Introduction to CFD

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Lecture - 49
Numerical Solution of One Dimensional Euler Equation for Shock Tube Problem (continued)

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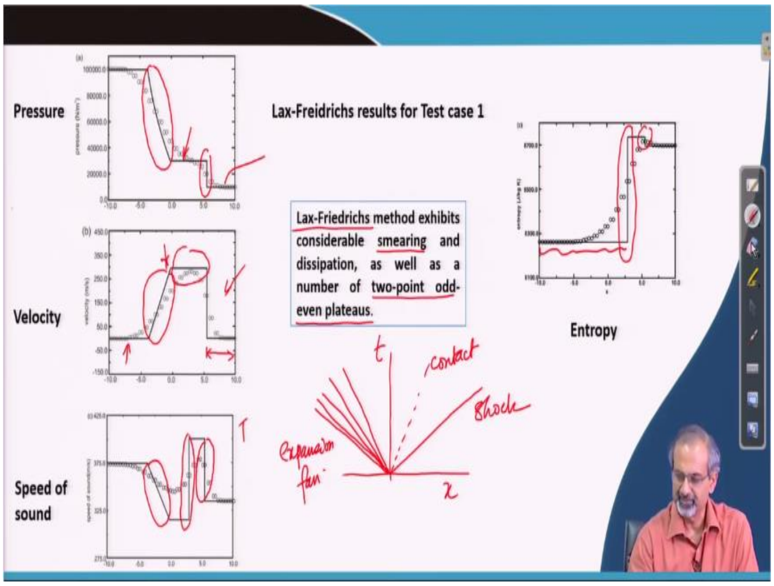

We now come to the last lecture on one dimensional Euler equations. So, in the previous lecture we had discussed briefly about the idea of shock tube and we know that we are looking at a specific problem which is called as the SOD’s shock tube problem test case 1. So, let us have a look at some of the results using the flux base schemes that we discussed in the previous lecture.

So, we first discuss about Lax-Friedrichs method and we see the different variations of different parameters like pressure velocity and so on. So, you may recall that in the shock tube problem after the diaphragm ruptures, there is a shock running towards the right. So, this is a shock, there is a contact surface behind it and there is an expansion fan developing on the other side.

So, this is how roughly the waves look like the xt plane, shock, contact, and expansion fan. Now, if you look at any of these plots, they are going to give you signatures of these different features. And remember that these are instantaneous pictures at a certain time which was fixed by the test case. So, you can see that there is a large pressure difference across the shock here.

This is the pressure in the low pressure region which has not been reached by the shock yet. And this is the region behind the shock. And this is the region over which the expansion fan is developing. The solid line is essentially the analytical solution to the problem while the dots are essentially the numbers coming from Lax-Friedrichs. Of course, there is smearing and there is this two point odd even plateaus which we are aware of and thereby the sharp corners have not been well captured.

It is also due to the first order accuracy of the solution. But incidentally, because of the over damping, there are no oscillations visible in the solution anywhere where those jump changes are occurring or slope discontinuities are occurring. So, if you look at velocity similarly, you will get more interpretation on the physics.

So, for example, in this region, the flow is still stagnant and this is the region which is being dragged by the shock and this is the expansion fan region where there is variation of velocities from stagnant here to the region which is complying with the velocity induced by the shock. So, thereby there these are kind of ramp type of behaviors of the different parameters. So, velocity is one pressure we have already seen.

And you will also notice the temperature would vary in the flow field like temperature would suddenly increase across a shock will gradually vary across an expansion fan and things like that. Alright, also try to explore why this jump is there is it due to the contact discontinuity. You already said that contact discontinuity can support entropy jumps. So, will that lead to temperature jumps?

Also, again, remember that wherever entropy is not changing, so, in a region like this, we have an isentropic flow field. That is why the entropy is not changing. So, is expansion fan going to be isentropic in nature? So, those are the question one has to ask. Again, notice that there is a sudden increase in entropy across the shockwave, which is known from the shockwave physics.

