Biomathematics
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Lecture No. # 18
Diffusion

Hi. Welcome to this lecture of biomathematics. We have been discussing various applications of vectors and as well as calculus in biology– how do we apply whatever we have learnt, so far, on vectors and some ideas in calculus that we learnt, we can apply together and learn few things that is relevant to biology.

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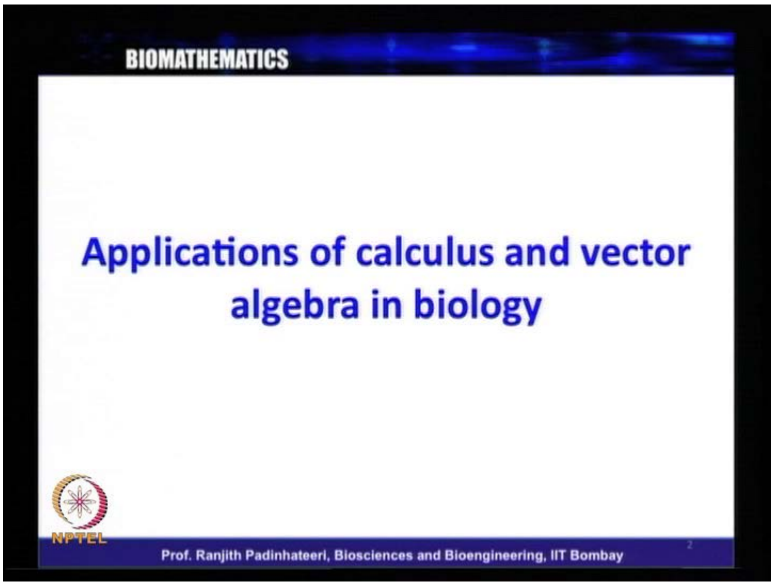

So, that is our the current topic, which is applications of calculus and vector algebra in biology. As we go along, we will be learning a few new things, but overall, this generally application of whatever we learnt, so far.

Now, last lecture, we discussed various things related to this. So, various things related to vectors, and various things to… related to calculus, and in this lecture, we will be discussing something about diffusion. So, diffusion is the process that is very common in biology. We know that proteins diffuse from one part of the cell to the other part; various… diffusion is the throughout the cell or any system that you… even if in vitro things we take, we want… we know that things will diffuse to the diffusion to the phenomena in biology as well as many other fields of science, but for biologists, it is important to understand many things about diffusion, and it turns out that diffusion– the phenomena of diffusion– if you discuss that– if you… if you think of mathematics as a language, we are discussing in beginning the phenomena of diffusion can be expressed through an equation– a mathematical equation.

So, whatever we see in diffusion, the process can be very precisely quantitatively expressed through an equation, and that equation is called diffusion equation. So, today, we will learn about this diffusion equation– what is the logic behind and where does it coming from. Why such an equation represents diffusion?

This will also give you some insight about what exactly is diffusion. So, that is the aim of today's lecture– to understand about diffusion and some… something that if we can learn from diffusion equation– something– new things that we can learn from diffusion equation that we probably did not know about diffusion, so far.

So, when you say diffusion, the first thing that comes to your mind is that if you put a drop of ink, for example, on paper or you put one drop of ink in water, that will diffuse. We can see that because this ink has a color and we can see it diffusing. So, diffuse… and you know that the proteins diffuse, and in general, diffusion is basically a process where things go from higher concentration to lower concentration.

That is something that we… many of you have probably, if at all, you know anything about diffusion, this what probably what you know– what you all… you might already know is that diffusion is a phenomenon where things go from higher concentration to a lower concentration. Whether you know this or not does not matter for today's lecture. Even if you know nothing about diffusion, we will… we will discuss from the beginning and go ahead and learn what diffusion equation is. So, I do not assume that you know anything about diffusion in this lecture. So, now, let us go to the topic which is today's topic, which is diffusion.

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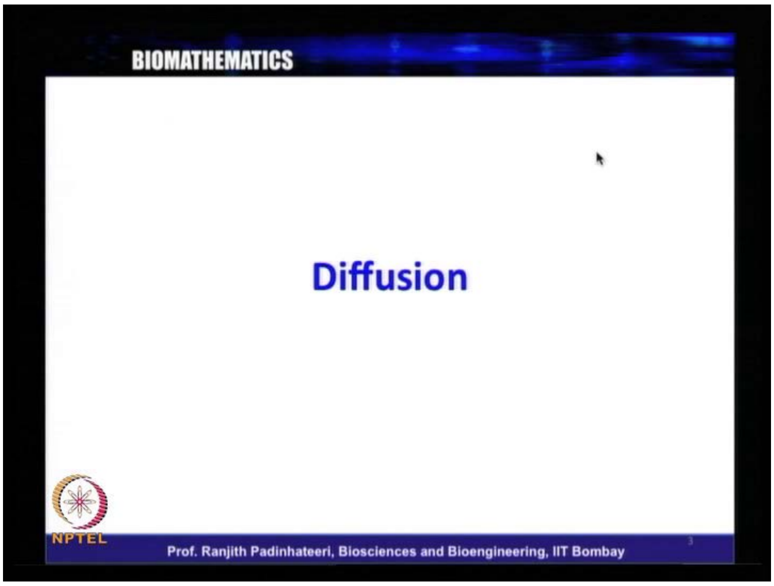

So, what we learnt or what we know is that if we have something which is having.... So, imagine that you have a container, and if you put a drop of… if you put some amount of protein molecules here, they will diffuse this way.

