Introduction to CFD
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Lecture - 48
Numerical Solution of One Dimensional Euler Equation for Shock Tube Problem (continued)

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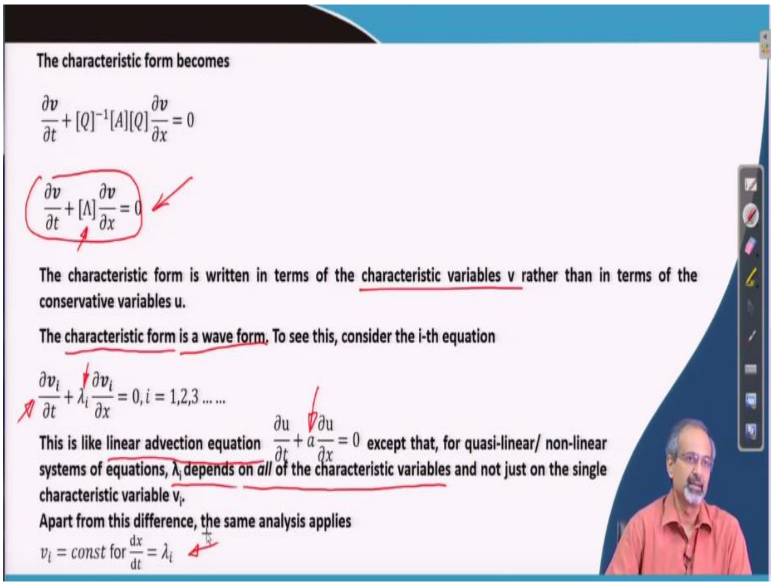

We continue our discussion on characteristics form of one dimensional Euler equation in this lecture. In the last lecture, we had come up with this form of Euler equation. And you recall that this was written with the idea of converting the equations from the concert variables u to the characteristic variables v space. And the interesting part is that we now have a coefficient matrix here, which is diagonal.

So, this gives a waveform to the system of equations. Why is it? Because, as you can see that if you look at any ith component of this system of equations to the system of equations is this. It has multiple components so if you are looking at any ith component. It can be written like this, which is like very similar to the linear advection equation in terms of structure, right. So, you have a certain speed, lambda, which is playing an analogous role to a in linear advection equation.

And different components can have different contributions so lambda 1 lambda 2 lambda 3 may have different values. So, in linear advection equation because it was only one equation a scalar conservation law, we had only one speed. Here we will have multiple speeds come from the multiple lambdas. But in principle, the different components can be individually represented like linear advection equation, so, that gives it a very clear waveform.

So, characteristic form of the equations, very explicitly show us the wave form of a hyperbolic system of partial differential equations. Now there is a slight issue over here in terms of how these lambdas behave. So, since we are handling nonlinear partial differential equations. These lambdas would depend on all the characteristic variables. That means if you pick any lambda i, it will depend on all the characteristic variables, say v 1 v 2 v 3 and so on.

It does not have its dependence only on the respective characteristic variable that means we cannot say that lambda is only dependent on v i. It will be influenced by other v's also. So, that is a distinct issue in nonlinear partial differential equations or even **(()) (03:31)** linear hyperbolic partial differential equations. But apart from that, the analysis is very similar. That means if we look back at the concept of characteristics that we were discussing in some of the previous lectures that the v i.

The characteristic variable will remain constant. And as long as you are moving along a certain characteristic which is defined by dx dt equal to lambda i.

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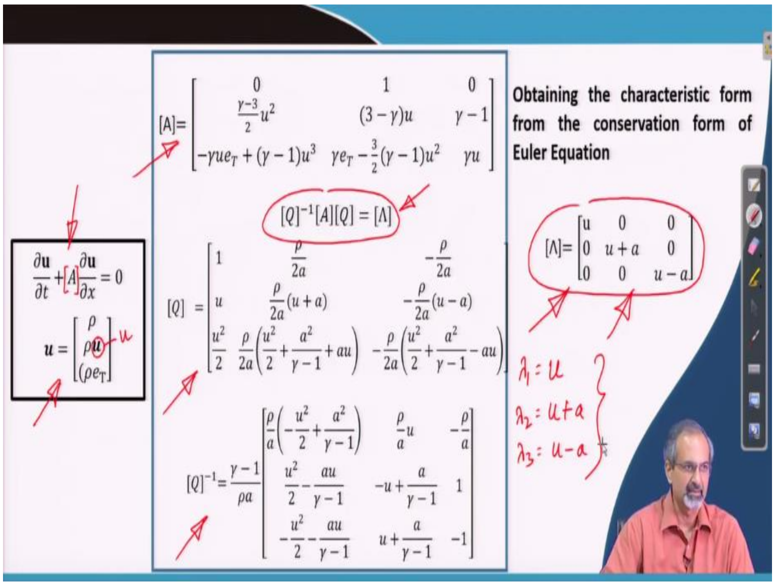

