Flight Dynamics II (Stability)

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Module No. # 09

Perturbed (Linear) Aircraft Model

Lecture No. # 27

Small Perturbation Method, Linearization of Equations

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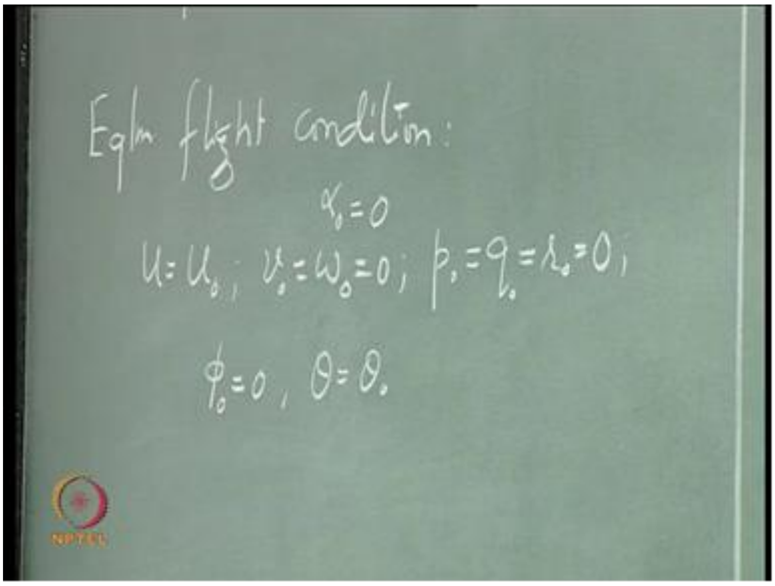

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Derivation of perturbed equations of motion. If you remember we said to look at dynamic stability of aircraft, at any particular equilibrium condition, what we need to do is, we need to linearise the equations of motion, and then, study the dynamics of the perturbed variables. See if the perturbed variables are decaying in time or not. If they all decay in time, then we will say that the aircraft is stable in that particular equilibrium condition.

So, first of all, you need to define one equilibrium flight condition (or steady-state obtained by setting time derivatives of states to zero in Equations of motion). So, in this case we will take flight condition: $u = u_0 \neq 0, v_0 = w_0 = 0; \alpha_0 = 0; p_0 = q_0 = r_0 = 0; \phi_0 = 0; \theta = \theta_0$

equilibrium flight condition, you know, this is only one such flight conditions; you will have many, so I am taking one as a reference flight condition and trying to derive these equations.

Let us say aircraft is flying with the speed, u equal to u naught. So, aircraft has non zero velocity only in the forward direction, other two velocities are 0, all rates are 0, wing level condition phi naught, bank angle is 0 in this reference flying condition and let us say theta is some theta naught. From here (slide below) you see that the angle of attack at this equilibrium flying condition is 0 right, because w naught is 0. This also means that alpha naught is 0. So, you understand what kind of flying condition it is? Theta is non 0, alpha is 0. So, this flying condition is actually describing, describing a longitudinal flying condition. And what all flying condition you know in longitudinal plane? Climb, landing and cruising.

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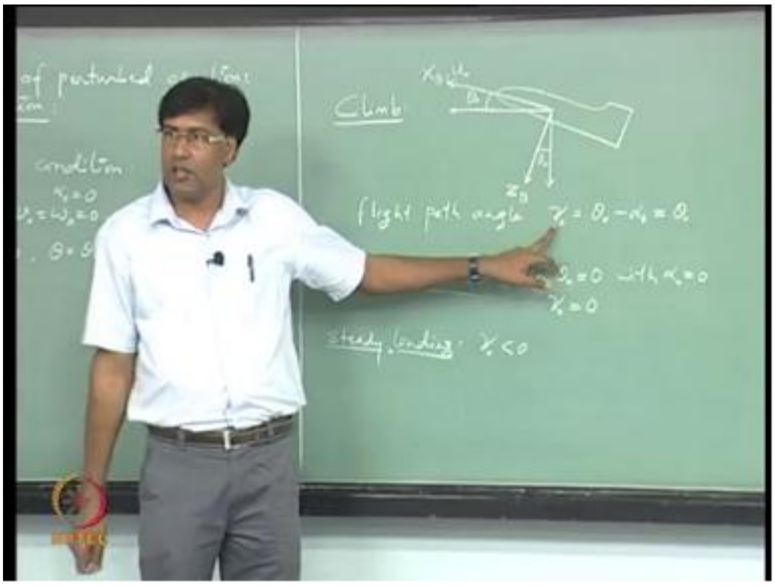

So, let us look at those flying conditions. So, one is climb.