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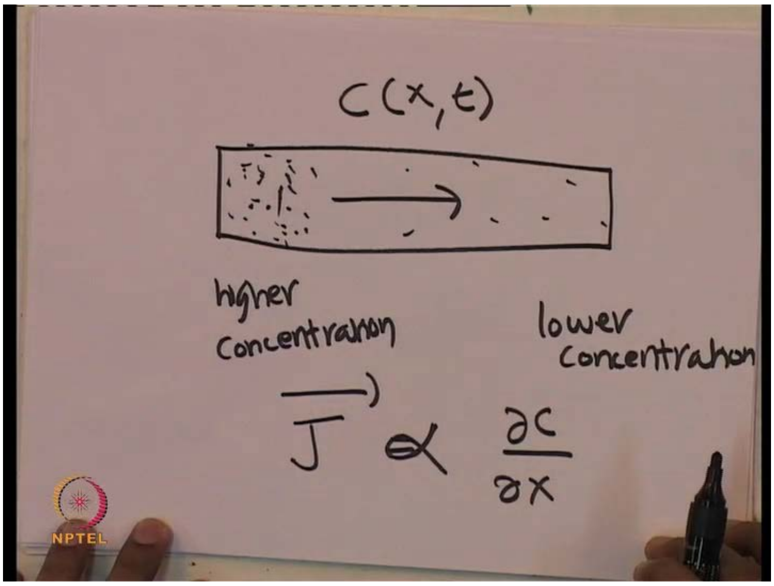

So, they will flow at least this way, because here, there is higher concentration; here the concentration is 0. So, it will go from higher concentration to lower concentration. Even if there are one or two molecules, still it will just keep flowing, and they will keep flowing until the concentration is same everywhere.

So, this phenomena– this phenomena that things will flow from higher concentration to lower concentration– can be written mathematically through an equation J. So, J– let us say J is the flow or the current. The current I does not… do not mean electrical current; it is just a flow– the current of the, as we say– call– flow the current. So, the current is can be is this flow is actually proportional to…, so that we can say that this flow is proportional to the change in concentration.

If the concentration is… there is a change in concentration, then there is a current. So, del C by del X, if there is a change in concentration. So, C is a concentration of… concentration of this protein or this ink molecule– this dot. You could imagine it could be either protein, it could be ink molecule, or it could be anything that you take, and you know that this a concentration and this concentration varies in space; that is, as we along the X axis, if this is a X axis, this concentration is the function of X axis– X– and this could be also function of time, because if you just wait, the concentration could change. So, concentration– the function of the concentration of this particular X will change with time.

So, concentration function of space and time and the current or the flow of the molecule is proportional to the change in concentration. If there is change in concentration, then there is a change in flow. So, this idea that it is proportional to change in concentration can be written– this is the mathematical way of writing– the things will flow from higher concentration to lower concentration. This is the way of mathematically stating that things will go from higher concentration to the lower concentration. So, now, what is proportionality constant here? So, you have something proportional to here. So, have a look at here: something is proportional here.

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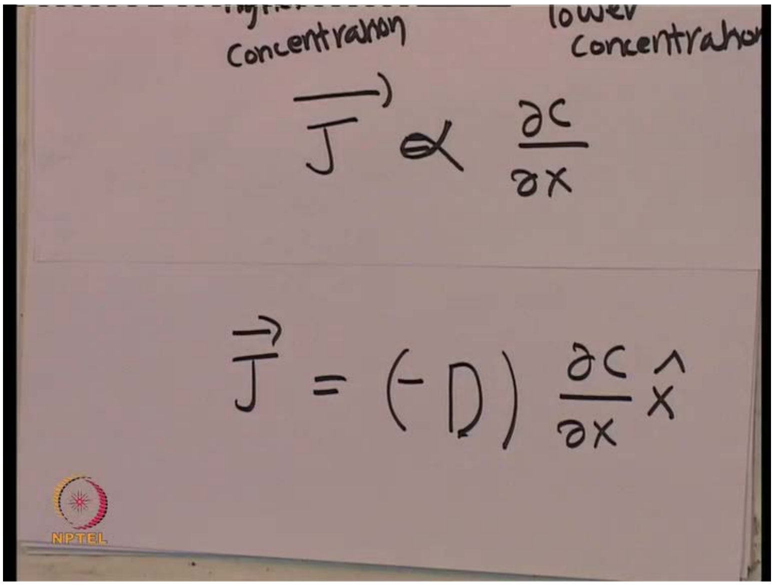

So, I can write J is equal to something– some… this some constant– into del C by del X, and we know that it will flow in the X direction. So, if you call this (( )) direction. So, this will flow in the X direction.

So, X cap. So, the X cap means flowing in the X direction. Now, if you look at here, again, the concentration is decreasing with respect to space. So, del C by del X is negative, because as the X increases, C decreases. So, del C by del X is negative. So, the… as of now, this part is negative, so, but that means it will flow in the negative X direction. But we do not want we want to flow the negative-positive X direction. So, you put a proportionality constant, which is D, and there is minus sign. So, this the mathematical form of the current, as you can see here in this.

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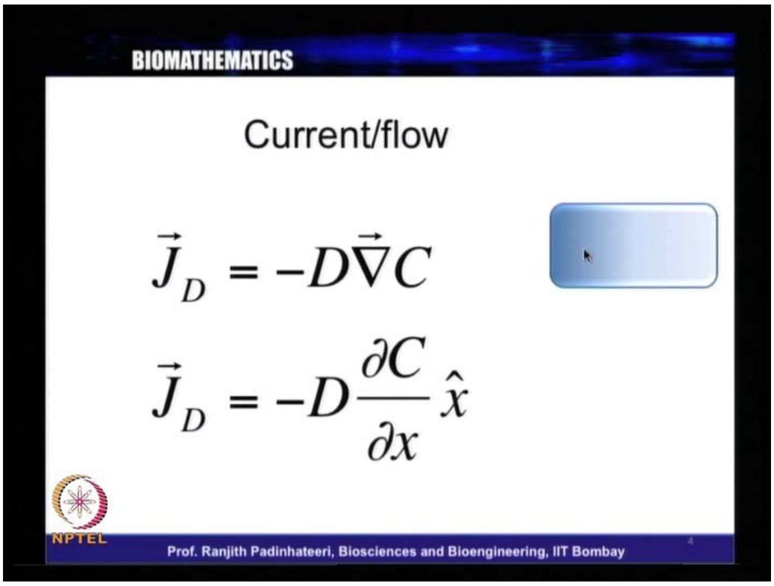

So, see this slide here. The slide shows how this flows in this mathematical…. This can be represented mathematically as del C J D, that is the current due to diffusion is minus D times del C, where del is the gradient as we discussed before.