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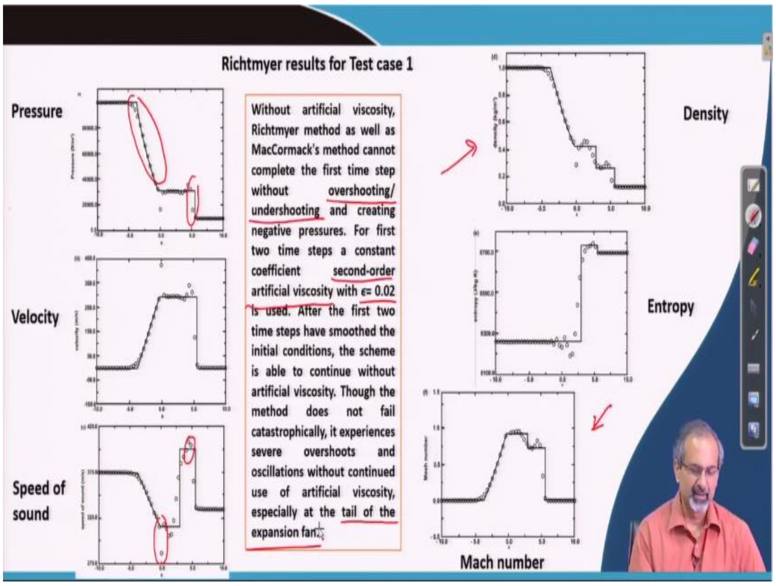

Let us look at the Richtmyer method results. We already understand what we expect in terms of pressure, velocity. Say, entropy, speed of sound, all these we had discussed earlier. Additionally, you will also find density plot here or a Mach number plot here. So, now, you should be in a better position to understand how the physics is working.

If you are not very well aware of the shock tube problem it is available in some of the standard texts on compressible flows from where the basic physics can be first understood before looking at these numerical results. Now, coming to the Richtmyer method because it is a variant of the Lax Wendroff, so, it is a second order method. You can see that it much better complies to sharp changes, much better complies to regions like this as well.

However, there are undershoots and overshoots like what we are very well aware in higher order methods. Of course, if you were to include a TVD kind of approach over here that may have done the job of damping out these oscillations. But without such tool, you would continue to face these problems of overshoots and undershoots. And it is found that you may not be able to even sustain the calculations right from the first time step until unless we include some amount of artificial viscosity.

So, these results are all done through incorporation of an epsilon 0.02 associated with the expression for artificial viscosity which was discussed in the context of Richtmyer method. So, only with that, it will sustain. Usually, these undershoots overshoots are particularly very critical, especially at the tail of the expansion fan where there are slope discontinuities.

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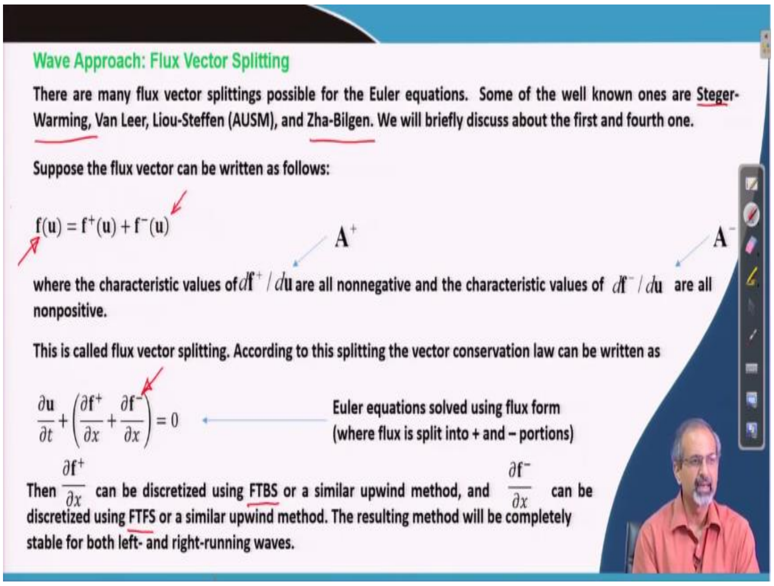

Now, we come to the second approach, we have already discussed the flux approach in the beginning. Now, this is the second approach the wave approach, which is also called as the flux vector splitting approach. There are numerous splitting techniques which are invoke which are in use today. And some of the earlier ones were Steger Warming, Van Leer. And there were quite complex formulations involved.