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So, from the local horizon, this axis X_B is making an angle which is theta naught. I have taken this theta to be theta naught, which is the reference flying condition in the longitudinal plane. This angle is theta naught and alpha naught is 0. u, v, w are all along body fixed axes. You will have u naught in this direction, w naught is 0. So, aircraft is actually flying along its nose but at an angle theta naught with respect to the local horizon, so it is climbing.

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In this case, your flight path angle and, that is, the steady flight path angle, because we are talking about a steady state condition, equilibrium flying condition, is nothing but theta naught minus alpha naught which is theta naught, because alpha naught is 0. If we have theta naught also 0, then gamma naught becomes 0 and then you have a cruise condition or level cruise condition.

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Landing is when this flight path angle is negative. So, theta naught has to be negative for us because I am assuming this alpha naught to be 0 ($\gamma = \theta - \alpha$). So, we are looking at these three cases when I am assuming theta naught to be non-zero. You can also have alpha naught which is non zero and look at what is this quantity, theta naught minus alpha naught, to get one of these conditions, condition in climb is that gamma naught should be positive and that can happen when alpha naught is positive, but smaller than theta naught, which is positive. So, let us look at this (our Equilibrium flight) condition and try to linearise our non-linear aircraft equations of motion around this equilibrium flying condition. So, I will go one-by-one. So, we will start linearising the force equations first. The first in that list is $\dot{u}$ equation.

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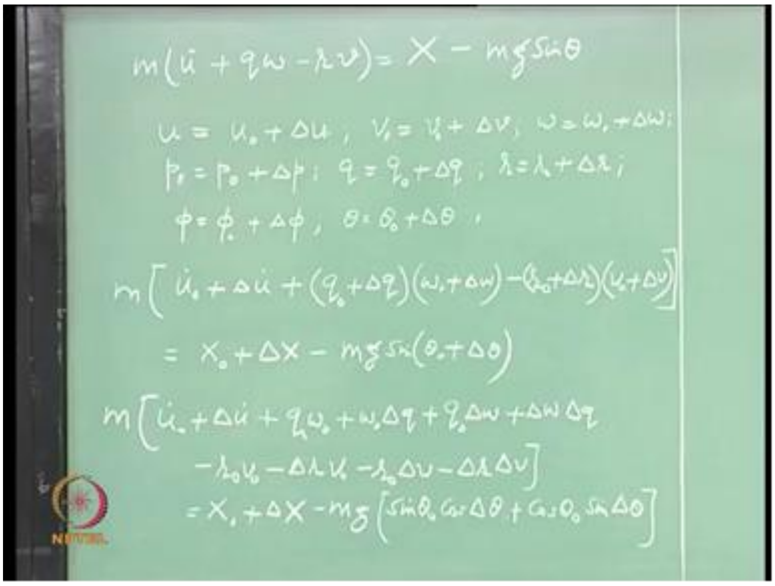

$$m(\dot{u} + qw - rv) = X - mg\sin\theta \quad (1)$$

$$u = u_0 + \Delta u, v = v_0 + \Delta v, w = w_0 + \Delta w, p = p_0 + \Delta p, q = q_0 + \Delta q, r = r_0 + \Delta r,$$
$$\phi = \phi_0 + \Delta\phi, \theta = \theta_0 + \Delta\theta \quad (2)$$

Now, let us take perturbations from these equilibrium conditions (Refer Eq(2)). So, I am now writing this u as u naught plus delta u.

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Where all delta quantities are perturbations in the variables. So, this equation (Refer Eq(1)) is for the aircraft motion along the X-axis, and is true for any condition, whether it is a perturbed condition or the equilibrium flying condition. Is it not? This is the equation of motion, you know, for the aircraft motion in any condition.