So, the gradient del can be written as del by del x x cap. So, J D can be written as minus D del C divided by x cap. As we just said, since del C by del x is negative, because as x increases, del C decreases– the concentration decreases. As x increases, C decreases; del C is negative. So, del C by del x is negative, and with this negative sign, it is the whole thing is positive. So, that means the J– direction of J– will be along in direction of x, that is along the direction of x cap. So, this… what does this mean? Now, let us take some particular… some particular functions for concentration. Just for fun– just for fun– so, now, let us take a simple example for C.

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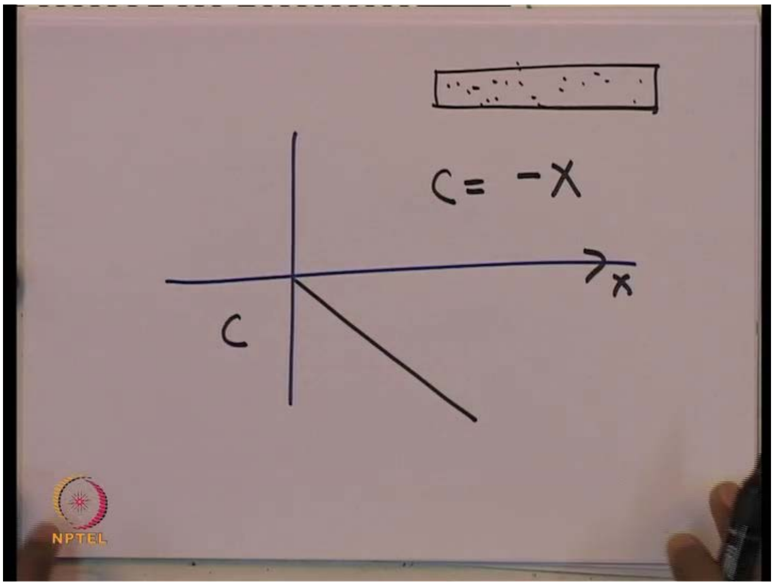

So, let us imagine that C is a… if you plot C is let us imagine that C decreases in a particular manner. So, let us say C decreases in a linear manner. So, let us… this is C and this is X. So, this the concentration decreases linearly. So, what does it mean? This means C is equal to minus X.

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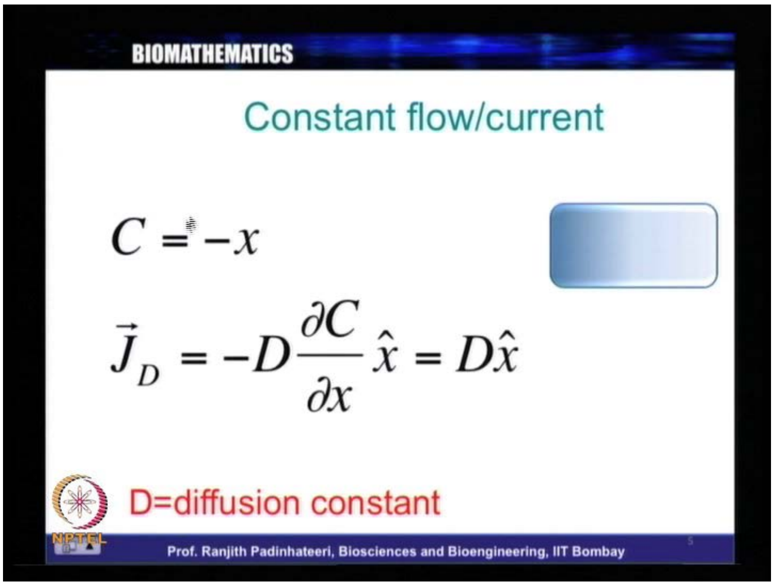

So, that is, if we have a container having concentration, the concentration decreases as you go along the x axis, and this decreases in a linear manner. If this is the case, let us have a look at here: C is equal to minus x. So, del C by del x with a minus D is a minus… is minus 1, and minus D del C by del x is… we know that D is a constant. So, that current is just a constant times the direction. So, this is a constant flow or a constant current; that means, if the concentration decreases linearly along x.

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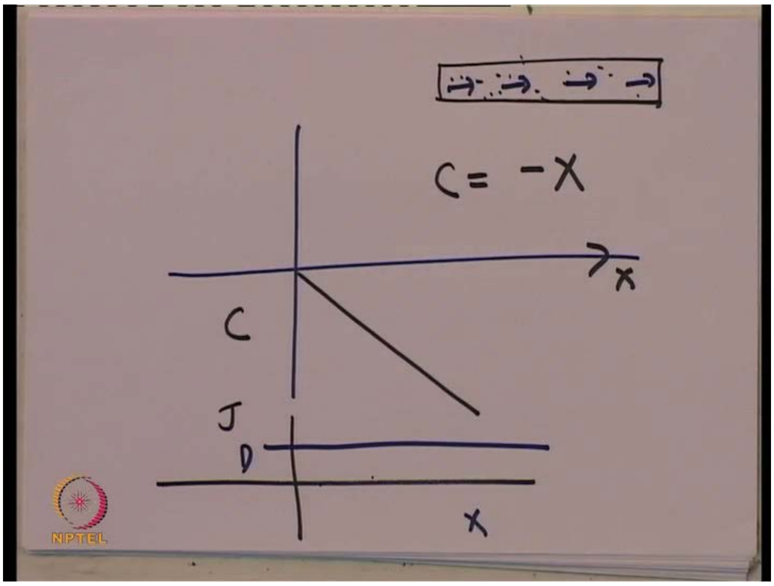

So, that is what if the concentration decreases linearly along x, the current the… if you plot the J, J will be just a constant; that means, the... So, this is J as a function of X. So, J here, here, here, everywhere. So, here, flow here will be the same as the flow here, will be the same as the flow here; this is what it means. So, J will be a constant then this constant will be D.