The recent ones are somewhat more simpler to implement. And we will discuss more on this Steger and Warming and Zha-Bilgen fluxes A in the next few minutes. Before we do that, let us first look at the concept of flux vector splitting. So, let us say we are talking about the flux vector f and we are splitting it into two parts f plus and f minus, how are we splitting it such that the characteristic values of df du plus are all non negative and characteristic values of df minus du are all non positive.

This is how we are splitting it. Now, if we do that if we can achieve that, then the system of equations may be expressed like this. And then the df plus dx or del f plus del x can be discretized using forward time backward space or a similar upwind method. And df del f minus del x can be discretized using forward time forward space or a similar upwind method because that would make the calculations robust according to the wave propagation direction.

Why is it that way? Because you have split the waves in that manner, you have segregated them in that manner that some of them are moving towards the positive direction, some others are moving towards the left in the negative direction and accordingly you are disposing your upwind scheme to capture that. So, inherently the calculations will be stable in that case.

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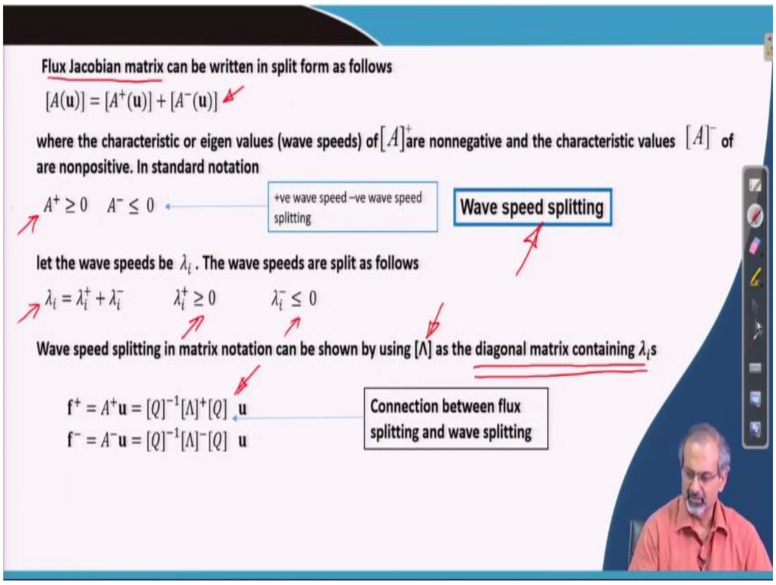

Now, let us take the concept a little further. So, we say that, let us express the Jacobian matrix in a split form like what we discussed even in terms of the flux vector we are trying to now apply it to the Jacobian matrix. And we say that the characteristic or the eigen values which are essentially the wave speeds of the Jacobian matrix in its split form should be such that the A plus carry all non negative wave speeds and A minus carry the non positive wave speeds.

So, we split them according. So, you can in a standard notation you can just represent it like this that A plus should be greater than or equal to 0, A minus should be less than or equal to 0. And what do these A plus and A minus going to contain? They are going to contain some lambdas essentially. Lambda based information will come from the A, is it not?

Because we found that you know on diagonalizing A, we had actually been able to extract that capital lambda vector matrix right and the diagonal entries of that capital lambda matrix were those small lambdas we have discussed it in the previous lecture. So, that way, we can actually look at A plus and A minus contributing to the positive and negative lambdas. So, if we put it that way, then the wave speeds can also be split like that.

We can have a general expression like this that all such lambdas can comprise off a positive part and the negative part. The positive part should always satisfy this. Negative part should always satisfy this. For certain cases, there will only be a positive contribution or only a negative contribution that can happen right. Now, wave speed splitting in matrix notation can again be shown through the capital lambda matrix **(()) (11:11)** and we remember what is the connection between A and capital lambda through the diagonalization concept.