Now, what we are trying to do is, we are trying to derive the perturbed equations of motion. So, I am going to substitute for u, u naught plus delta u, the perturbed condition from the equilibrium flying condition. Let us do that and see what we get.

$$m[\dot{u}_0 + \Delta\dot{u} + (q_0 + \Delta q)(w_0 + \Delta w) - (r_0 + \Delta r)(v_0 + \Delta v)] = (X_0 + \Delta X) - mg\sin(\theta_0 + \Delta\theta) \quad (3)$$

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The perturbed motion is because of this change in force. From somewhere we are getting input, so that we are getting into the perturbed condition. So, we will have to see all that. We are right now looking, this is the force, which is going to give me disturbed motion. We will see where this force is coming from. ..... And now, let us try to write this equation in the expanded form.

$$m\left[\dot{u}_0 + \Delta\dot{u} + q_0 w_0 + \Delta q w_0 + q_0 \Delta w + \overbrace{\Delta q \Delta w}^{small} - r_0 v_0 - \Delta r v_0 - r_0 \Delta v - \overbrace{\Delta r \Delta v}^{small}\right]$$
$$= (X_0 + \Delta X) - mg(\sin\theta_0 \underbrace{\cos\Delta\theta}_{\approx 1} + \cos\theta_0 \underbrace{\sin\Delta\theta}_{\approx \Delta\theta}) \quad (4)$$

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So, let us also make another assumption and that is let us say these perturbations are small.

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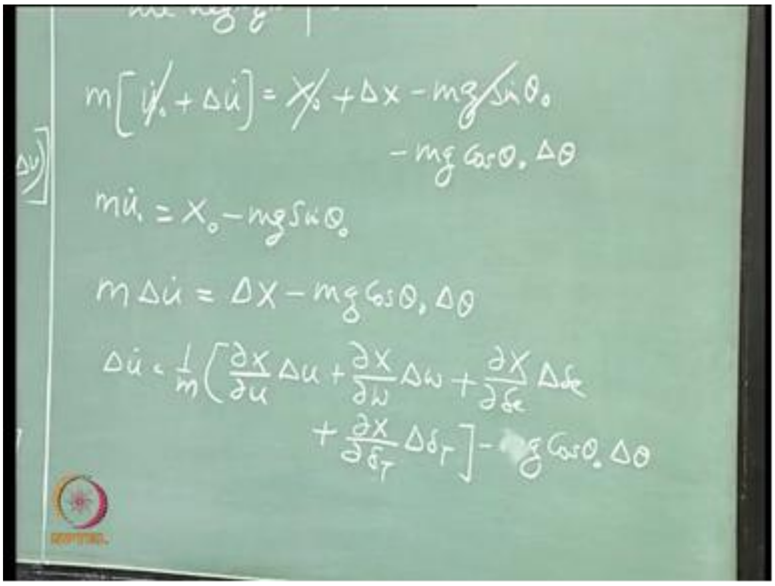

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(Refer Eq(4)) When we say that, this is 1, sin delta theta is delta theta, and some of these terms will disappear, when we are multiplying two variables which are small, then they are negligible as compared to other terms. So, we can drop them. So, this is small and that is why we are dropping it.

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So, that is one thing. Now, look at the equilibrium flying condition and see if we can drop further terms from here. So, this is 0, because of the equilibrium flying condition that we have taken.

$$\left.\begin{aligned} m\left[\dot{u}_0 + \Delta\dot{u} + \overbrace{q_0 w_0}^{0} + \Delta q \overbrace{w_0}^{0} + \overbrace{q_0}^{0}\Delta w - \overbrace{r_0 v_0}^{0} - \Delta r \overbrace{v_0}^{0} - \overbrace{r_0}^{0}\Delta v\right] \\ = (X_0 + \Delta X) - mg(\sin\theta_0 \underbrace{\cos\Delta\theta}_{\approx 1} + \cos\theta_0 \underbrace{\sin\Delta\theta}_{\approx \Delta\theta}) \end{aligned}\right\rangle \tag{5}$$

$$m[\dot{u}_0 + \Delta\dot{u}] = \underbrace{(X_0 - mg\sin\theta_0)}_{=\dot{u}_0} + \Delta X - mg\cos\theta_0\Delta\theta \Rightarrow m\Delta\dot{u} = \Delta X - mg\cos\theta_0\Delta\theta$$

(No audio from 18:13 to 18:29) So, what we have finally is this ....(Refer Eq(5)).