So, if we assume the C is the linear function of X is minus X is decreasing linear function of X, then the J with the current the flow is the constant; that means, the same flow everywhere– whatever the flow here, the same amount of flow here; same flow everywhere– this is what this means. So, let us see what the consequence are: if you look at the consequences, so, what we have is J is equal to… minus is equal to constant times X, and where D is a diffusion constant. So, here, with some constant, which is… which is important for diffusion, we will discuss what the D is in the coming lectures. So, but let us look at the consequence of this J, D being a constant.

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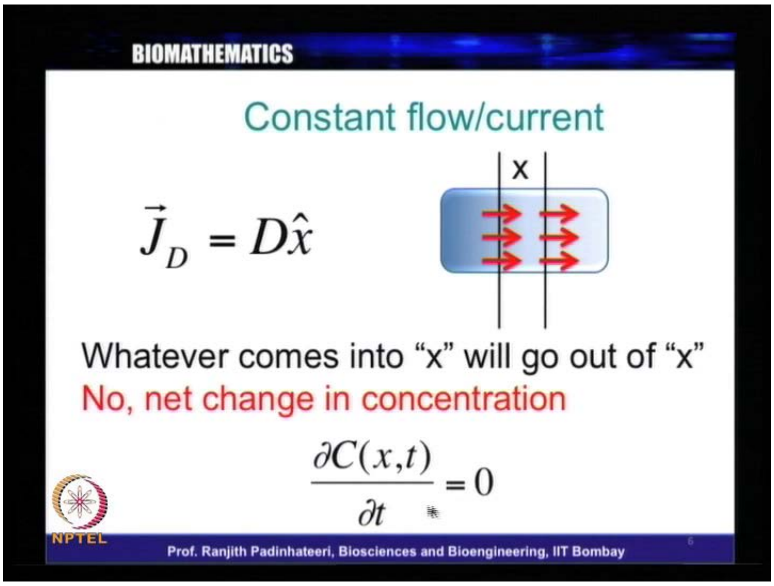

So, let us discuss a bit about constant current. So, have a look at this slide, and here, there is a larger concentration; here, there is a small concentration; and I am looking at this particular area, which is in this particular position x, which is in this area. So, this a area, this is two lines are marking some area in this… in this container.

Now, imagine J is equal to d x; that means, direction of J is along the x axis and the amount magnitude of the current or the flow the magnitude of the flow is a constant, which is D; it is like a number: D could be 3, 4, 5, 8. So, what does it mean? That means that if some amount of molecules comes in to this area in one second, the same amount of molecules will go out of this area in the same second; that is what does it means. Whatever comes in, goes out. So, if three molecules come in, three molecules will go out in one second. So, whatever comes into x will go out of the x. So, in one second, if three molecules come, in one second three molecules will go out.

So, that is that is what it means by constant flow– that constantly it is flowing. So, as far as if you look at the concentration in this area, the area between these two lines that we are showing here, if you think about it, if something comes here and the same amount of things goes out, the concentration here will remain a constant; the concentration here will not change. If the concentration was 3 micro molar to begin with, if something which comes in and same amount was taken out, the concentration here will remain 3 micro molar. In other words, this C at x for all time is equal to a constant. Whatever be the time, it will be a constant because same amount of particle is coming in and the same amount of molecule is going out. Same amount of molecule is coming in and same amount of molecule is… molecule is going out.

So, this can be written mathematically as del C by del t equal to 0; that means, the derivative is 0; what does it mean? It means C is a constant. We learn that when you say derivative is 0, that functional derivative of function is 0– that function is a constant. So, that… this means here is del C by del t is 0, that only means that C is the constant.

So, what we have is constant flow or a constant current. So, if the constant current or if we have constant flow or a constant current, what we will have is the constant concentration at a particular place. If you look at any place in the… in the container where there is a flow, the concentration there will remain a constant. So, this is a simple common sense that if same amount of things comes in and same amount of thing go out, the concentration in that area will remain a constant; this is just common sense.

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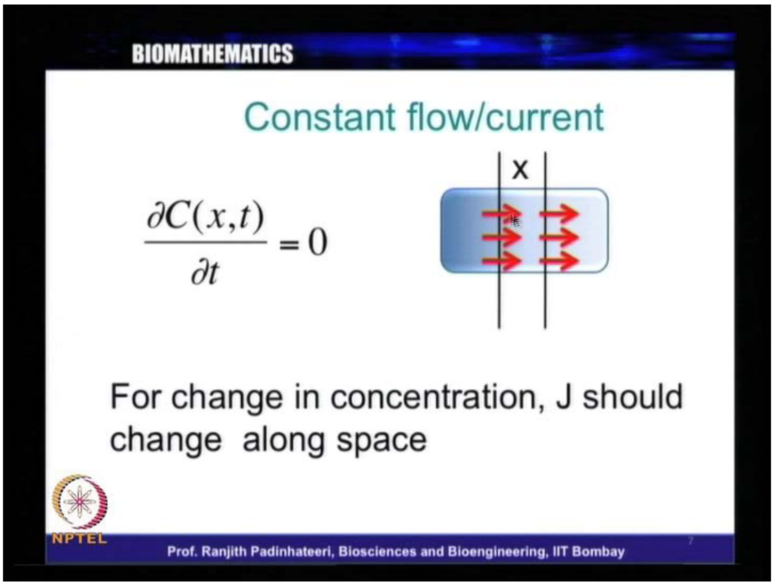

So, now that we know, what is… what does it imply? That implies that if you want to change the concentration in this place, if you want the concentration here to be changed, the J should change. So, that means, whatever should come in should not go out– whatever comes in should not go out. So, the del by…. So, that means J should change along space; that means, the J– the flow here– should be the different from the flow here. If the flow here is different from the flow here, let us say the flow here is more compared to the flow here. So, what does it mean? More things will come in and less things will go out. So, then there will be a constant concentration here.

