Introduction to CFD
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Lecture – 63
Structured and Unstructured Grid Generation - Continued

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$\cancel{\Delta x}\ \cancel{\Delta y} \longrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad u = u(\xi,\eta)\ ,\ v = v(\xi,\eta).$

$$\frac{\partial u}{\partial x} = \xi_x u_\xi + \eta_x u_\eta$$

$$\frac{\partial v}{\partial y} = \xi_y v_\xi + \eta_y v_\eta.$$

$$\longrightarrow \xi_x u_\xi + \eta_x u_\eta + \xi_y v_\xi + \eta_y v_\eta = 0.$$

metrics of transformation

We continue our discussion on the calculation of matrix in the context of structured grid generation. So last time we were talking about transforming the continuity equation and we just introduced the concept of matrix of transformation. So let us expand on the idea of what the matrix are all about.

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Transformation derivatives or metrics of transformation — Metrics.

$$\xi_x = \frac{\partial \xi}{\partial x} \cong \frac{\Delta \xi}{\Delta x}$$

Represents the ratio of arc lengths in the computational space to that of the physical space

$$d\xi = \frac{\partial \xi}{\partial x}.dx + \frac{\partial \xi}{\partial y}.dy$$

$$d\eta = \frac{\partial \eta}{\partial x}.dx + \frac{\partial \eta}{\partial y}.dy.$$

So these are essentially transformation derivatives or matrix of transformation, often just referred as matrix and if we look at any one of those terms, then you can approximately write it as delta xi by delta x. So what does it represent? It represents the ratio of arc lengths in the computational space to that of the physical space. So this is a small arc length in computational space, this is a small arc length in physical space.

So we are talking about a ratio essentially and in the limit it becomes the derivative. Now, let us try to find out a differential change in xi how it will work out in terms of the matrix which are involved. So we can write it like this. Again, a differential change in eta can be written like this involving the respective matrix.

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$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad (A)$$

Reverse the role of dependent & independent variable

$$dx = \frac{\partial x}{\partial \xi} . d\xi + \frac{\partial x}{\partial \eta} . d\eta = x_\xi d\xi + x_\eta d\eta$$

$$dy = y_\xi . d\xi + y_\eta d\eta .$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} \quad (B)$$

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Once we do that, we can write it in matrix form. Now, you could also reverse the role of dependent and independent variables. What does that give you? Through this you can write expressions for differential changes in x, similarly differential change in y. We will write it using the simplified nomenclature. So from that, you can get another matrix form. So what do we have from these two? If we call these two equations as say A and B and if we put them together, we realize that if these two are multiplied, they are 2 by 2 metrices.

They will produce an identity matrix. Why is it so? Careful review of the two equations will clearly show that if you were to just substitute this equation on the right hand side of equation A, then you will get the product of this matrix and this matrix what we have written over here times dx dy, sorry times d Xi d eta and you already have d Xi d eta divided on the left hand side, which means that that product produces an identity matrix.

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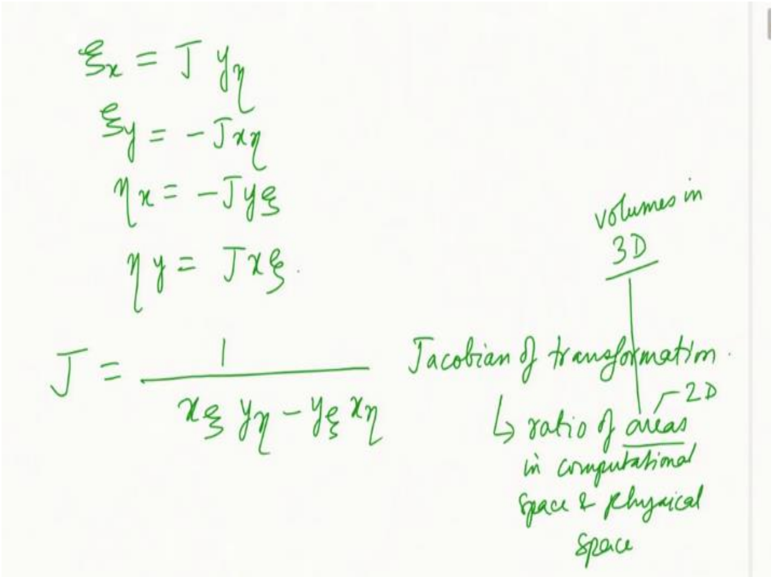

Now, from there we can obtain some relations which connect these matrix to matrix inversion, whereby a term J comes up which is called the Jacobian of transformation. Physically, how is it interpreted? It is a ratio of areas in computational domain or computational space and physical space. Incidentally, it is areas for a 2-dimensional problem it would become volume, that means a ratio of volumes for a 3D problem. So, matrix were comparing arc lengths, while Jacobians are scaling the areas in 2D or volumes in 3D.

