Gas Dynamics

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Module - 16
Lecture - 35
Non-Isentropic Flows- Crocco's Theorem

Hello everyone welcome back, from today onwards you are going to discuss analyses of flows that are no more isentropic. And before we going to it we neglected a whole bunch of terms, when we derived the basic equations of flow mass momentum energy consideration equations. So, will go revisit them and then include those terms and see how the equation looks like, and what that is telling us that is where we are going to start today of course, this not much change in mass equation.

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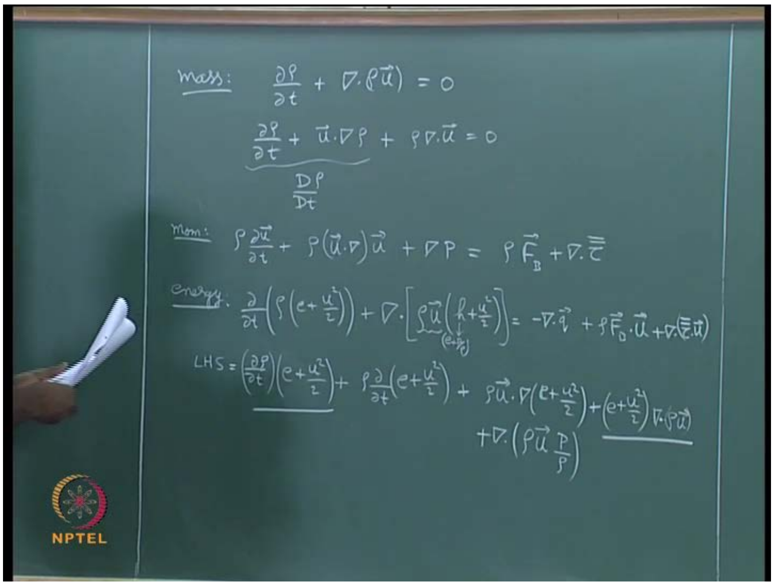

Mass consideration equation we derived this form, this still remains the same except for I will write it in one more form, where this can be club together as total derivative this is of course, fluid mechanics I will just avoid going into details of this total derivative. So, much this is called the total derivative will be using it in derivation because, it makes our life simpler writing terms will decrease in number.

So, can be written in this form also, and if we say the flow is incompressible I am going to say density does not change at all and, so this term goes to 0 and that is how we got to incompressible flow divergence is 0 kind of term. But, of course, we are not in incompressible world we are very much incompressible world, so this need not be 0, next is momentum conservation where we got an expression this is all I am referring to like say the 5'th lecture or 6'th lecture somewhere at the beginning, where we derived all the equations of motion.

I will put this arrow for vector u everywhere, I have written these extra terms now which we said 0 in the previous time and we derive, I will just added these two terms. Basically this is telling you the total momentum conservation, and the very first time when we derived it we had this term as dou by dou t of rho u inside the derivative. And I have used this expression inside that to simplify it to this form, I will not go and in details of its fluid mechanics when you go derive equation for the first time it will come out to be that and, so will keep only this form.

Now, previously when we were deriving we said that we do not have any body forces, we said this equal to 0. And then we said that we do not have any viscosity, so every term in the shear force will have a mu inside it, which is said to be 0, so this term was 0, so we neglected this. Now, we do not want to neglect it we will just keep it as is, so basically with this becomes our momentum equation, next is energy I will write what we had before and then from there we will add two more terms similar to this.

I have written this form here, the left hand side is exactly same as what we derived at the beginning. And we had this term already we did not have F b and tau terms before, we said that it was 0 and we never wrote an expression for these, we just removed at the beginning itself in the contribution stage itself. And later we said there is no heat transfer and we neglected this also that is how we came to this particular form of the equation.

One more thing I need to tell you is this F b is body force per unit mass it is roughly an acceleration term F b is body force per unit mass, but everything else is per unit volume basis. So, I am multiplying by density to get to per unit volume basis that is how we get this form, now I want to rearrange this energy equation, so that we can get more information out of this, what is it telling is not very clear if I write it like this, this perfectly correct. But, it does not give me full information we will try and rearrange the terms here.