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So, let us see now. Let us compare this equation with this. Now, this equation is satisfied at any condition (Refer (Eq(1)). So, look at the equilibrium condition when you have m u naught dot equal to X naught minus m g sin theta naught. Other variables are 0. You could have kept them here and still cancel them from both the sides. Is it not? So, this is also satisfied.

So, we can drop further these three terms from this equation. So, finally what you get is this (Refer Eq(5)).

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Now, let us try to expand this (ΔX), that is going to be, this force is going to be a function of velocity as you said and also of the control inputs. So, let us try to expand this now. ....Refer Eq(6)

$$\Delta \dot{u} = \frac{1}{m}[\Delta X] - g\cos\theta_0 \Delta\theta = \frac{1}{m}\left[\frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta e}\Delta\delta e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T\right] - g\cos\theta_0\Delta\theta \qquad (6)$$

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Anything else? In general, if you are talking about the post-stall flight region, then these forces are also going to be functions of many other non-longitudinal variables, but we are talking about pre-stall flights. So, there this force is going to be, this force is the sum of aerodynamic and propulsive forces along the body fixed X-axis. So, it is not going to be a function of the non-longitudinal variables in the pre-stall flight regime.

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So, these are perturbations in control inputs ($\Delta\delta e$ $\Delta\delta_T$) from the reference condition. This ($\Delta\delta_T$) is the perturbation in the throttle from the reference flying condition, perturbation in the elevator from the reference condition ($\Delta\delta e$), and these derivatives have to be evaluated at the equilibrium flying condition dou X by dou v, that is what I told you. We are looking at pre-stall flight where X will not depend upon the side velocity. If it affects, you should include that. You can have another term here.

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We will assume that this force is not being affected by the side velocity, perturbation in side velocity in the pre-stall flight regime.

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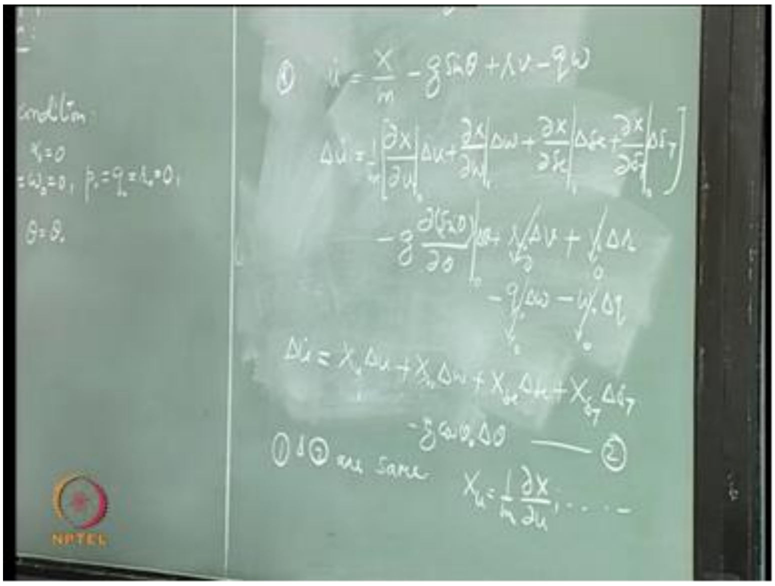

$$\Delta\dot{u} = \underbrace{\frac{1}{m}\frac{\partial X}{\partial u}\bigg|_0}_{X_u}\Delta u + \underbrace{\frac{1}{m}\frac{\partial X}{\partial w}\bigg|_0}_{X_w}\Delta w + \underbrace{\frac{1}{m}\frac{\partial X}{\partial \delta e}\bigg|_0}_{X_{\delta e}}\Delta\delta e + \underbrace{\frac{1}{m}\frac{\partial X}{\partial \delta_T}\bigg|_0}_{X_{\delta T}}\Delta\delta_T - g\cos\theta_0\Delta\theta \qquad (7)$$

$$\Delta\dot{u} = X_u\Delta u + X_w\Delta w + X_{\delta e}\Delta\delta e + X_{\delta T}\Delta\delta_T - g\cos\theta_0\Delta\theta$$