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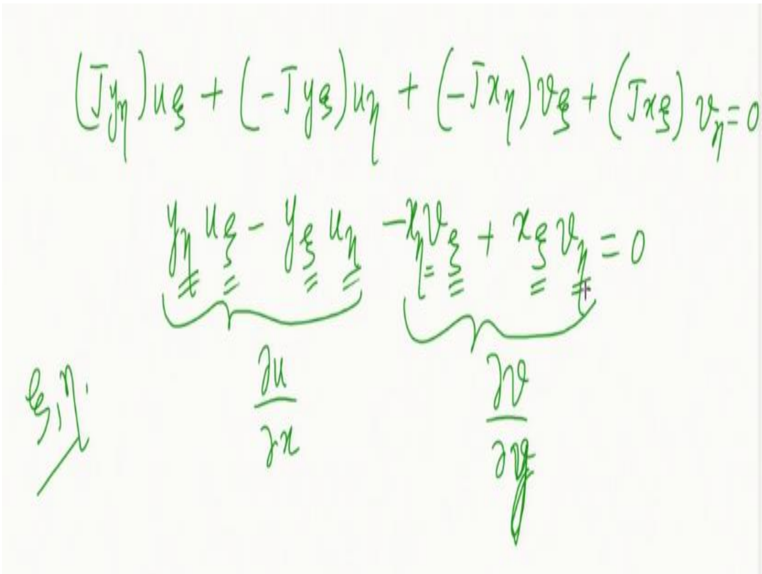

Now, it is apparent that if you revisit the continuity equation, the transformed continuity equation, you can now write it as like this. Once you cancel out J, you can clearly realize that this was contributed by the del u del x, this was contributed by del v del y. So in the transformed space, it looks much more complicated, but nevertheless in this form, you can compute all of the terms because all are differentiations with respect to either xi or eta each one of the terms, and therefore they can all be computed in the uniform grid that you have created in the computational space.

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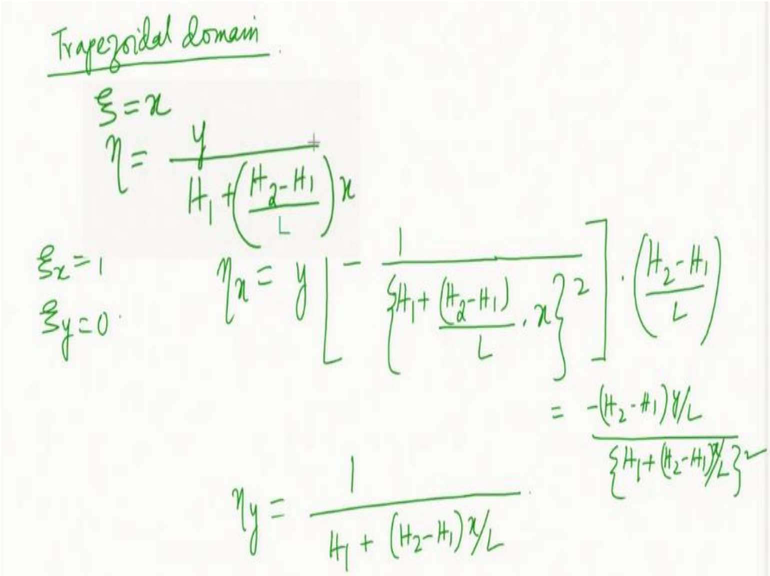

Now, if you look back at the trapezoidal domain problem that we had discussed in the previous lecture, so in that problem you remember that we had applied the transformation xi equal to x, eta is equal to y by the yt. So now you know that because these are analytical expressions, you can actually derive all the matrix, for example xi x will be 1, xi y will be 0. Similarly, eta x will be, so there will be x here like this, eta y will be like this.