So, let us see where we stand at this point. So, we had shown how the vector model equation can be expressed from the conservative form to the non conservative form involving the Jacobian matrix. This should be lowercase. And then we showed how to obtain the different entries of the Jacobian matrix A, we discussed about the diagonalization property of matrix A, which makes it a hyperbolic system.

And then we were talking about Q and Q inverse, comprising of the right and left Eigen vectors of A, they are dependent on a. And then if you obtain the characteristic form from the conservation form of Euler equation, you can show that the capital lambda matrix which is a diagonal matrix on the right hand side of this equation takes up this form. We are not going into the details of the calculation, but this is available in many more elaborate texts on the subject.

We are just trying to obtain the information straight away and state it here. So, we find that the entries in the diagonal matrix R u u + 1, and u minus sorry u u + a, and u – a. So, you can call them as lambda 1 lambda 2 lambda 3. So, like we were saying before, in the case of linear advection equation, the lambda was equal to A. In the case of inviscid Burgers' equation, the lambda was u.

And in the case of Euler equation in one dimension we have multiple lambdas because it is a vector model. And they happen to be u u + a, and u – a. That means we can expect that information through the domain would percolate at different speeds along different characteristic directions.

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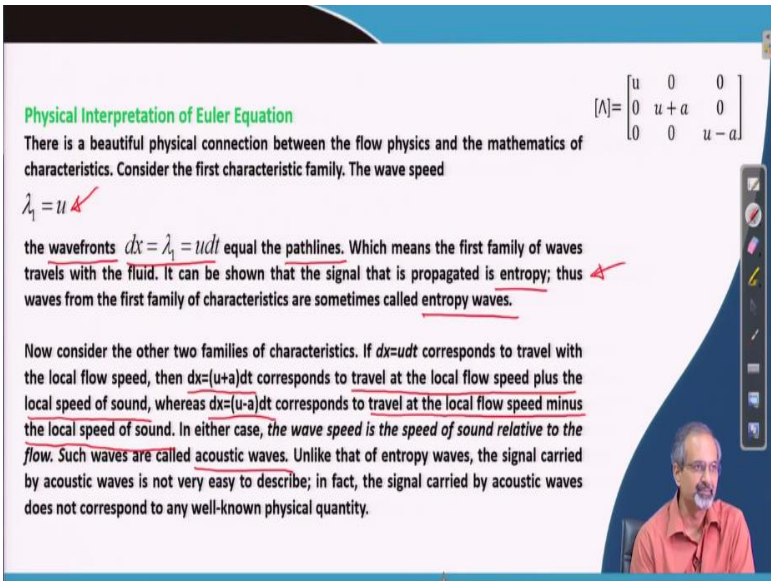

So, that brings us closer to a physical interpretation of Euler equation. So, there is apparently a very elegant connection between the flow physics and the mathematics of characteristics.

So, if you consider the characteristic family and take the different components into picture so the first value of lambda, the wave speed, lambda 1 is equal to u. Now that creates wavefronts which are defined by this equation. So, u is equal to dx dt.

From there we can define these wavefronts, the equation of these wavefronts. And they are nothing but the spotlights on the flow which comes from our basic fluid mechanics which means that the first family of waves they would travel along with the fluid. And what information or signal do they propagate? They are supposed to be propagating entropy information. So, they are called as entropy waves.

We mentioned about it earlier in a previous lecture on Euler equation. But now we come up with a more concrete information that how or where from this entropy wave emerges. So, it comes from the first component of the lambda vector or lambda metrics. And now if you look at the other two families of characteristics, we find that their wavefronts will satisfy these equations, the one will be dx equal to u + a times d t another will be dx equal to u – a into dt.

Now, if it is u + a, it corresponds to traveling at the local flow speed plus the local speed of sound. So, your local flow speed, if you add the speed of sound to that there is a wave front propagating at that speed. And there is another wave front which along which if you travel, you are going to travel at the local flow speed minus the local speed of sound. So, it is wave speeds which are traveling at the speed of sound relative to the flow in two different directions.