This (Eq(6,7)) is the first perturbed equation along the body fixed X-axis. So, we have to repeat this exercise, you know, for all the equations. There is another way you can do this. Use Taylor series expansion. So, let us see how we can do that. So, I am re-writing this equation as .....

$$\underline{\dot{x}} = \underline{f}(\underline{x},\underline{U}); \quad \underline{x} = [u\ v\ w\ p\ q\ r\ \phi\ \theta]'; \underline{U} = [\delta e\ \ \delta T\ \ \delta a\ \ \delta r]'$$

$$\underline{x} = \underline{x}_0 + \Delta\underline{x}, \underline{U} = \underline{U}_0 + \Delta\underline{U}$$

$$\underline{\dot{x}}_0 + \Delta\underline{\dot{x}} = \underline{f}(\underline{x}_0 + \Delta\underline{x}, \underline{U}_0 + \Delta\underline{U}) = \underbrace{\underline{f}(\underline{x}_0, \underline{U}_0)}_{\underline{\dot{x}}_0} + \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_0 \Delta\underline{x} + \frac{\partial \underline{f}}{\partial \underline{U}}\bigg|_0 \Delta\underline{U} + \underbrace{Higher Order Terms}_{negligible} \quad (8)$$

$$\Delta\underline{\dot{x}} = \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_0 \Delta\underline{x} + \frac{\partial \underline{f}}{\partial \underline{U}}\bigg|_0 \Delta\underline{U}$$

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You know this? So if you have a function which is $\underline{x}$ dot equal to f 1, and this f 1 is a function of several other variables, then this delta $\underline{x}$ dot is .... and so on.

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And you have to evaluate these derivatives at the equilibrium condition. So, my equilibrium condition is denoted by this subscript '0'.

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$$\dot{u} = \frac{\frac{X}{m} - g\sin\theta - (qw - rv)}{f1}$$

$$\Rightarrow \Delta\dot{u} = \left.\frac{\partial f_1}{\partial u}\right|_0 \Delta u + \left.\frac{\partial f_1}{\partial w}\right|_0 \Delta w + \left.\frac{\partial f_1}{\partial q}\right|_0 \Delta q + \left.\frac{\partial f_1}{\partial v}\right|_0 \Delta v + \left.\frac{\partial f_1}{\partial r}\right|_0 \Delta r + \left.\frac{\partial f_1}{\partial \theta}\right|_0 \Delta\theta + \left.\frac{\partial f_1}{\partial \delta e}\right|_0 \Delta\delta e + \left.\frac{\partial f_1}{\partial \delta_T}\right|_0 \Delta\delta_T$$

$$\Rightarrow \Delta\dot{u} = X_u \Delta u + X_w \Delta w + X_{\delta e}\Delta\delta e + X_{\delta T}\Delta\delta_T - g\cos\theta_0 \Delta\theta \tag{9}$$

So, this is cos theta naught, and we can drop these terms, because their values at the equilibrium flying condition is 0. So, what you get is the same equation as Eq(7).

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So, 1 and 2 are same.

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Is this alright? So, if you have understood this, we can go on writing other perturbed equations.

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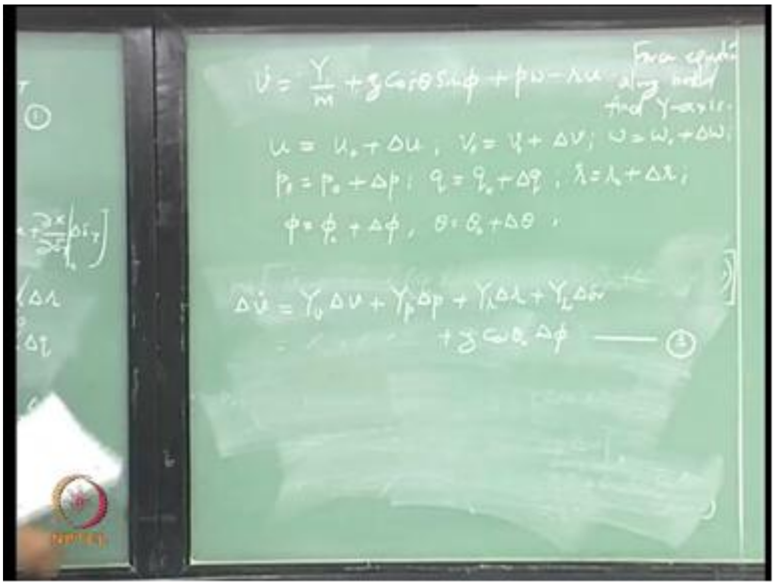

So, this is the force equation in $\dot{v}$.....