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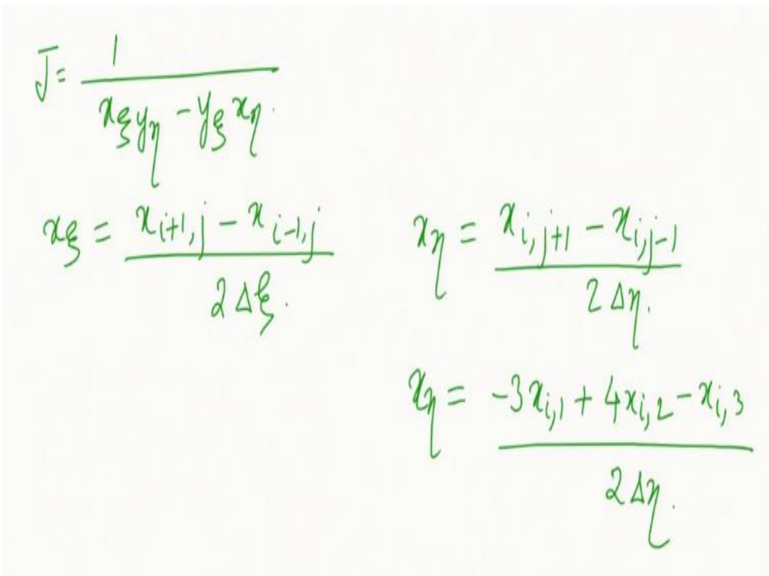

When it comes to the Jacobian, these can be computed numerically in the computational domain. So, say for example x xi can be calculated like this or x eta may be calculated like this and of course there are issues near the boundary. So, for example, if x eta has been calculated near the top or bottom boundary, you will actually have to bring in one-sided differences put it in the second order accuracy.

So this is how you would handle the trapezoidal domain problem in the transformed space using matrix and Jacobians because that is the generalized approach and we now understand both of the ideas.

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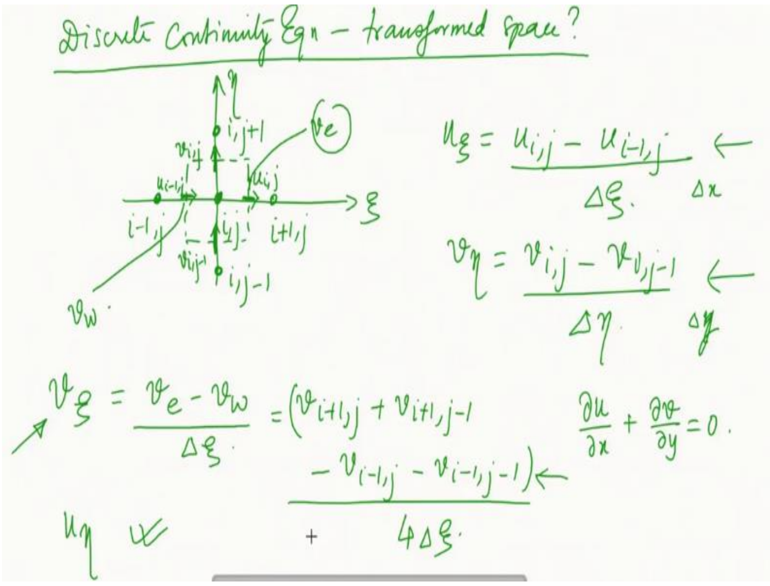

What about the discrete continuity equation in transform space? Because we already understood that transformation is not only for the grid but also for the governing equations, so when it is the computational plane, we are talking about points like this. Remember that these grid locations though they are in a regular manner over here, in the physical plane they are irregularly displaced, but nevertheless the mapping is identical.

The ij which we see in a computational domain exactly maps with the corresponding point in the physical plane where a particular xi line and a particular eta line has intersected. Now u xi remember you are handling velocity, so you have to be careful. If you are using a staggered mesh, you are doing it at the point ij means you are imagining that there is a scalar control volume here surrounding that point and the uij is here while vij is here.

So that is the staggered system that we have discussed about earlier, right. So when it comes to u xi, it is uij – ui–1j. It sits here and similarly if you were to do a say v eta, so it is vij – vij-1 by delta eta, right. How about v xi? So that will be like ve – vw by delta xi where ve has to be defined here, vw has to be defined here. These were not there if you remember when we were dealing with continuity equation on a Cartesian mesh in the physical domain.