You can look at it that way. And these waves are called as the acoustic waves. And, incidentally, though we understand the concept of acoustic waves, the signal that is propagated along an acoustic wave is not very easy to describe. We can just talk by saying that these are acoustic waves. And we know at what speeds they propagate. Now let us try to make a small sketch to take it forward further.

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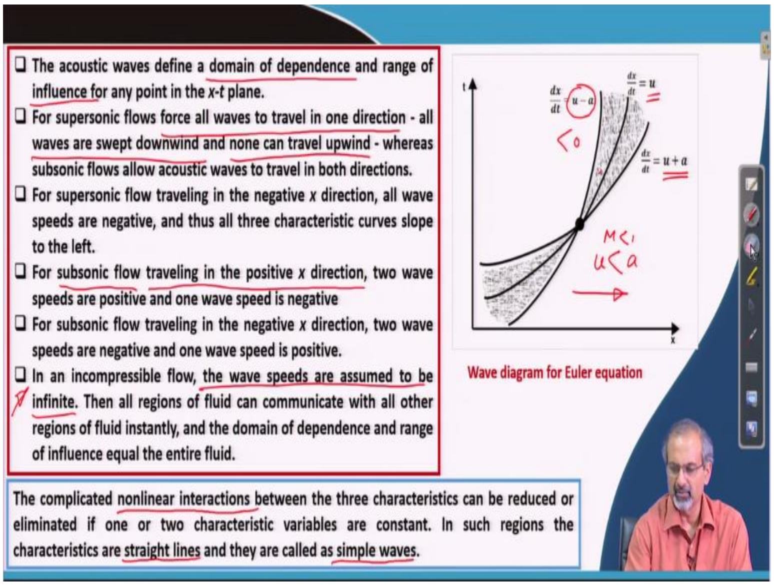

So, we make a small x t diagram for Euler equation, like we have been doing for the previous scalar conservation laws. And we find that if we take any point x t in that plane, then there are three characteristic lines which will or characteristic curves which will pass through the point x t. And each one of them will have a separate slope. So, if you look at the three, you have one carrying information dx dt equal to u and other carrying information, the slope equal to u + a, and other is u – a.

And what happens is that because, there are nonlinear interactions like we were saying that the lambda i’s are not dependent on only v i’s, but all the different v’s. So, these nonlinear interactions between the three characteristics make them curved. However, you can reduce or eliminate them, if one or two characteristic variables become constant. So, we are talking about those v i’s, the characteristic variables.

So, if one or two characteristic variables are constant, then you will see that these characteristic lines will become straight lines. And then we have the so called simple waves formed. So, depends on how the characteristic variables are behaving in a certain region of the flow. Depending on that these characteristic curves, the nature of these curves will change. So, in general, due to these nonlinear interactions, they will be curved.

Now, of course, if you look at this point x t will always be able to define a domain of dependence and the range of influence for this point x t. So, information propagates into the point from here and out of the point into this region. So, accordingly those domains get defined. Now, if you consider different kinds of flow speeds. That means you are now trying to define u and its direction.

So, you can have the value of u being less than or greater than a. And again, it is the direction of u, whether it is moving towards the positive x direction or negative x direction. Accordingly, the characteristic slopes will be decided. So, let us say, if there is a supersonic flow moving from, say, left to the right. What will that do to all these characteristics? What you will see is, it makes all the waves to travel in one direction.

It forces to sweep them towards its flow direction. So, all the waves are swept downward and non can travel upward or upwind travel. Now, when that happens, there is a very interesting thing which will occur. If you are standing at a point like this, what will happen is, if there is a noise source upstream of you, then, you will hear the noise source. But if you create a noise at your point, you will never be heard by that location from where the other noise is coming.

So, this is not possible. So, there is only one directional signal propagation, which is possible. Now if you think that supersonic flow is traveling in the negative x direction. Again, all the waves will sweep towards the negative direction. But the physics remains very similar in nature, what we discussed regarding the system of waves when the supersonic flow is moving towards the positive x direction. Only the directions just become opposite.

Now, the things are slightly more difficult, more involved when it is subsonic flow. The moment it is subsonic flow, you have to remember that if you look at these values, then these values will remain positive as long as you are moving along the positive x direction, and remember that your u is less than a. Because your subsonic your Mach number is less than 1. Now what will happen to this? This will become less than 0.