I will just take the left hand side, left hand side of the equation I will expand it where I will say h is e plus p by rho I will write it like that, I will see what I am getting I will do one more thing I will use differentiation by parts. So, that I will cut down one statement here, so I am taking rho as one quantity and this bracket as another quantity and I am derivative taking derivative by parts from here, this is coming from the first term alone plus the next term again, we have to do similar thing I am going to group this rho u as together and the bracket is another term will keep.

If I keep the rho u term together I am going to have ideally I should have del dot of h plus u square by 2, which I want to write I will write this right here e plus p by rho I will put h as e plus p by rho I will take this e and this u plus u square by 2 together and write it here. Will remember that there is one p by rho term which is left out inside this whole equation, this is one term plus the other term where derivative is on this rho u del dot of rho u.

Now, will remember that there is a term del dot of rho u times p by rho u which is left, which I will write separately plus del dot of we will just keep it like this I know I can cancel the density will do that later. Now, look at this expression I want to simplify it a little bit, I will keep the terms with e plus u square by 2 as a common factor together, if I look at that, that will be this term and this term. If I look at that term I take this common factor out, what is remaining is dou by dou rho by dou t plus del dot of rho u that happens to be equal to 0 from mass equation. So, I will just remove that two terms from there, so my left hand side is only these three terms.

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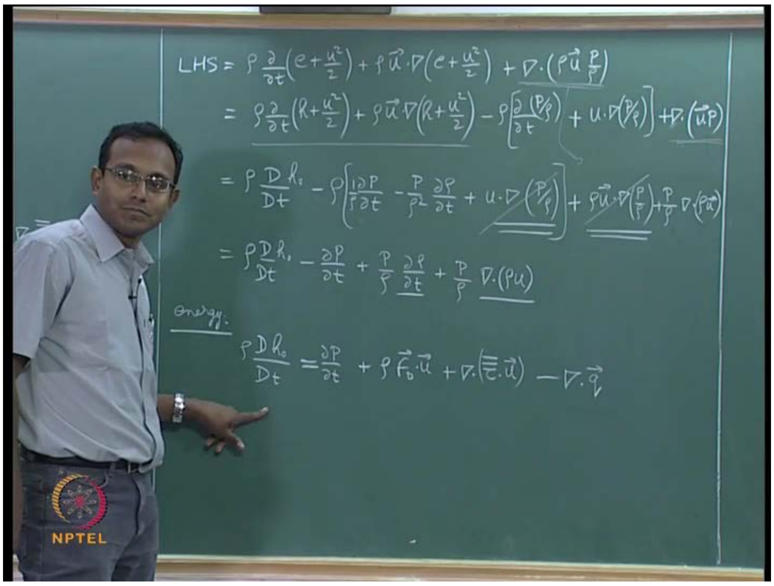

So, I will continue from there by the way that is the same exercise we did for getting the momentum equation from the previously derived form to present form. Now, I will rewrite this again I want to get to a point where I will go to h naught the stagnation enthalpy. So, I will rearrange this now, now that it is in a better form I will rearrange this such that it will be h instead of e in both the places, so that I can write this whole thing as total derivative.

So, I will directly write it as h plus u square by 2 will figure out what is missing after this, now will put all the missing terms I added a p by rho here. So, that will come as the next term sitting there minus rho into rho by dou t of p by rho I will put a plus inside here plus rho dot del of p by rho plus the final term there del dot, since I do not have space I will cancel the density and write it there, I have these terms here. Now, I can club these two terms together and I write it as rho into total derivative of h naught.

These two terms come together to form this one term minus, the remaining terms will expand this such that I will have just dou p by dou t and dou rho by dou t. So, if I do that it will be density into 1 by rho dou p by dou t minus 1 by rho square p is also there dou rho by rho t. These are the terms that comes from differentiating this p by rho that is this first two terms plus I will keep the this term as is u dot del of p by rho as is, now I will expand this term actually I like the form this particular form I will expand this form here next I will close this and I will open that one next.

Where I want to split this term again rho u together and p by rho together, so that will come and cancel this particular term that's what am looking for finally. So, now, I will write it as rho u dotted with del of p by rho that is one term plus p by rho outside the derivative times del dot of rho u, this is the other form. This is actually coming from this term, but I am using the previous form which is nicer to expand and it comes to this particular case.