Alright, so finally, it boils down to the informations that we are obtaining from the capital lambda matrix through the small lambdas which are essentially the wave speeds, right. And we are going to apply the splitting concept to the wave speed. So, this is what is called as the wave speed splitting concept. Now, what is the connection between flux vector or flux vector splitting and wave speed splitting? So, in order to connect them together you can very easily show that they can be obtained by an equation of this kind. We are not going into the derivation of this.

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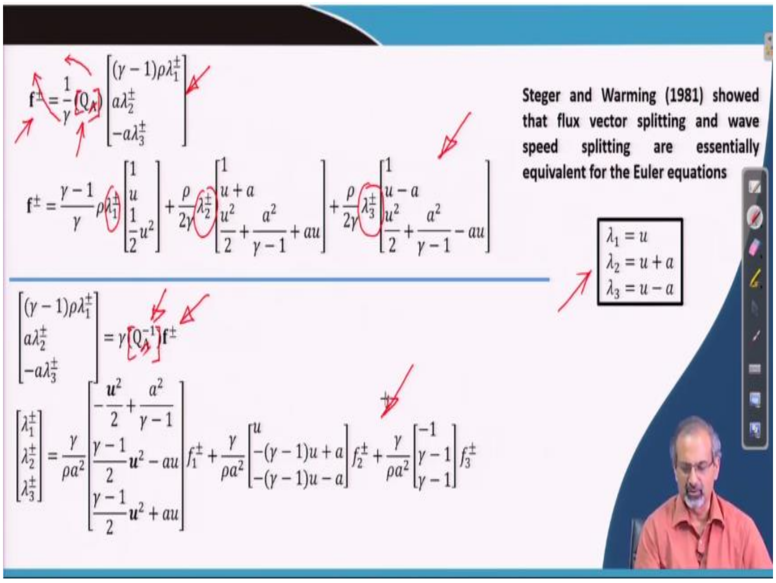

Now, Steger and Warming, they had showed that flux vector splitting and wave speed splitting are essentially equivalent as long as you are applying it to Euler equations. We know beforehand that for Euler equations, these are the lambdas that we have. Alright, now, the flux vector f in split form can be connected with the lambdas using an equation of this kind. This is merely Q.

And if you apply the different components of Q, then you can show that f plus can be represented in terms of the lambda 1 plus minus lambda 2 plus minus and lambda 3 plus minus in this form. Whereas you remember that the plus portions can always be positive greater than or equal to 0 and the negative portions have to be less than or equal to 0 that way.

Again, you can alternatively represent it represent the lambdas and f’s through an equation of this kind. You can very easily show that if you multiply f plus minus by gamma and take the Q matrix to the other side, you will be able to generate this equation. So, this is just changing the dependent and independent variables. So, that way whichever information is available to you, accordingly you use the particular equation or it also depends on the kind of formulation you are looking at.

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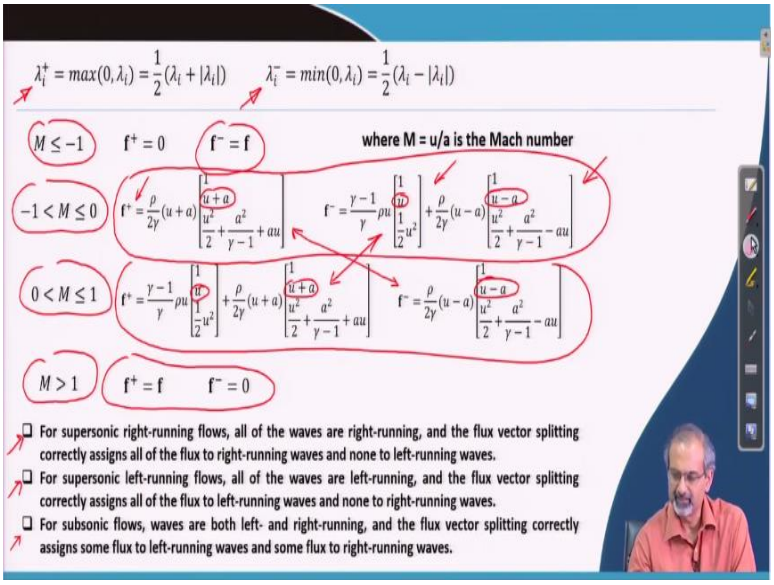

Now, in the Steger Warming approach, the splitting of wave speeds is done using max and min functions. And then for different ranges of Mach numbers, the flux splitting is expressed through the equations which we showed in the previous slide. So, as you can see that if Mach number is less than equal to –1 that means, of course, you can understand the flow is proceeding towards the negative x direction and it is supersonic in that direction.