So, naturally, two waves will slope along the flow direction while one will slope the other direction. So, that is what happens when you have a subsonic flow traveling in the positive x direction. So, two waves speeds are positive one wave speed is negative and accordingly their slopes. Again, similar things will happen when there is a subsonic flow proceeding towards a negative x direction. So, you have to think for yourself how it behaves.

In that case, how the wave speeds will their science will be decided, how the slopes will be decided. When it comes to very low subsonic flow where the compressibility effect is now lost altogether. That means disturbances can propagate at infinite speed into the domain, the sonic speed limits to infinity, then what will happen. What we have mentioned over here in the last bullet point. The wave speeds are assumed to be infinite.

And then all regions of the fluid can communicate will, with all other regions of the fluid instantly. And the domain of the dependence and range of influence equal the entire fluid. So, there are no regions specifically defined which will just be able to communicate or not communicate. Every part of the domain will communicate with every other part. So, that is how incompressible flows will behave. So, now we are able to connect things together from a more global perspective which this wave concept has brought about.

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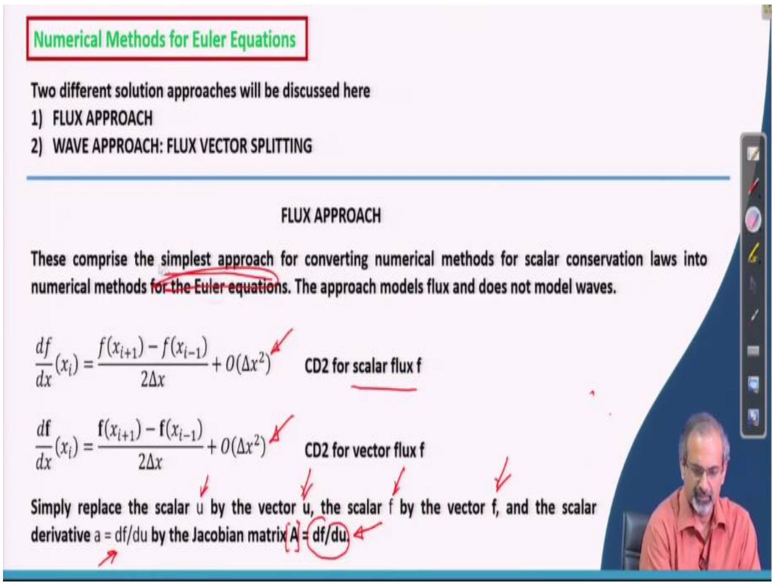

Now, we will start looking at some of the numerical methods for Euler equations. We will first look at an approach called as the flux approach, which is a comparatively simpler approach. In fact, one of the simplest approaches for application of either scalar conservation laws or vector laws like Euler equations. So, we look at the derivative definition using CD2 scheme using a scalar flux f in this first equation which is a very routine equation.

We have seen in our finite difference exercises and we just apply it to a vector f. The law remains identically the same. If you try to do numerical integration, like, which you need for say finite volume approaches the rules also remain the same there. So, that means the regular rules which we have seen applicable for scalars are extendable to vectors in this manner. And that is the basis on which the different terms would be discretized in the vector model problem.

So, what you are doing is simply replacing the scalar u by the vector u, the scalar f by the vector f, and the scalar derivative, a df d u. where f is a scalar, u is also a scalar by a Jacobian matrix. So, that is the only big difference which is coming up, but functionally it looks very similar. So, this is how the flux approach calculations are essentially done. We will just try to elaborate on it and look at a few schemes in this domain, some of which we have already started earlier.

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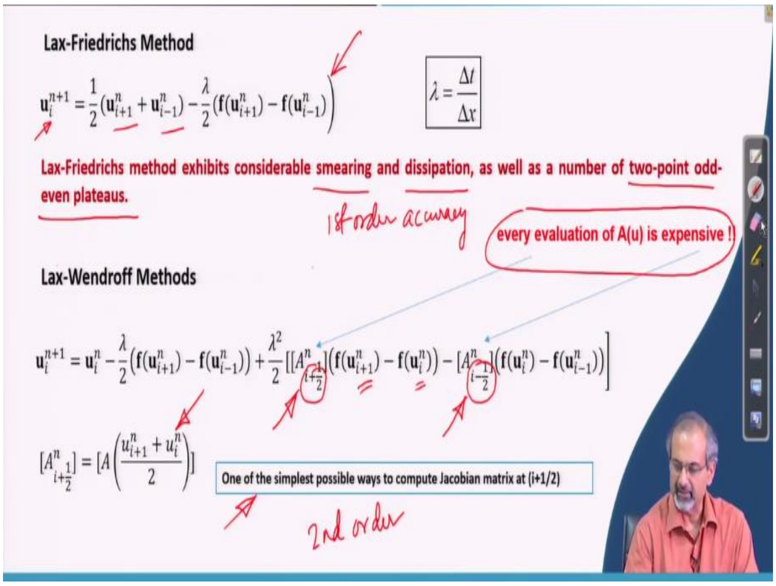