There will be a concentration change here if more things come in and less things go out. There will be a change in concentration here. In other words, if less things come in and more thing go out, still, the concentration there will changed; the concentration will decrease. On the other hand, more things come in and less things go out, the concentration here will increase. So, that much is clear. Now, how do we say this mathematically? How do we say that the current– the flow– should change in space?

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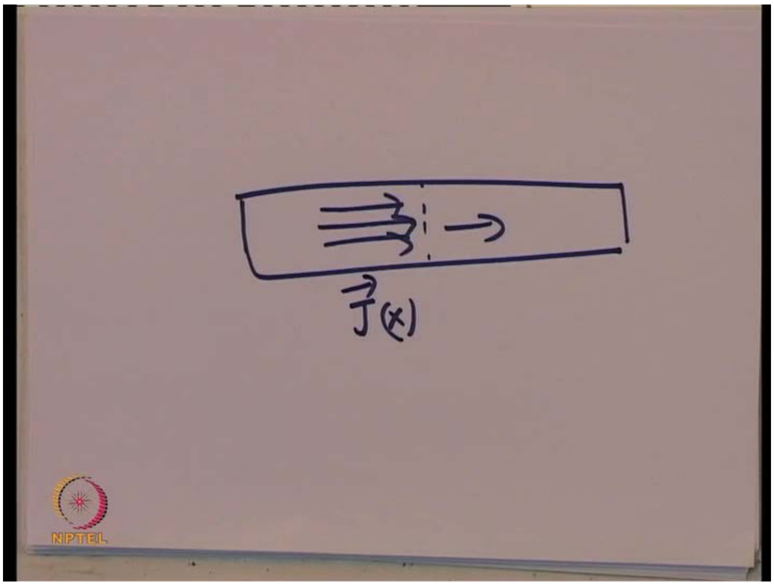

So, let us have a look at the container again have a. So, let us say you have a pipe where things flow, and if you look at this particular point and the flow here, so there should be more flow here and less flow from here; that means, J– the… if the what we call the J should change along x. So, the derivative of J should be… some should be non zero– derivative of J should be non zero. Now, we learn something called gradient. So, let us look at what we learnt– gradient.

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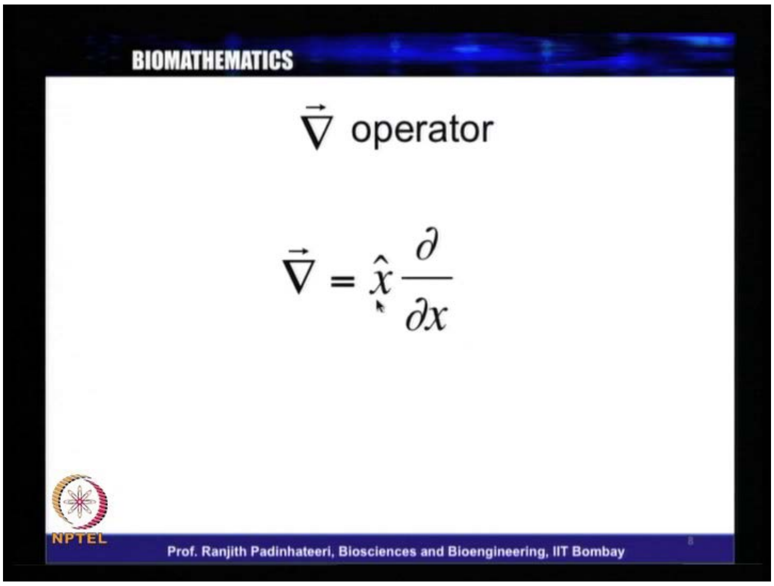

So, this called… we called the del operator the gradient. The gradient del is defined as x cap del by del x. So, this is something that in vector algebra, this is something that defines change in space. Del by del x is a vector, and we have here J in this; have a look here– J is a vector. So, J is a vector del, which is x cap del by del x is the vector.

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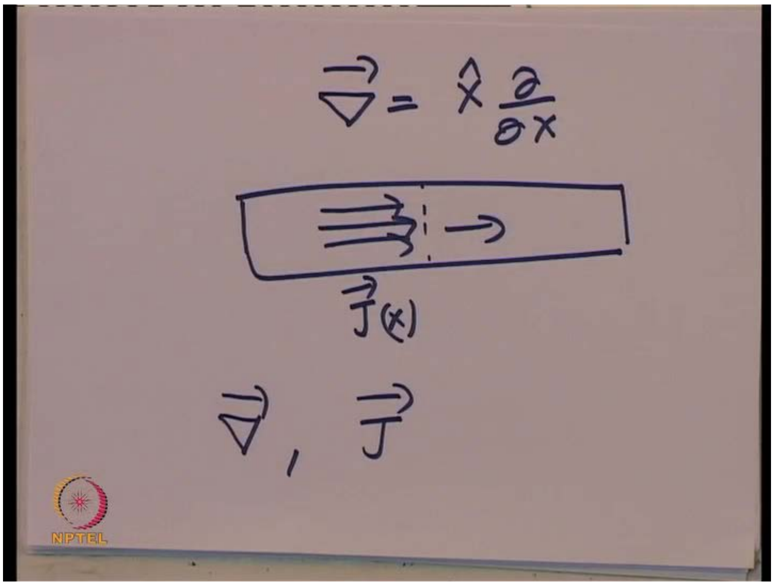

So, this is two vectors. So, have a vector del and have another vector J, and we want the derivative of J. So, this del should act on J. Now, if we have two vectors, we learn that there are only two ways of finding the product, or we learnt about two ways of finding product: one is we call scalar product and other one we call vector product. So, now we have two vectors, and let us see what is the product of this.