Then all the contribution comes from f minus and f plus is 0. That means all the waves are swept towards the left. Right, and then if you have it in a subsonic form, but proceeding towards the left, then this is how the system works. But there is only one wave which sweeps towards the right. And therefore, you have one contribution in the f plus while there would be two components, which are going to sweep towards the negative direction.

And as you can understand going by these expressions, you can figure out why the respective portions of f are sweep are swept towards respective directions. So, if it is a subsonic flow moving towards the left obviously, u is negative u – a is negative, but u + a is going to be positive. Alright, so likewise you have similar expressions here you can understand that basically gets altered to this f plus part now, and this kind of goes to the other side.

But of course, you have to keep track of these elements in the vectors. Accordingly you have to understand which goes to the positive part which goes to the negative part. When it is supersonic and moving towards the right everything gets contributed to the f plus and nothing to f minus. So, finally, for supersonic right running flows, all the waves are right running and flux vector splitting correctly assigns all the flux to right running waves non to left running waves.

Similar thing happens for supersonic left running flows because everything goes to the negative part nothing to the positive part. And in subsonic flows, it will always be mixed. So, you have to accordingly decide that which contributions go to the positive part which contributions go to the negative part.

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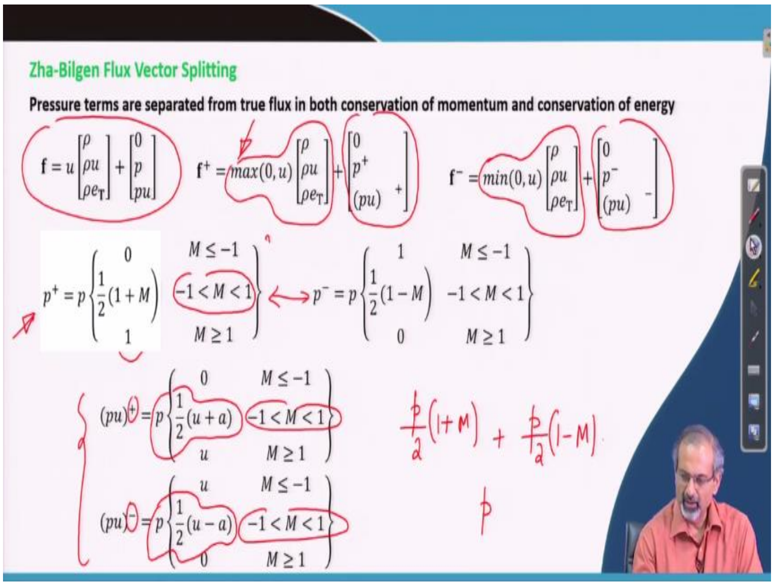

So, we come to one more flux vector splitting technique which is called the Zha Bilgen flux vector splitting, which is one of the most convenient to implement recent flux vector splitting technique, while Steger and Warming was one of the more earlier techniques. So, here the pressure terms are separated from the true flux terms in both the momentum as well as the energy conservation equations.

You may recall that we had talked about it in an earlier lecture that such splittings are possible and such splittings are done based on certain numerical conveniences associated with them. And thereby, you can see that the flux is also split that way. So, there is a pressure pot segregated out and you have a positive or negative associated with that, which has been provided here and the max and min operating on the other part.

So, the pressure part works out like this for the range of Mach numbers. So, the p plus has a smooth changeover in this region. If you look carefully that if you have a Mach number anywhere in the region –1 to 1, then the p plus provides you a relation p by 2 into 1+ m and p minus gives you p by 2 into 1 – m. So, if you just sum them together you retain p. So, such consistencies are always ensured when this splittings are done.