So, the Lax-Friedrichs method when it is applied to Euler equation would take up a form like this. Good exercise to do at home would be to try deriving this form yourself because you are aware of the Lax-Friedrichs method as applicable to scalar conservation laws, especially linear advection equation. Now from our prior experience we know it exhibits considerable smearing and dissipation.

That is one property the other is that it has two points odd even plateaus. This is because of the decoupling issue. And as you can understand that from the stencil itself it is clear that you have i + 1 i – 1 influencing i. And that is where from this whole problem arises. And then you have the issue that the large amount of dissipation because of the first order accuracy. If you go to second order methods, you look at the Lax Wendroff methods.

Here you can find that the Jacobian matrix comes into that calculation. That means you have a matrix getting multiplied with a vector that usually makes the computational scheme quite costly because you have large number of components repeatedly to be multiplied. So, every evaluation of A u is also expensive because as you can notice that in this scheme, we have used two grids. One is the main grid comprising of i's and i + 1's.

There is also a staggered location at which you are calculating the Jacobian matrix. So, one thing is you have to evaluate the Jacobian matrix at the staggered locations. The other is that you also do want to do the matrix vector multiplications. So, both of this can make this scheme quite expensive. There are numerous ways in which you have to you can calculate this value of the Jacobian matrix at staggered locations we have seen a very simple one has been shown over here.

This is one of the simplest possible ways. There are more better and more accurate ways of doing it by examples like **(()) (20:00)** averaging. But we are not discussing that here. So, second order methods of this kind would have better accuracy, but will suffer from higher computational costs. Moreover, there may be issues of numerical oscillations which has to be tackled very carefully and damped carefully so, that we do not lead the calculations to become completely unstable.

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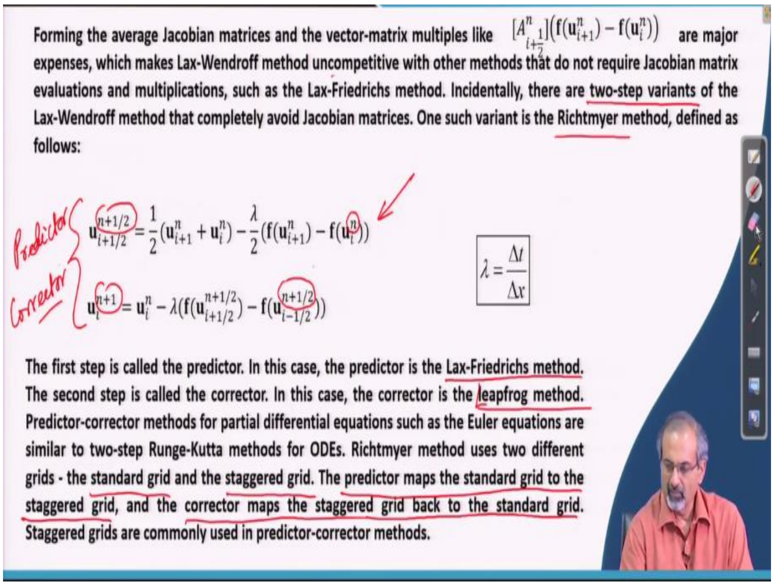

Now, forming the average Jacobian matrices and carrying out these vector matrix multiplications can add to the expenses significantly. And they can go to an extent where Lax Wendroff method can become uncompetitive with other methods. So, in order to keep it competitive still certain other variants have been evolved over time. And one of the more popular variant is the Richtmyer method which is shown over here.

It is essentially a two step method. So, these variants of the basic Lax Wendroff scheme are essentially two step schemes. And they are somehow able to tackle this problem of eliminating the Jacobian matrix out of the calculation, and therefore making it less costly. And thereby, even the matrix vector multiplications are also eliminated. So, we generally say that such step schemes are comprised of a predictor step and a corrector step.