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And now, you can repeat this procedure and we can .... write down the final form.

$$\Delta\dot{v} = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + +Y_{\delta r} \Delta \delta r + g \cos\theta_0 \Delta\phi; \quad Y_{(.)} = \frac{1}{m} \frac{\partial Y}{\partial (.)}\bigg|_0 \tag{10}$$

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Again, we are assuming here that the side force is not changed much because of the aileron deflection. We have to look at the, the order of magnitude, and then if the order of magnitude is large, then you have to keep that particular term. If it is very small as compared to other terms, then you can drop them.

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Remember, whatever this derivative may be, this has to be evaluated at the equilibrium condition. Is it not?

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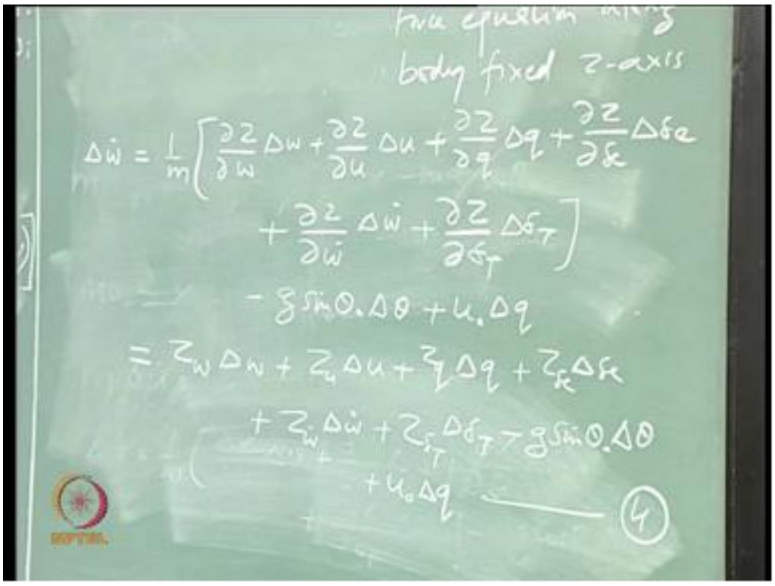

The force in the Z direction is also a function of the pitch rate and this quantity is not small as compared to other terms in the equation. That is why we have to retain this also (Refer Eq(11)).

$$\Delta\dot{w} = Z_u \Delta u + Z_w \Delta w + Z_q \Delta q + Z_{\dot{w}} \Delta\dot{w} + Z_{\delta e} \Delta\delta e + Z_{\delta T} \Delta\delta_T - g\cos\theta_0 \Delta\theta + u_0 \Delta q \qquad (11)$$

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This force can also change because of the acceleration in the Z direction and this term is may not be small.

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Any question here? (()) How do you get this?

(No audio from 37:00 to 37:41)

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So, let me just explain you what you, giving a general example, the way we started with. So, you have the equations written in this form where $\underline{x}$ is the vector of state variables in our case u, v, w, p, q, r, phi and theta.

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Now, what I am saying is, we have to keep the derivatives which are not insignificant. You can have w dot also as a variable. In that case, we can write it in this form (Refer Eq(11)).

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Let us say I say that this delta w dot is the normal acceleration and that normal acceleration is a state variable. I am only trying to expand this force. This force can be a function of these variables. It can also be a function of this normal acceleration. (()) We will come to that. We will look at these derivatives. The important derivatives we will look at them separately.