Now, I will look at this term this and this they will get cancel, why this is having a minus sign here, this minus sign minus rho u dot del of p by rho this is plus rho u dot del of p by rho they get cancel. Now, I will look at the remaining terms I will write it one more step rho D by D t of h naught minus dou p by dou t, and this minus and this minus will become plus and density one of them will get cancel plus p by rho dou rho by dou t that is only thing plus p by rho del dot of rho u, this is my left hand side.

Now, again I am seeing that this plus this together is 0, why again continuity equation, continuity equation says this plus this is equal to 0. So, p by rho multiplied by 0 these two terms will go away now, so I finally, have left hand side to be just these two terms. Now, I will bring all the other terms and write it together my full energy equation, now be rho D h naught by D t, now I want to write it this on the other side.

So, I will put it as rho D h naught by D t the I have taken the dou p by dou t to the other side, and I have the remaining terms I will write it as plus rho F b vector dotted with u plus del dot of tau tensor dotted with u vector, this one more term minus del dot of u. Finally, have this particular form we would not do any more changes to this, this gives you some more physical field.

All this time we said, total enthalpy is a net energy content of the fluid, this includes kinetic energy as well as internal energy, thermal energy including the pressure energy it is having all forms of energy, thermal energy, pressure energy, and kinetic energy complete set is inside this. So, this is the net content of energy in that particular fluid element, now we want to see the change in that particular quantity, if I look at it may change the simple term is this one.

It may change, because of if I call it a del dot q is divergence of heat, this is a heat conduction, heat flukes q basically talking about divergence of heat away from this point. So, if there is divergence of heat away from this point, that is going to have energy loss, so it is negative or you can think about it as minus of divergence is the convergence of heat, opposite of divergence right it is converging of heat into that point that is increasing energy, whichever way you think about it that is one term.

The next term is here shear stress tensor doing work with that particular fluid is going to finally, generate some amount of energy because, it is increasing energy of the system by adding work to that particular fluid that is going to increase energy and that is this. This is the same kind of work, but done by the body force the bulk on the bulk fluid itself and that is given by this term, work done per unit time that is coming out to be this, everything is in terms of per unit volume basis.

And finally, there is one more term here which is talking about if I suddenly have unsteady pressure rise that can also add energy to my fluid, where have you seen this we have seen one example, where we said there is no heat transfer, there is no body force, there is no shear, but we have a moving shock. If we had a moving shock, then when the shock crosses my current point of interest, then when the shock crosses I am going to have pressure rising, and the pressure rises I am going to have t naught increasing that is how it is show right.

We found that when a moving shock crosses my point t naught increases that is directly come in front. These are the various mechanisms by which we may change energy of the fluid, till now we thought about only simple things, now will go and look at little more details of what all other things can change the net energy of course, in particular we are going to think about, we will still neglect this body force term of course, you can add it its simply in of term, you are going to think about heat transfer and viscous forces.

We will not deal with this complicated viscous force term, I did not explain what this tau tensor is really, I am just going around it is not really important for this course will just go around it. Because, we are going to work in typically one dimension it will come out to be much simpler than this, if I send an expansion wave what happens, this will automatically become negative pressure is going to drop that will automatically pull out energy from the system.

That can also happen I may have if there is a moving expansion it is going to cool down the gas, it is going to pull out net energy from the gas also that can also happen any of these can be happen in there.

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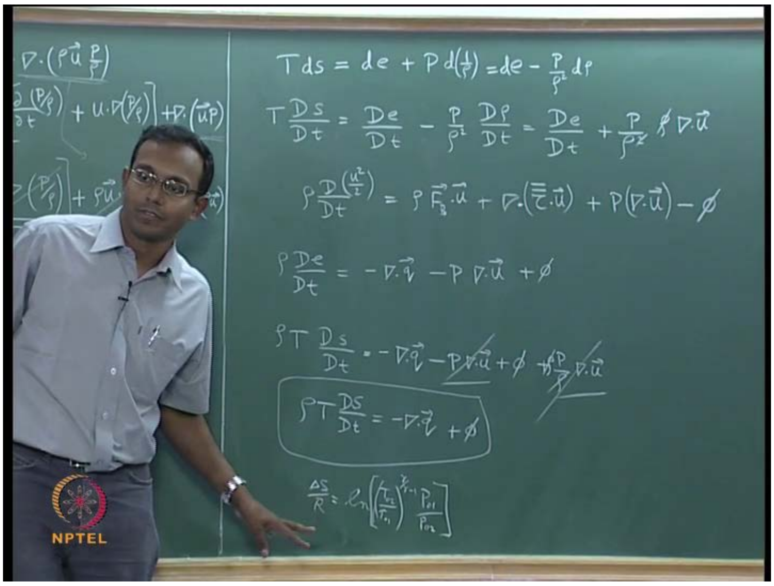