Things were much simpler because we just dealt with del u del x and del v del y. If that was the case, you could have actually gone out and done something like this, but only difference was that this would be delta x and this would be delta y on a Cartesian mesh in physical domain, but now that you are in the computational domain, you also need to handle terms like this v xi or u eta which were not there in the physical domain.

So when it is v xi for example, how do we go about doing it? We will get an expression of this kind. I leave it as a small homework problem for you to figure out how we are doing this. It is of course based on interpolation to get an appropriate expression for ve and vw based on neighboring values of v which are available and then you will find some terms will cancel out to finally give you an expression like this. You can similarly work out an expression for u eta. So in this manner, the transformations can be done.

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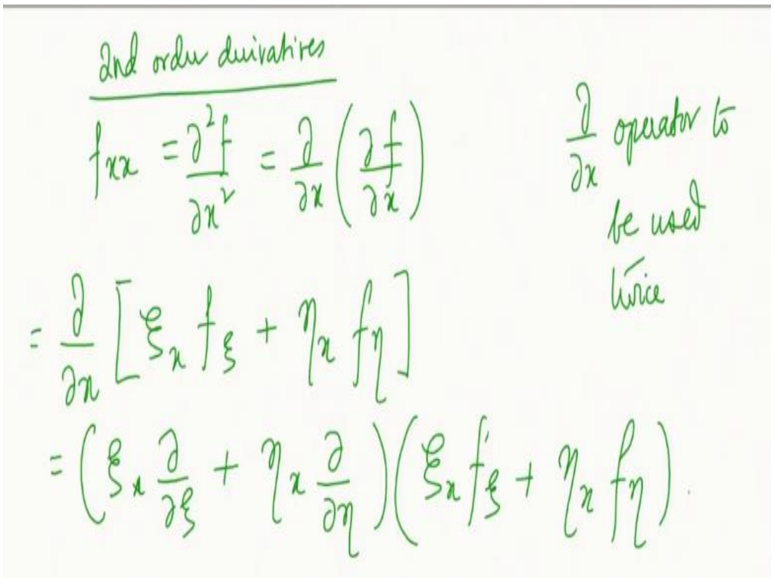

Now, if you remember that in Navier-Stokes equation, you would actually have to deal with second order derivatives coming from the viscous term which makes it a little more involved than what we have already dealt with. So let us try to find out how we will tackle that kind of problem. So let us see if you are trying to find out fxx that means del 2f del x square. So we use the del del x operator to be used twice on the function f.

So we first work out an expression for del f del x and then apply the del del x on it once more. So del f del x can be written as this. So it is like you are applying the operator del del x on del f del x. So what will that give you?

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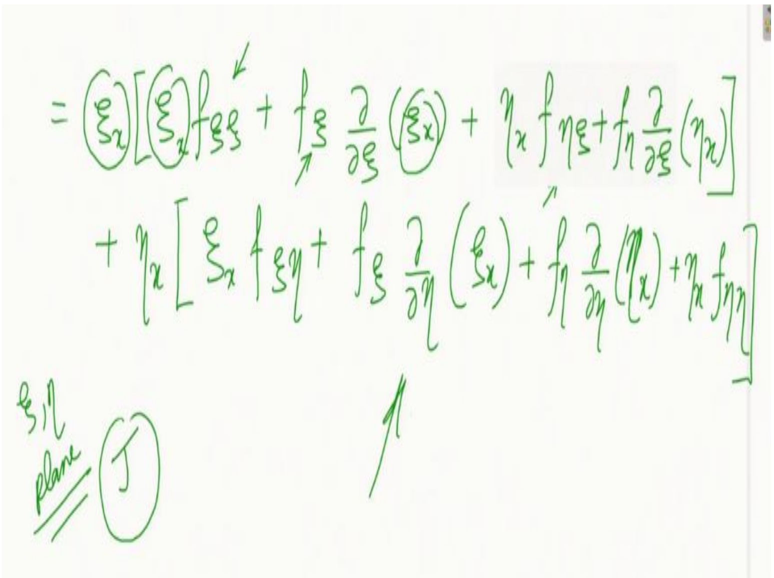

If you try to work it out, it will give you so this will be xi x, this will be eta x. Now, we have taken it to a certain level where you find that many of these derivatives are with respect to either xi or eta, but again many of them are not. So terms like this will have to be further worked out so that you can bring all such derivatives, all such terms to the xi eta plane and how would you do it? You would have to involve the Jacobian based expressions whereby all derivatives with respect to x and y get transferred to xi and eta.