And when it comes to the product, p into u, then again you can split it into the positive and the negative parts. And then the main changeover occurs again in this range of Mach numbers. So, in this case, it is u + a. And in this case, it is u – a. So, this is how the splittings are done in this Zha Bilgen flux vector splitting. So, we will quickly look at 1 or 2 numerical results using the Zha Bilgen scheme in the next 1 or 2 slides.

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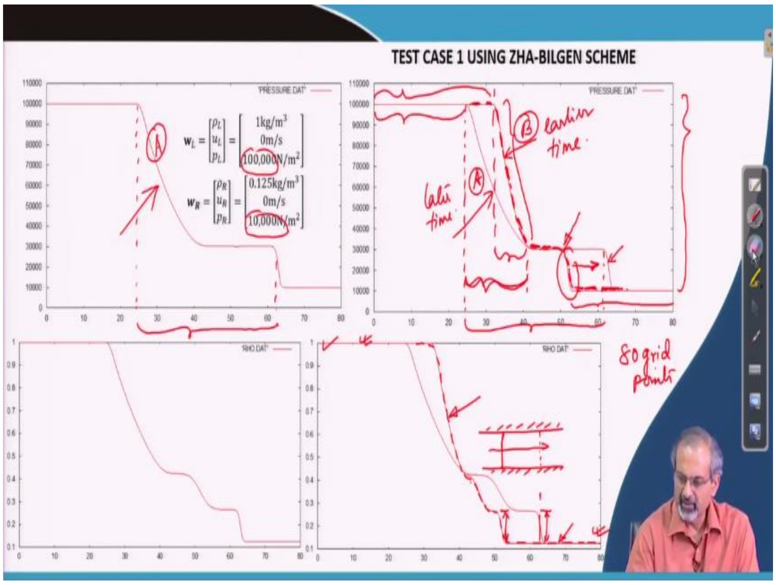

So, we continue to discuss about the test case 1 which it is the SOD's problem as we mentioned earlier. And this is from an in house simulation, where you can see that this is the pressure distribution at a certain instant of time and just to show how things change with time. If we call this curve as A, then you can see that the maximum pressure is somewhere here beyond 20. So, just to identify the same curve over here, it is essentially this, right.

And what we have done over here is we have super imposed the result coming from another time step so that you can see how things are changing with time. So, one of the curves is A which is actually at a later time, comparatively later time. And the curve B is it at a comparatively earlier time. Now we will try to understand how it works. Just to make things a little clearer, because we have incidentally used the same color here which is probably hindering the clarity a little bit.

So, we try to identify the two curves clearly. So, this is the region for the curve A and I am trying to dot the curve B separately. So, we can identify this dotted region it reflects an earlier time instant. So, what do we find? At an earlier time instant, we find that the shock has formed it has started moving towards the right across which we can see the jump. And then we have a larger undisturbed region ahead of the shock.

And then we see expansion fan developing over here. And we have an undisturbed region which is longer beyond the expansion fan on the left end of the shock tube. Now, as time progresses, the expansion fan moves further to the left. And now, the new location is represented by the curve A where the shock has moved further downstream towards the right. So, this is the shock location at the later time and then the expansion fan has grown wider.

So, this is the spread of the expansion fan the earlier spread of the expansion fan was this much. Now, we have a wider expansion fan, and because it has reached out further towards the left, the region beyond the expansion fan has become smaller. So, these are the changes which occur with time. As time progresses the shock tube problem as you keep the computations going for longer time periods, this is how the waveforms will emerge with time.

And as you can understand that it has been discretized with 80 say grid points along the x direction. It is a one dimensional problem, so you need only along x direction and you have the highest pressure as 100,000 Pascal’s the lowest as 10,000 Pascal’s. So, it is operating between these 2 pressure values. And the features are emerging as you compute and **(()) (24:02)** the solution in time using the Zha Bilgen flux vector splitting, which has been shown in this diagram.