Next thing we want to do is entropy, so I will start from the standard thermo dynamics form of entropy, when I write this small s I am going to say it is per mass basis. We just work with that form, we had the same convention from the beginning will maintain that d e which is the internal energy plus p d v where d v I will write it as d of 1 by rho because, that's per mass basis volume. So, it is 1 by rho will rewrite this now this can be re written from calculus as minus p by rho square d rho, it can be re written like this.

Now, will use this particular definition and we want to write D s by D t total derivative of entropy, right entropy change in my fluid element, if I write that using this particular derivative notation I can immediately jump to this particular derivative, and I am going to say it is D e by D t minus p by rho square D rho by D t. Now, I will look at this minus D rho by D t, I will go and use my continuity equation, which is out here this is why I wrote this here D rho by D t if I put a minus sign, then I it is equivalent to taking this part to the other side of the equation.

So, it will just come out to be rho times del dot u, I will use that in there, so I will simplify this to D e by D t plus p by rho square into rho del dot u one rho gets cancel. I have this particular form here, now I need to write a D e by D t which is just talking about the internal energy of the fluid and not the total enthalpy, which is what we being writing till now.

So, I now I need to split this total enthalpy from my full energy equation, for this I have to think about what are the components of my energy equation, I am going to say I have my enthalpy plus kinetic energy as my total energy equation, where enthalpy is having internal energy plus p by rho. So, if I write conservation equation for kinetic energy alone, which I can obtain by multiplying the momentum equation by another u vector dot product of it.

And then finding out the what the final expression is that comes out to be the kinetic energy conservation equation, we would not do that here I will just give you the final expression of course, this is again been simplified by using continuity equation. So, that density is outside even though it is used for compressible flows, I am putting u square by 2 because, it is that particular form of kinetic energy form per unit mass basis.

This term comes out to be rho F b dot u plus del dot of tau tensor dot u vector plus pressure times del dot u minus there is a free term, which I did not want to go into details of it. But, since it is in the equation I have to talk about it a little bit, it is call the dissipation of energy due to viscosity, dissipation of kinetic energy due to viscosity in this case. A written a equation for kinetic energy conservation alone, where we did not talk about internal energy of the gas we just that kinetic energy part.

What we have finding is kinetic energy change is due to external force doing work on the system, on the particular fluid element or shear forces doing work on the fluid element or pressure forces doing work on the fluid element, a this is actually pressure work on the fluid element, this is the divergence of the fluid element, and this is the expansion of the fluid element multiply by the pressure that is like p d v work that is forming here. And then this is here energy removed from kinetic energy, so that it enters into thermal energy will see that soon.

This is the part, which causes heating of the gas when there is viscosity, we would not go into details of it will also have some terms like I believe it is u dot del dot tau that kind of form will be there, I do not want to get into the details of it will just ignore it for now, it is not needed except for this one particular place. So, if I subtract this from our original energy equation which we had before, then it is going to come to point where I can write it as rho D e by D t is equal to minus del dot q minus p times del dot u plus p.

Only these will be remaining, will find that the viscous work is going to be internal and it will just sit in the total energy also. So, only thing remaining is this particular part, what we are now seeing is this is only the thermal part of the energy, and we look at the thermal part of the energy we are finding that if there is net convergence of heat into the fluid element, then it increases. If there is net contraction of fluid element del dot u is related to rate of expansion of the fluid element minus of that is like compression of the fluid element that is external work done on my fluid that can cause heating of the fluid.