So only then, you will get an expression for the second derivative, which can be entirely computed in the uniform computational domain. So we have taken it up to a certain level. You can do the rest as a small homework problem, which will take some more time, but it can be easily done by being in the Jacobian expressions.

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Algebraic functions $\rightarrow$ grid clustering / grid stretching.

logarithmic or hyperbolic functions can also be used.

$$\xi = x$$

$$\eta = 1 - \frac{\ln\left\{\left[\beta + 1 - (y/H)\right] / \left[\beta - 1 + (y/H)\right]\right\}}{\ln\left[(\beta+1)/(\beta-1)\right]}.$$

parameter which can be tuned.

β - clustering parameter

$1 < \beta < \infty$. very large finite value

$\beta \rightarrow 1$ more grid points get clustered near the $y=0$ boundary.

Earlier, we were talking about using algebraic functions for grid clustering or grid stretching. There are other means of doing that also. You can use logarithmic or hyperbolic functions can also be used for grid clustering. For example, if you have a transformation xi is equal to x and eta is equal to 1 minus natural log. So you can see that there is a ratio y by H, which means that the total height of the domain is H, y is any location of a horizontal grid line and beta is a parameter which you can tune.

If you tune that parameter, you will get different extent of clustering along the y direction. Notice that there is no clustering along the x direction because xi is equal to x that gives you uniform mesh along x. So this is typically going to produce a grid clustering which is suitable for wall bounded flow and wall is essentially horizontal. Now, this beta parameter is often called as a clustering parameter and it can be within a range like this.

Infinity in the sense a very large value in practical calculations. So as beta tends to 1, more grid points get clustered near the y = 0 boundary. So this is another means by which you can do grid clustering.

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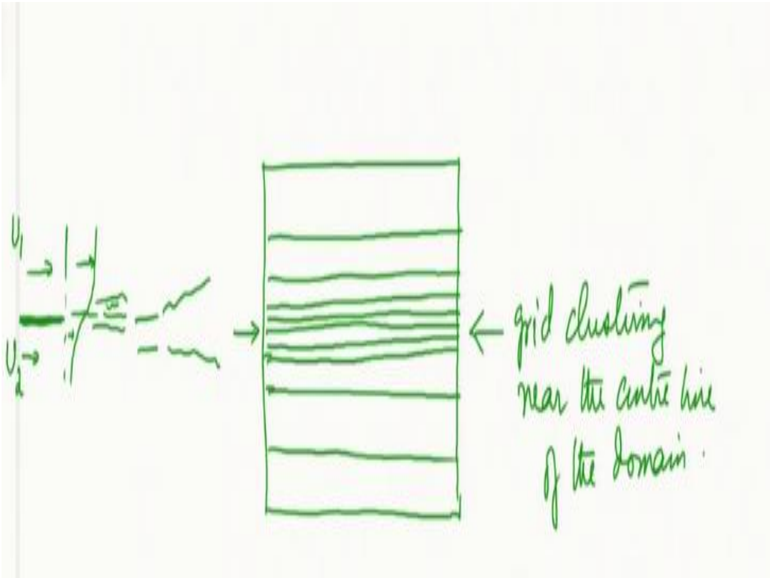

Also in certain problems, the grid clustering may actually occur not at any one of the boundaries of the domain, but maybe in some other intermediate region of the domain. For example, you may like to have highly clustered mesh in the central portion of the domain for certain problems. Let us say you are trying to solve a mixing layer problem or an open jet, so most of the activities are happening in the central region here.

So either you have a flow emanating from a jet or you have a small thin boundary across which the velocities, the tangential velocities are different. So when you allow these two velocities to leave the surface and go into the flow, there will be a mixing of these two regions and thereby there would be formation of a mixing layer downstream. Now these kinds of problems would have to be handled with grid clustering near the central line of the domain.

So we can understand that it is very much problem specific, we do not have a generic definition as to how you should go about doing the grid clustering. We have to understand the geometry of the problem, we have to understand what physics we are trying to capture from the problem and accordingly we try to deploy the mesh for the problem. So with this, we finish this lecture. Thank you.