So, predictor step gives us certain prediction about the vector u and then we try to correct it before we take it to the next time step. So, essentially, you are taking the solution from n to n plus half through a prediction. And then from n plus half that is at the prediction level to the corrected velocity or rather the corrected vector u at the next time step n + 1. So, that is how the whole setup works.

So, in the first predictor step as you can see it is a Lax-Friedrichs method which has been applied. And in the corrector step is the leapfrog method which has been applied. This is somewhat analogous to the two step Runge-Kutta type of methods which we are aware of in the context of solving ordinary differential equations. And even in the case of Richtmyer method, you can see that there are two grids which have been used.

One is the standard grid the other is the staggered grid. The interesting thing is that the predictor maps the standard grid to the staggered grid, and the character corrector step maps the staggered grid back to the standard grid. So, these kind of back and forth between standard and staggered grids also sometimes add to the numerical robustness of schemes. Very often we find the use of both standard as well as staggered grid schemes in these predictor corrector kind of methods.

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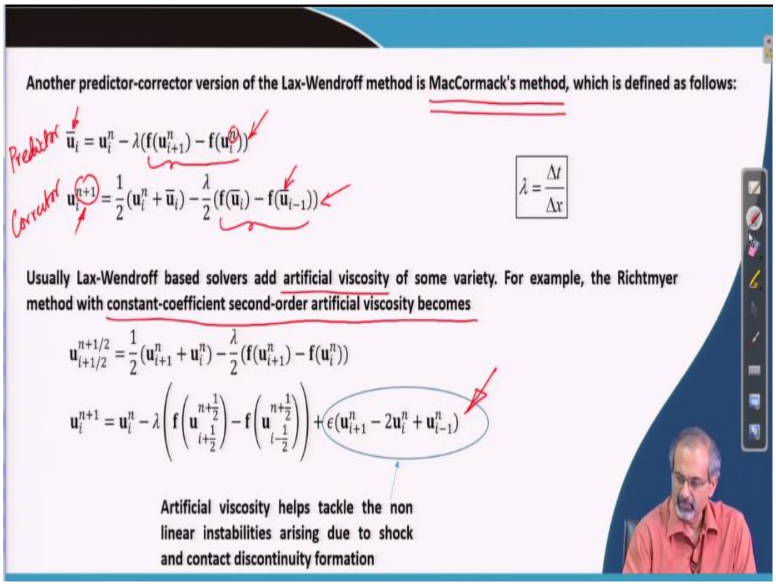

Another very widely used and popular predictor corrector best method, which has been around for a long time in solving Euler equations is the Lax Wendroff method, modified version known as MacCormack’s method. So, it is again a predictor corrector version of the Lax Wendroff method which is known as MacCormack’s method. So, here once again you find that there is a predictor step and a corrector step.

And the predictor step values are usually represented by over bar, the corrector step values are of course represented by the next time step value. So, once again you are going from the nth time step values to the predictor values which are indicated by an over bar and the over bar field taken to the corrector step which is the n + 1th time step. Incidentally, MacCormack does not use a mix of staggered and standard grid. It works on the standard grid alone.

But you need to notice that in the predictor step, when you are taking a difference of the fluxes, you are using the grid points i + 1 and i, while in the corrector step you are taking the grid points i and i – 1. So, you can imagine that waves which are traveling from right to the left will be better tackled in this step. Waves which are traveling from left to the right will be tackled better with the second step.

So, that way with the mixing and matching it manages to capture different kinds of waves which are possible in a given flow field through the change in the direction of the bias in the stencil. Now usually Lax Wendroff base solvers at artificial viscosity in some manner. Now, for example, the Richtmyer method, it uses constant coefficient second order artificial viscosity which is represented by a term like this or an expression like this.

So, as you can see, it is applied in the corrector step. So, this artificial viscosity would help tackle the nonlinear instabilities which may occur due to the discontinuities, which form in the flow field as computations go on, which is due to either shockwave or contact discontinuity.