And this is the energy added from kinetic energy to thermal energy due to viscous effects. If we look at this previous equation and this equation, in the kinetic energy equation there is a minus v I did not write the term, but it is going to be negative it is pulling out energy from kinetic energy. And here it is plus v it is adding that energy into this thermal energy.

If we look at that total energy equation which we had before I will go back to it once more that is this one, we find that there is no phi in this particular thing why it is just a internal rearrangement, it is taking from one pocket and putting it to the other pocket. So, it is not there at all, this is the net energy over all how much do I contain that kind of thing is this. In here if we look at this part this is sitting only in my internal energy equation, this part is sitting only in my kinetic energy equation this also sitting only in my kinetic energy equation.

This term is hidden somewhere inside, it is sitting in that p dot del u kind of term we would not worry about that part will see it will come up now. So, this is my internal energy equation, now I want to put this D e by D t inside here, and I have a rho here will multiply this entropy equation with rho, so that I will get rho D e by D t. So, I am going to finally have, form rho into temperature D s by D t is equal to I will put rho into D e by D t instead of that I will write this expression, it should be minus del dot q minus p into del dot u plus phi I have this particular term.

Now, I will write this particular thing p by rho times del dot u, this is my net entropy equation what is that last term should have a multiply by a rho also. Because, I multiplied the whole expression with rho while rho D e by D t was this term that one should also be multiplied by rho that p will get rho will get cancel. So, I have this term now you will find that this term and this term gets out, they are getting cancel, so I am finding that for my entropy to increase, it has to be through convergence of heat into the fluid element or through frictional heating of the fluid element.

These are the two modes where I can have entropy increase in my fluid flow, all this point I always used compressible density was changing it is, so happens that I used my continuity equation. So, that a density always comes out of my derivatives, but I know that it is compressible flow, so the essential part of this particular statement is that if I do not have heating of my fluid, either by heat conduction or by viscous forces I will not have any entropy change in my fluid flow.

These are my predominant changes in entropy, entropy changes are mainly due to this of course, we also know that there is one more way we can change entropy that is through shocks in the flow. Which is just sitting inside this which we could not talk about here, that will come as delta p naught that will not be from here that should come from, the direct expansion of this term.

If I expand it we did this long back where we I do not know where I have that expression right now, but we found that delta s by r was equal to log of p naught 2 by t naught 1 to the power gamma by gamma minus plus 1, multiplied by p naught 1 by p naught 2. We had one such expression this is going to take care of that fluid mechanical way of increasing entropy, this is one more form of writing it this is actually derived from this top equation we would not go into the details of that right, now it was done I believe just after moving shocks discussion.

So, we want to look at those particular two characteristics that is one is heating of the fluid element or viscous forces causing entropy change. So, we are going to deal with two types of flow fields, now where there is entropy change important first one will pick up friction and friction flow next one will pick heating. But, before that I just wanted to give you this small snipped of analytical work by Italian engineer called croccos. It is a nice point to give it here it has to be given, so that you will understand the connection between vorticity and entropy. Since, we are deriving gradients of entropy in my flow etcetera it is a good point to give it this point.

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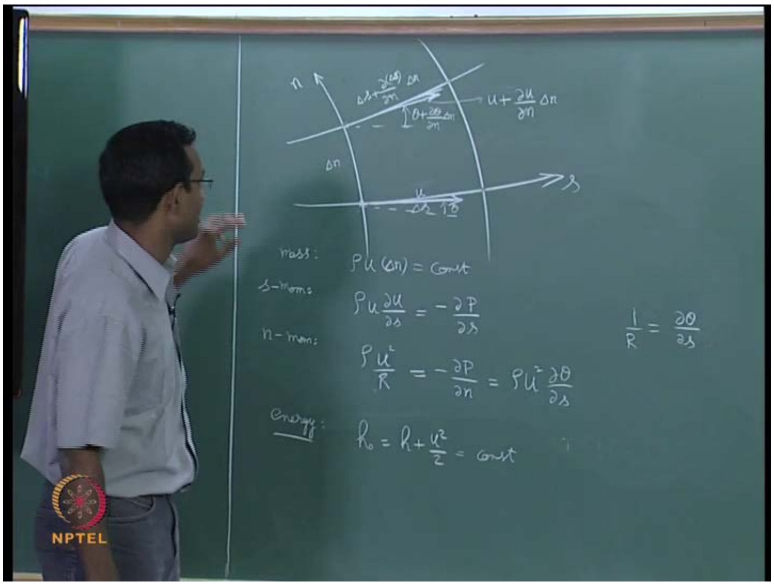

So, will derive it the simplest way first then will give you the most complicated expression, that is the final result. It is also going to introduce you stream wise coordinate system, I am calling this as my stream line I am using this kind of s small s while if I use the other s this is entropy per mass, this is my stream wise distance coordinate system. So, I am having this is s let us say this is my reference point am picking this is my n, the normal stream normal coordinate system.