So, do we want scalar product or do you want a vector product? How do we know? So, let us think what we want we… what we want is that we just said that if the current varies along space, if there is more flow in and less flow out, if the J– the flow– varies along space, the concentration will change with respect to time; the concentration will keep changing; that means, we said that del C by del t is proportional to change in flow along x.

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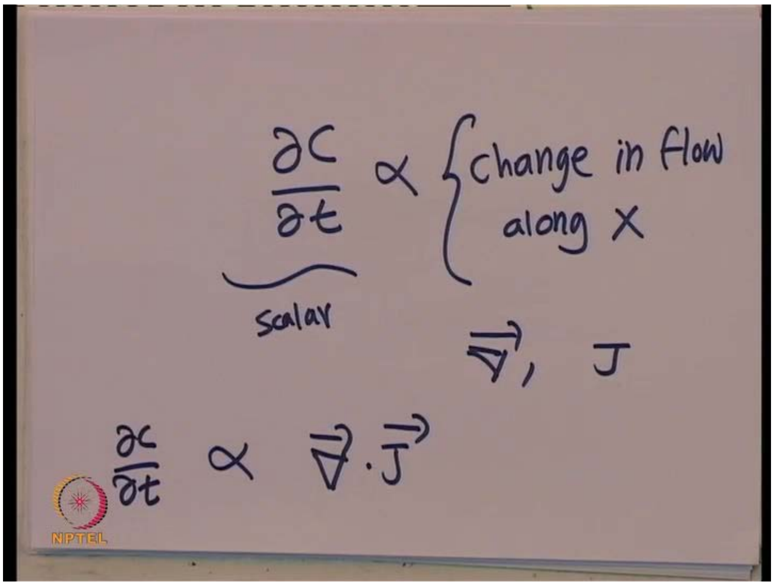

So, we said that if there is change in flow along x, there is a del C by del t. So, del C by del t is a concentration in the scalar. So, time derivative of a scalar is again a scalar. So, this is a scalar. So, whatever here, the change in flow along x must also be a scalar, because you can only equate this scalar to a scalar; you cannot equate this scalar to a vector.

So, since this is scalar, this has to be a scalar. So, the scalar with del and J, the only scalar you can produce is the scalar product del dot J. So, del C by del t has we proportional to del dot J. Now, what does it mean? Let us have a look at it.

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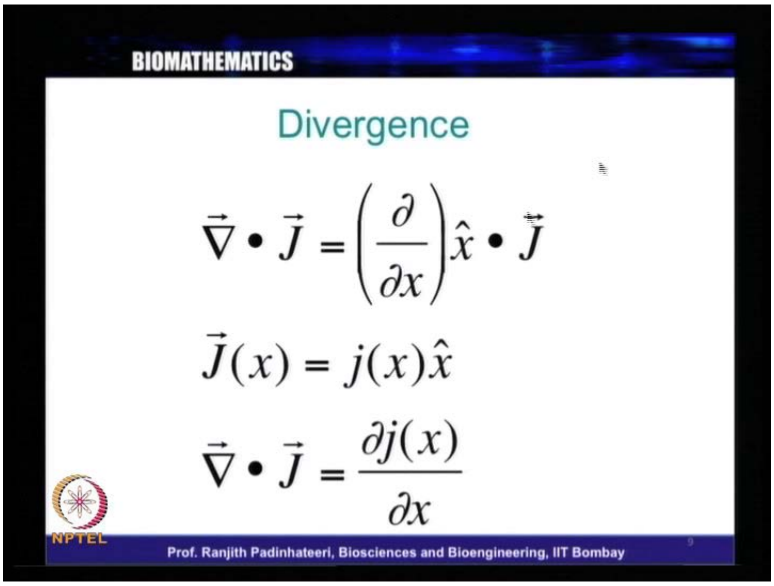

So, let us look at here. So, we can define something on del dot J, as we said, and the del dot J is something called divergence. Then, in vector algebra, del dot J is called divergence. So, divergence of the function J is defined as del dot J, and this is del by del x x cap dot J, and let us… let us imagine that J is some function, which is little J of x have x cap, where little J is some function– it could be x, x square, x cube, x power 4, minus x, minus x square, whatever be it– if this is J of x, del dot J can be written as del J by del x. So, this is how the flow varies along x axis, the change in flow– the change in current– along x axis. Now, we said– we saw that there has to be a change in flow now, and this is (( )) concentration to change.

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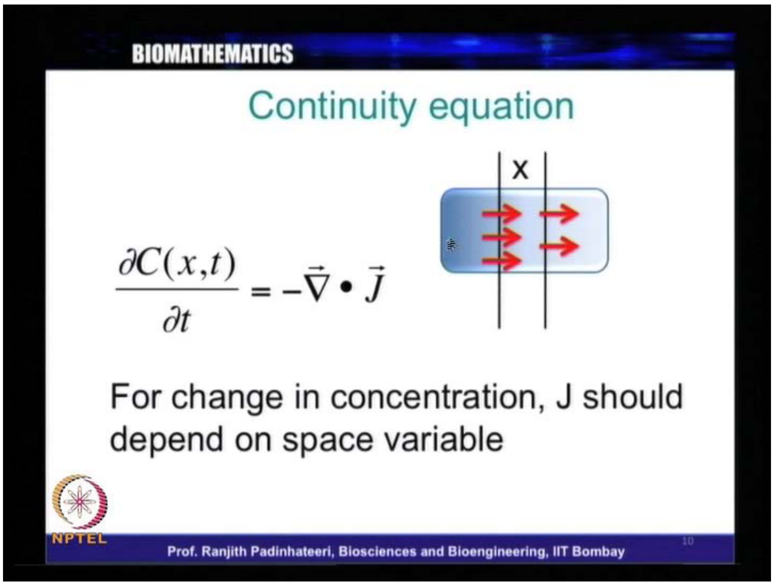

