, once you define matrix Q, Q inverse in terms of A and you are able to come up with this equation, then you can say that the system is hyperbolic.

Now, having said that, we will look at one step mode where we multiply the above system of equations which we have on top by Q inverse. So, Q inverse is the matrix which comprises of the left Eigen vectors of A, we remember that. So, we are multiplying the vector model equation by Q inverse. So, this is the first term; this is the second term. Now, we are proceeding towards forming the characteristic form of equations.

So, that means we are converting the equations from the plane of conserved variables u to another plane where we will talk about some derived variables which are the characteristic variables and we call these characteristic variables as v. So, there is going to be a vector comprised of characteristic variables. So, having the components if v 1, v 2, v 3 like you have u 1, u 2, u 3. So, how are the characteristic variables defined?

They are defined using this equation. So, dv is Q inverse du. Okay which also means that the du is Q dv. So, keeping this in mind, we are going to look at the full form of the characteristic equation.

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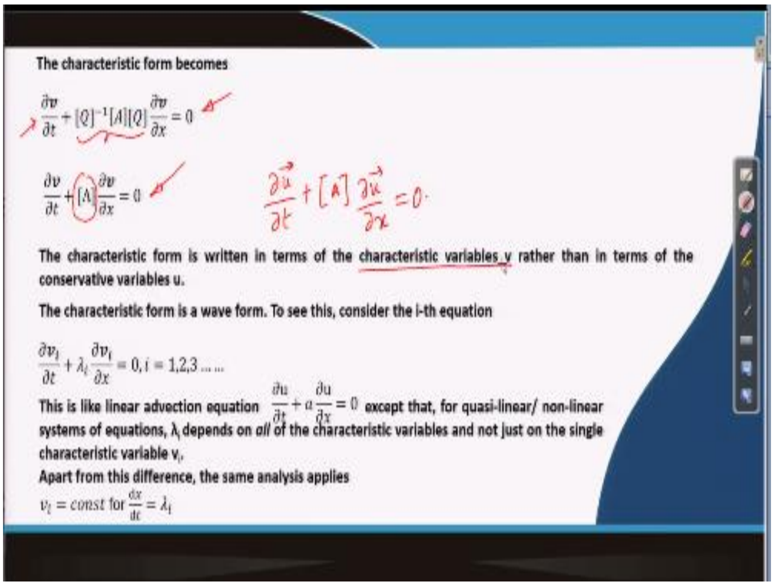

So, you have it here. So, you have dv dt then the product of these 3 matrices times dv dx or other del v del x. Even this was del v del t is equal to 0 and do you recall that this product should give you the diagonal matrix capital lambda. So, now, what do you have? You have the system of partial differential equations represented in the characteristic variable space. So, earlier it was in the space of conserved variables, while now you have written down the equation in the characteristic variable space.

So, we will finish this lecture here. We will continue our discussion on Euler equations in the next lecture. Thank you.