So, I will say that if this is my point n comma s this is going to be a delta n there, so this point will be a delta s here. This will be n comma s plus delta s while here it will be n plus delta n this will be my delta n, n plus delta n and s this will be n plus delta n s plus delta n, that kind of coordinate system I am going to have. Now, let us say I have some particular velocity direction with respect to my previous original rectilinear coordinate system.

Let us say this is having a theta x y coordinate system this where theta with respect to x y coordinate system, that is my velocity vector that is the case then I go from here to the next n point. It is going to be again velocity vector will be parallel to that local stream line which is this line another s curve and that angle is going to be theta plus dou theta by dou n times delta n, just a generalize form of writing it for a small change in n how much will be the change in theta is what this represents and will just keep it, that way very generalized form and here this gap was delta s.

I am going to say this is delta s plus dou by dou n of delta s times delta n change in delta s with delta n is what is shown there, that is the this point to this point the gap is there. Just taking care of full details of course, we may not use all of them and what will happen to velocity here. This velocity will be u which is the value here this is my velocity u.

It will become u plus dou u by dou n times delta n this will be the velocity at this point should I think about components of velocity, there is no components here. Because, it is a stream wise coordinate system which means all the velocity is going to be only along this s, there is no velocity component perpendicular to it because this is the stream line. So, I can just talk about velocity always along this direction there is no perpendicular component.

So, I have basically brought my problem down I do not need to think about u as a vector, I will just think about u and it looks as if it is a one d problem finally, because, it is going to be varying only inside this stream tube, between any s and s plus delta n, n and n plus delta n that particular thing is one set of stream tube and my problem is one d problem inside this which means, now I can use all the one d equations.

So, will write the one d conservation equation for this whatever we are used all the time will just write it here, rho u times delta n this suppose to be rho u a is constant and I say delta n here, this is the delta n is this length and unit depth inside. I am picking two d problem its unit depth inside, so delta n times one is what is given as area here. Similarly momentum, now you have to be a little bit careful forces can act along this n direction also.

So, now I have think about s momentum first only along the s direction s momentum equation like, we had x momentum and y momentum way see here it is s momentum equation. Now, it can be written as rho u dou u by dou s equal to minus dou p by dou s, if you look at this is your acceleration of your fluid element inertial forces here, that has be compensated by the pressure force from here to here.

The fluid is accelerating this way because, I am going to say pressure here is higher than pressure here. So, there is d p by d s that is negative and what is a net force acting, that will be minus of d p by d s times that area. I have divided by area throughout that is this, this is basically here force balance, momentum increase is equal to the force that is what is written here or you think about it just f equal to m a kind of formula, next one is a normal momentum equation.

Where, now I have to think about this stream line is curved and it is curving if it is curving then, there is force perpendicular to the velocity vector that is when a flow stream line will turn it is like a centrifugal force centripetal force, so I will write an expression for that that will be rho u square by r, this is your centripetal component this should be given by minus of dou p by dou n.

The normal force if my pressure here is higher than pressure here then my stream line will curve up, as the pressure goes from lower n to higher n pressure decreases which means my stream line will turn up, that is what is this is showing here minus of dou p by dou n is the net force upward, that is going to causes centripetal acceleration that is this terms, now I will just write since we have this u by r, r is here radius of curvature.

I will write this as in terms of theta which we are given in the drawing, rho u square went to 1 by r can be represented as dou theta by dou s that, happens to be the curvature. 1 by r is given as curvature which is equal to dou theta by dou s, how does theta vary as I go this way that talks about the curvature of this, this particular curve s coordinate system now.