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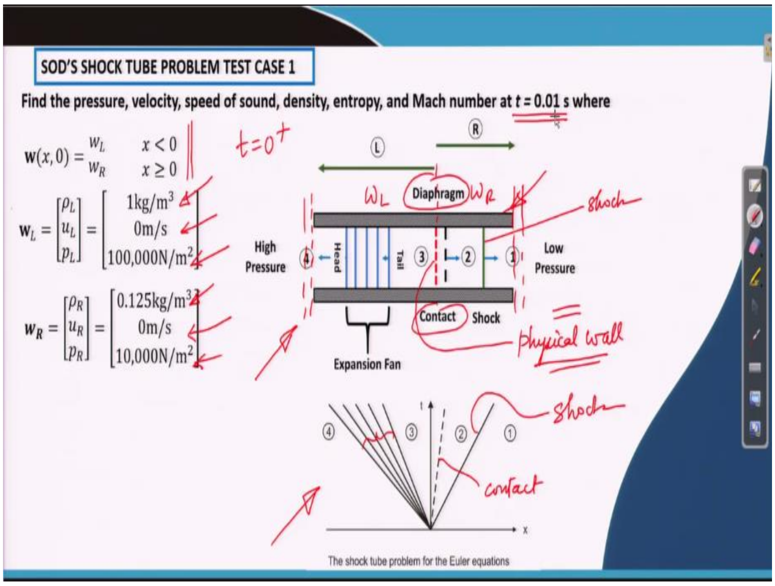

So, we have discussed a few methods, under the umbrella of flux approach. Remember that in the flux approach we never talked about the wave aspect of the problem. So, there is another set of schemes or approaches which can broadly be covered under the wave approach which we will discuss subsequently.

Before we do that, we introduce the shock tube problem where we have a long tube which is usually a thick metallic tube inside which we initially have a diaphragm that means a physical wall. It is usually not too thick a wall. It should be breakable by some means mechanical or electrical means at a point of time precisely where you would like it to break and then you try to look at how the shock tube operates.

Now, what we need to understand is that when the wall exists, then across the wall, what is that we have? So, we usually have two gases, or it could be the same gas but at different pressure or could be even different density, could be even different temperature. And then once we break the wall, we find how these two gases, interact with each other.

So, we will talk about one of the standard benchmark problems in the domain of shock tubes which is called as the SOD's shock tube problem test case 1 where what we have stated is that at time t = 0, we have two different states w L and w R existing across the physical wall. This is w L, this is w R, this is the wall, or the diaphragm. So, in the literature, it is more often referred as diaphragm which are separating the two states.

And in the left region we have a slight higher density than the right state. And the gas is stagnant, both in the left and right region because of the barrier. And the pressures is significantly larger on the left side compared to the right side. It is 10 times more. And you have to imagine that there are in a physical setup there have to be barriers at the end of the shock tube. You cannot have an infinitely long tube.

So, you prepare a length, which is suitable for your kind of work. And then once you are able to break the diaphragm at say t = 0+, you then see how the gases interact with each other. So, physical intuition will lead us to believe that the high pressure gas will rush into the low pressure gas, and that is precisely what happens. And then we can show that though initially the gases were stagnant, the moment they are allowed to interact, a very strong pressure wave populates into the low pressure region.

And that leads to the formation of a discontinuity, the shock. So, they are the normal shock which runs into the low pressure region, which is indicated by this green region. So, this is a shock, it is a normal shock and it is a moving shock in terms of the laboratory frame of reference.

So, if you are standing in the laboratory and you are looking at a shock tube equipments in front of you, then if you have some means by which you can either measure the movement of the shock or seat by some means you will see the shock running away towards the right till it reaches the end of the shock tube there. Of course, once it reaches the end it will reflect back and so on. But we are not going to go that far.

We just look at the initial picture after the diaphragm has been ruptured almost immediately after that. And then we find that as the shock runs away towards the right, it pulls with it the flow behind it. And somewhere in that flow, there is also a contact discontinuity which is running behind the shock. And this contact discontinuity usually can sustain certain jumps across it for example entropy can change across this contact discontinuity.

But it cannot have difference in pressure for example. And then beyond the contact discontinuity to the left end of the shock tube we will find an expansion fan running away, which is shown through this region. Of course, as you can understand all those lines are essentially characteristics and this dotted line is the contact. So, you have the physical picture in the upper part of the diagram, and the kind of wave diagram in the t x plane or x t plane in the below diagram to understand or explain what the shock tube is all about.

And we will try to see very soon that you know how pressure, velocity, speed of sound, density, entropy, Mach number, all these things would change over a very short time, even by t = 0.01 seconds after the diaphragm ruptures. So, we will discuss more about this in the next lecture. Thank you.