So, if we write this properly, what you essentially get something called the continuity equation, that is del C by del t is minus del dot J; del C by del t is minus del dot J is the continuity equation. What does it essentially say? It tells you is that the concentration will change if the flow here is different from the flow here. If more things flow in and less things flow out, the concentration here will change and the fact that it is more things are going in and less things are flowing out is represented by del dot J. As we saw, del dot J is del J by del x. So, for change in concentration, there has to be a del dot J; del dot J cannot be 0. So, as we just saw, we saw that we saw a few minutes ago that del dot J is can be written as del J by del x where J is some current.

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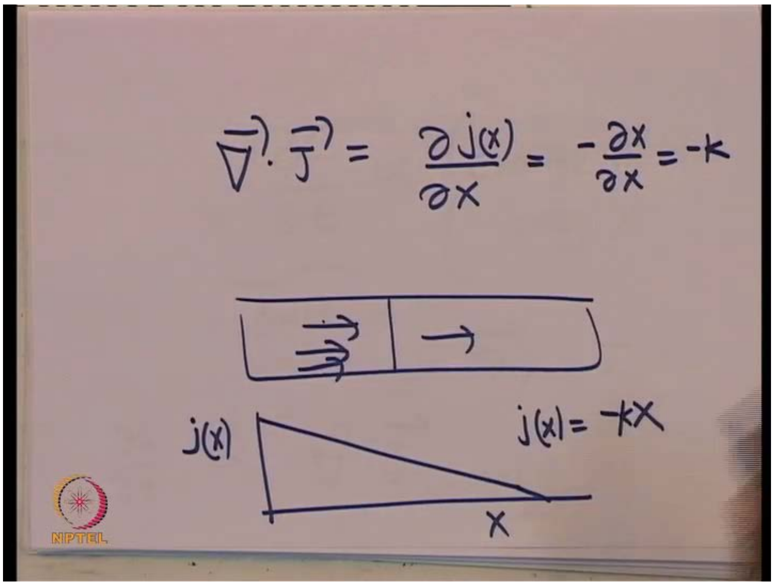

So, just imagine J is let us imagine that J is some particular…. So, if you imagine such a tube, again, let us imagine that are this particular point more things are flowing in and less things are flowing… less things are flowing out, that the flow here is more compared to the flow here. So, how do we plot this? So, we can plot J as again, a linear function. So, if you want J of x versus x can be plotted like this. So, J let us call J of x equal to minus x; let us say J of x is minus x. So, del dot J is minus del x by del x is minus 1; this is minus 1. So, if we want again write this k X here, some constant k is a slope. So, this will be minus k.

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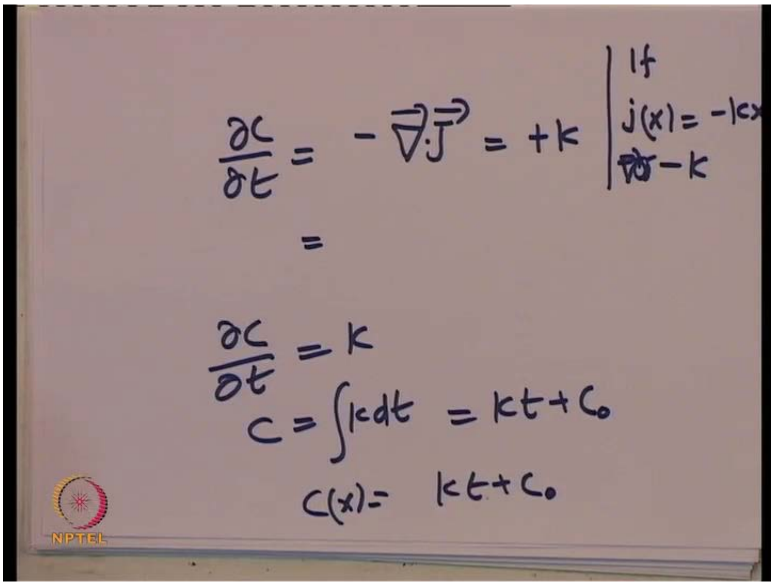

So, there is a constant– there is a constant decrease del dot J. Now, if we look at this equation that we just wrote that del C by del t is minus del dot J, and we found that del dot J is minus J del dot J is minus k. So, the J is plus k. So, we found that del dot J, if J is minus k X, del dot J is del J by del X, which is minus k.

And minus minus plus. So, what does it mean? Del C by del t is equal to k. So, it means C is equal to k d t integral. So, which means K t plus a constant this means the concentration will increase. So, what did? We say… we say that if more things flow in and less things flow out at this particular point x, we found that the concentration at that particular point x increases; C of x is equal to k t plus C. At this particular point x, the concentration increases with terms sometimes there this is a constant C 0. So, this is what essentially this is the equation that we learn, means that whatever we… if we decide the flow, we will tell you the how the concentration changes in time.

So, this is called the diffusion equation. Sorry. This is called the continuity equation. this essentially says that whatever things come in, if few things– if few molecules come in and only less molecules flow out, the remaining will stay there and that will change the concentration of the flow– the local concentration or the concentration locally, that is what it all says.

So, it essentially talks about the continuity of the flow. So, that's why it is called continuity equation, is the very famous equation and fluid mechanics and any concepts– any things– that is related to flow– the blood flow or anything that things anything flow related a few we have to understand this particular equation. Now, let us go the next step.