The only other thing left is energy will write the simplified form simplest form where will say that there is no heat transfer, there is no viscosity just for this crocco's theorem derivation alone. When we say that we know that h naught is constant this is the old 1 d derivation I am just writing it again. In fact, of in today us derivation also you can say when you said those terms v equal to 0 you just automatically get to this of course, I can, now introduce it may have a non stationary flow where pressure may be changing with time, I could get that kind of dou p by dou t term here.

If I want to currently we will keep it simplified when we give you the generalized expression I will give you the full thing, one more thing I need to tell here is this telling h naught is constant for every stream line s it may. So, happen that this stream line can have a different h naught from this stream line, I did not tell anything about h naught is constant for all s I just said h naught is constant along a particular s, that all we know 1 d gas dynamics.

We just know that inside a stream tube h naught does not change it may change from one stream tube to the next one that can still happen.

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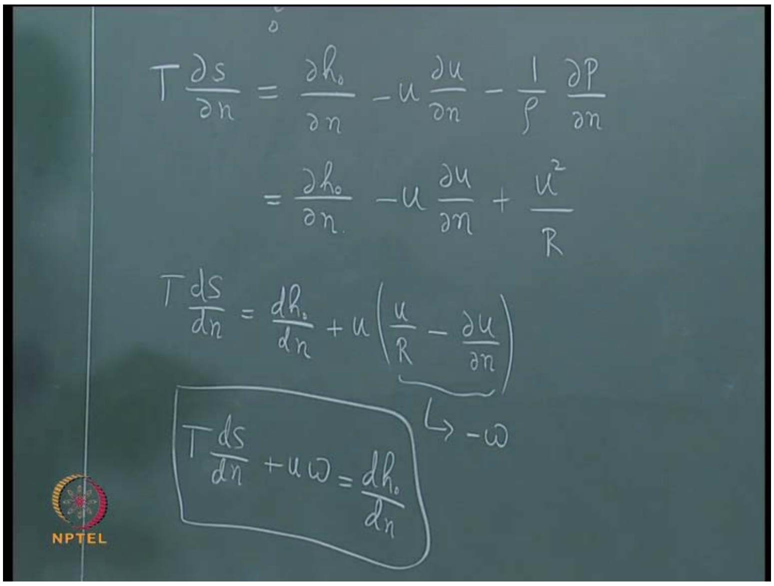

So, now I will go and write again entropy relation where this s is entropy per unit mass, I will just directly go into p d v v d p sorry d h minus v d p. Which can be express in terms of d h naught from the expression written there, d h naught minus u d u will be your d h minus 1 by rho d p I can write it like this. So, from here I want go and look at individual variation of entropy along s direction and n direction.

So, I will take this definition this is for any particular change in s, so now, I will go make it change only one particular direction, t dou as by dou stream wise coordinate dou small s is equal to I know that along stream wise direction dou s dou by dou h naught as, I will just write it also h naught by dou s which is 0. Because, that is what we said is constant minus u dou u by dou s minus 1 by rho dou p by rho s this, what we get.

And, now I will write the other one stream normal direction dou s by dou s this going to be equal to dou h naught by dou n minus u dou u by dou n minus 1 rho p by dou n. I have this, now I will just go back to this dou s by dou stream line this any problem, this is dou s by dou n. Now, we look at this dou s by dou s what we see is from my momentum equation s momentum equation, that is going to become 0.

If I take this rho 1 by rho here this is equal to that other term and the one is negative of the other these two will give out to be equal to 0. So, I just say equal to 0 from s momentum equation, from s momentum equation that is 0 that cannot be said about the other term. If we look at the other term, I can write something for dou p by dou n that will be rho u square by r, even if I write it that is not going to be 0 that term will just take.

So, this is going to be equal to dou h naught by dou n minus u dou u by dou n plus u square by r, just have this particular form. Now, since dou s by dou s is 0 and s can vary only along s or n, entropy can vary along stream wise stream normal direction and we say it does not vary along stream not stream wise direction. I can rewrite this partial derivative as total derivative, now same thing can be done for h naught also this can also be written as total derivative.

If I do that t d s by d n is equal to d h naught by d n plus u into u by r minus dou u by dou n, I will just say here that this term can be shown to be equal to minus of omega. The virtuosity vector this is actually should be a vector, but since it is 2 d problem, the vector is also always going to be pointing out, of the board and its going to be having a negative value because, in this case it will be pointing inside the board into the board.