So, like we did for pressure similarly, we have tried to show the density variations also over 2 time periods. So, now, I guess you will be a little more familiar to understand how to distinguish between an earlier time and a later time. Earlier time in the sense that it is only after a short time after the diaphragm has ruptured and the later time is more time has elapsed after the diaphragm has ruptured.

So, that way we are trying to explain earlier time and later time, right. So, very soon after the diaphragm ruptures, you can see the density variation may be something like this. Again, try to see that the highest density is of the order of say 1 kg per meter cube, lowest one is around one tenth of that. So, that remains in this undisturbed region ahead of the normal shock.

As you can understand that the normal shock is propagated propagating along this direction. So, this is a shock tube and ahead of the normal shock it is undisturbed region. So, the density will be undisturbed and at a low value. Across the shock, there is a large change of density. And then you see more changes as you go across contact surface and the expansion fan. Now, as more time elapses, the shock will move ahead, it will be somewhere here.

So, that is where you can see the foot of the shock now. As you can understand that the shock speed remains invariant that is why the jump of pressure or density across it also remains invariant. But in the intermediate regions, the changes are occurring. So, the highest and the lowest densities remain unaltered the highest and lowest pressures remain unaltered.

But you have variations occurring in between where the features are coming, the shock, the contact discontinuity and the expansion fan. So, across each feature, there is a change. And that is how the gap between the extreme values is breached. And this happens in the form of a time evolution. So, as you **(()) (26:36)** the solution, these time evolutions will occur gradually. This is of course, a very, very rapid event.

Because you remember that in the SOD's shock tube problem definition itself, it was a very small time that we talked about, after which we are looking at the solution. It is a fraction of one second a very small fraction of one sec. So, one second is actually a very long time in terms of shock tube operations. We will look at only one more plot before we finish the lecture.

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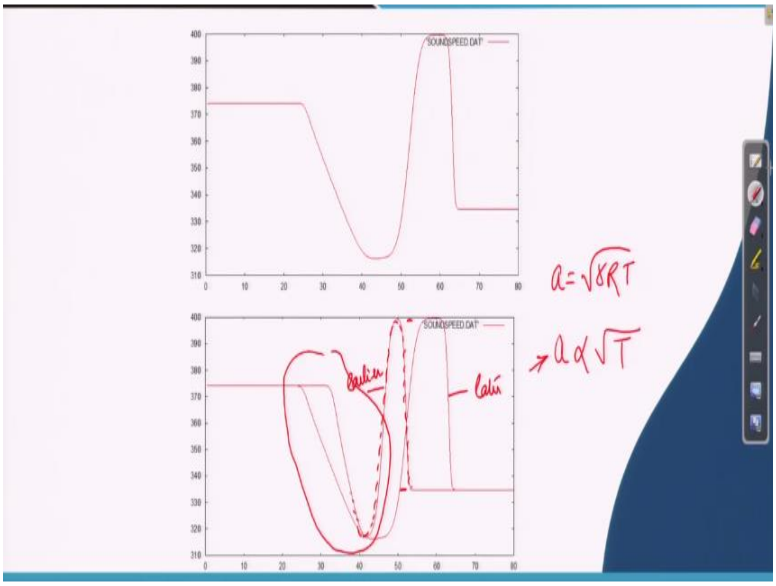

So, here we are talking about variation of sound speed. Again, you can figure out now that we have some more experience, we can figure out that this is the plot at earlier time. And this is the one which is at a later time. And if you remember that, you can define speed of sound as under root gamma R T that means the speed of sound is directly proportional to T.

So, directly proportional to under root T that means as temperature increases the speed of sound will increase and vice versa. So, across the shock, temperature changes, it increases, the static temperature jumps, the static temperature plus the dynamic part of course remains conserved that is the total temperature remains conserved. But the static jumps and that is what influences A and therefore, you see a large arise here.

So, that is why the rise across the shock. But then you will see variations in the other regions of the flow where you have to think for yourself, why those changes are occurring? What might happen across a contact surface? What might happen across an expansion fan? So, that way you can better explain how things are changing in this region for example. So, that I leave as a small homework problem for you to think over.

And with this, we would like to close our discussion on the shock tube problem where we have discussed at length about application of Euler equation. Thank you