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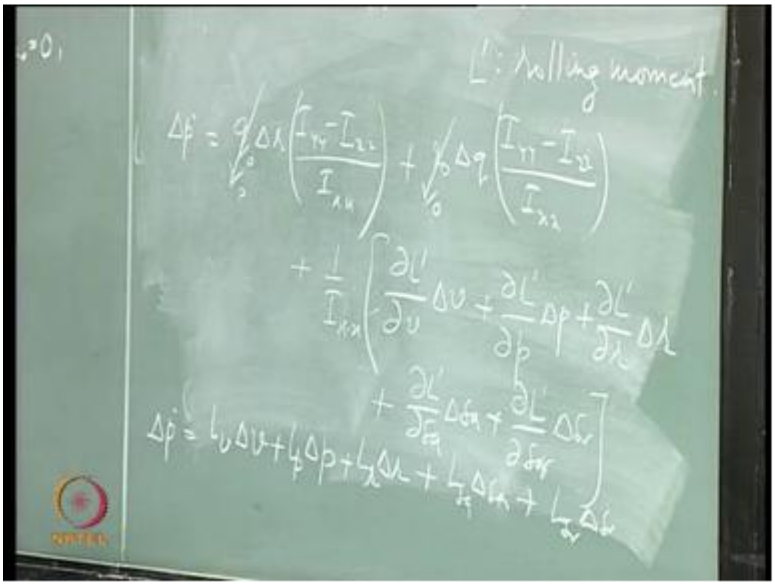

See, you think I should carry out this?

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Now, let us look at the moment equations.

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$$\Delta\dot{p} = \left(\frac{I_{yy} - I_{zz}}{I_{xx}}\right)(q_0 \Delta r + r_0 \Delta q) + \frac{1}{I_{xx}} \Delta L'$$

$$= \left(\frac{I_{yy} - I_{zz}}{I_{xx}}\right)(\underbrace{q_0}_{=0} \Delta r + \underbrace{r_0}_{=0} \Delta q) + \frac{1}{I_{xx}} \left(\left.\frac{\partial L'}{\partial v}\right|_0 \Delta v + \left.\frac{\partial L'}{\partial p}\right|_0 \Delta p + \left.\frac{\partial L'}{\partial r}\right|_0 \Delta r + \left.\frac{\partial L'}{\partial \delta r}\right|_0 \Delta \delta r + \left.\frac{\partial L'}{\partial \delta a}\right|_0 \Delta \delta a \right)$$

$$\Rightarrow \Delta\dot{p} = L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta a} \Delta \delta a + L_{\delta r} \Delta \delta r$$

(12)

This moment is a function of sideslip, rolling moment we are talking about. It is going to be a function of the side velocity or sideslip angle, roll, rate, yaw rate, aileron deflection, rudder deflection. So, this can be expanded now (Refer Eq(12)).

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It can be expanded as this. So, here the q naught is 0, r naught is also 0, so delta p dot is .... .

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Is this all right? (())

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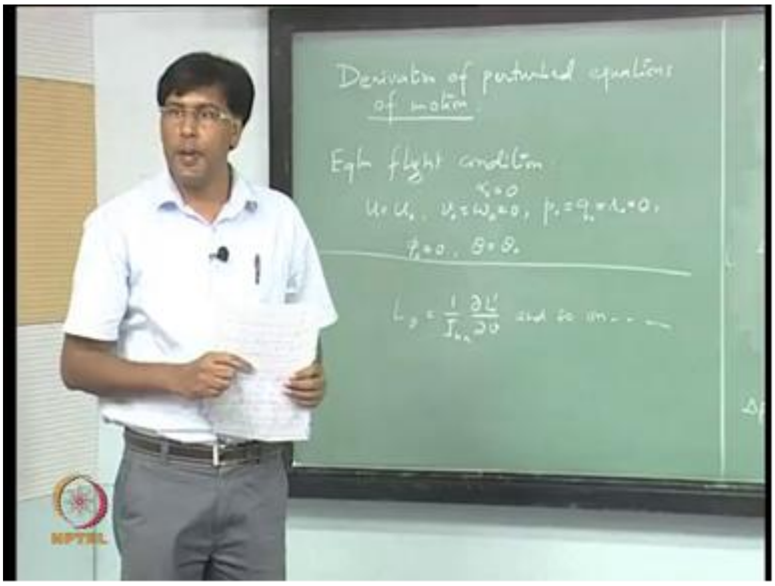

Here (Refer Eq(12)), this is L delta r, delta delta r. Remember, we are talking about perturbations from the equilibrium condition, so you can be flying with non 0 delta r. So, delta r naught can be non zero, but here, when we are deriving the perturbed equations of motion, you are talking about perturbation from that trim condition. So, we will write other four perturbed equations in the next class.