So, I will keep it as is this is from definition from virtuosity in stream wise stream normal coordinate system, you will get it to be this will just write it like this. So, I finally end up with the form I may check whether I made a mistakes or not sure of this whether it is plus or minus as of, now I will assume this plus it could be minus just a minute will just go through this d s by d n t d s by d n, I will just keep it as minus there.

So, it will become plus u omega is equal to d h naught by d n we get to this expression its look like I made one small mistake in my derivation, but I am not sure. So, I believe this should be minus omega here, but expression seems to be correct finally. So, maybe I made a mistake here or not made a mistake I am not sure about this, whether it is minus or plus I am not sure, but this is the final expression I am getting here.

But, the physics I want to talk will not change here as of now there might be a this could be a minus or a plus currently will just leave it there physics, I want to talk about is if I have enthalpy gradients in my flow from one stream line to other stream line that is in the normal stream, normal direction then I may have entropy variation that is one way of looking it or if I starting with constant enthalpy in all the stream lines.

If I have constant enthalpy in all the stream lines then if I have entropy gradient, then I will definitely produce virtuosity in my flow. This is a important statement of croccos theorem this is called the croccos theorem typically most of the fluid flows which we start or work with will always have d h naught by d n 0 and we also have d h naught by d s 0, that we already saw it is h naught does not change along the flow as stream line.

So, I am going to say that since we started with enthalpy constant at the beginning it is going to be remaining the same all through except, when there is a if that is again I made this 0 then I will produce virtuosity in the flow. If there is entropy gradient across stream lines d by d n of s not equal to 0, if I have this not equal to 0 then I will definitely produce virtuosity this is the main statement of croccos theorem.

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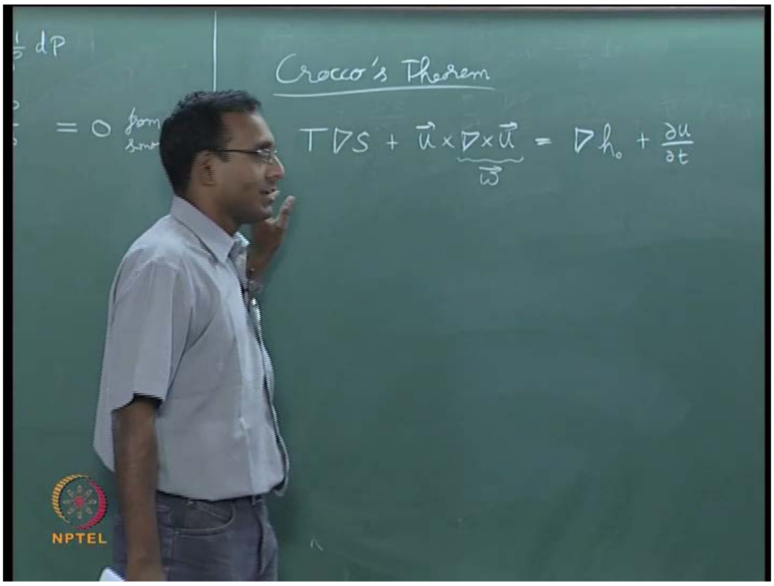

Now, I will just write the full or more complicated expression for croccos theorem then will stop here and we look at the implications of this next class. The full equation of croccos theorem is t gradient of s plus u cross del cross u is equal to gradient and h naught plus dou u by dou t this is, the full croccos theorem expression. Where what we derived did not have any non stationary term we removed that part and this is the most complicated form, where this is your virtuosity omega vector this case.

We have this particular term what we are seen again is if I start with the isenthalpic flow or home enthalpy flow you are having referring into some other books, isenthalpic flow then and I am having stationary flow then, I do not have these terms that 0 then if there is a entropy gradient in my flow I will produce virtuosity in the flow. This is the main statement from croccos theorem will keep, it will look at examples of virtuosity produced in the flow next class and then start going into frictional force and how to analyze flow with friction. This one small part I wanted to discuss also by today as not enough time and I do not want to hurry will do it next time. So, any question see you people next class.