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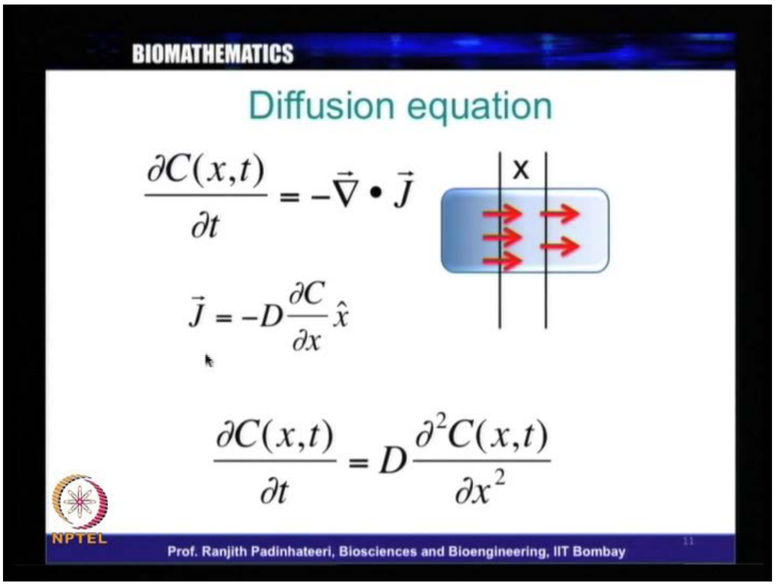

So, now we have… we have this equation– continuity equation. Now, let us… we also had… we also had that the J is minus D del C by del x this is something we said in the beginning; that means, the flow is essentially the change in concentration– the J is... if there is no change in concentration, there is no flow. So, if there is a change in concentration del C by del x, then there is a flow. If this is such a flow– the flow due to change in concentration, if J is given by minus del C by del x, and if we substitute this here in this equation, we will get something called diffusion equation, which is essentially this: del C by del t is equal to D del square by del x square– del C by del t is D del square C by del x square. So, this is what this equation is called diffusion equation. Let us quickly check how we got this– how exactly we got this.

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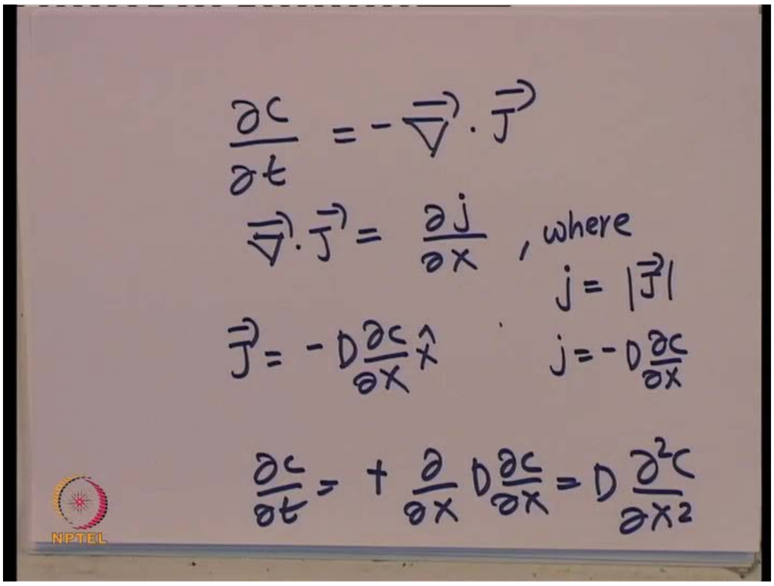

So, we what we had is that we had a continuity equation, which is essentially said that del C by del t is minus del dot J, and we said that del dot J is del J by del x, where J is the modulus of J. Whatever the magnitude of the J vector, we call J, then this is del dot J. Now, this is… We also learnt that J is minus D del C by del x, that is J is…. If there is a change in concentration, then there is a flow. So, this implies that…. What does this imply? This implies that little J here, so, there is an x cap here. So, little J here is D del C by del x with a minus sign. So, if you now substitute this little J here, what we get is del C by del t is del dot J, minus is minus plus del dot J del dot J is del x of D del C by del x.

So, this is D del square C by del x square, where D is a constant and this constant is called diffusion constant. We will discuss about this later. What are the significance of this, if etcetera…– we will discuss, but this is a constant that depends on the property of the of the medium that things are flowing; it depends on the temperature; it depends on viscosity, and so on and so forth. So, we will see what D is later, but this immediately gives you this particular diffusion equation which is called the diffusion equation– del C by del t is D del square C by del x square.

Now that we know diffusion equation, so the diffusion equation first derivative; this is also a differential equation. Actually, it is a partial differential equation, because these derivatives are the partial derivatives. What does that mean? As we said, partial derivative means that the derivative is concentration depends on x and time position and time.

So, the derivative here only depends on the time here and we are… and here only depends on x. So, these are partial derivatives. So, del C by del t is D del square C by del x square. So, this is the diffusion equation. Now, what does this equation describe? So, essentially, we found that this is… if you… if you take a continuity equation, which is…, which is common sense, which is saying that whatever things come in and minus whatever things go out will remain there– remain in this particular area– this common sense. So, if you take this and take that, the… if there is a concentration change, that will lead to a current. So, these two ideas if we combine immediately, we get this substitute for this J here; I immediately get this. So, this is called the diffusion equation.

Now, what does this equation describe? We said… we saw that it describes the flow– the concentration independent flow– there has to have a del C by del x for the flod conflod to happen, and flow itself should be depending on face, so that there is a diffusion. There is a… there is a concentration change in time. Now, let us… let us think afresh– let us think from the beginning a little– from the some… from a different angle; from a different perspective: let us think about what does this equation describe. So, this equation describes the following.

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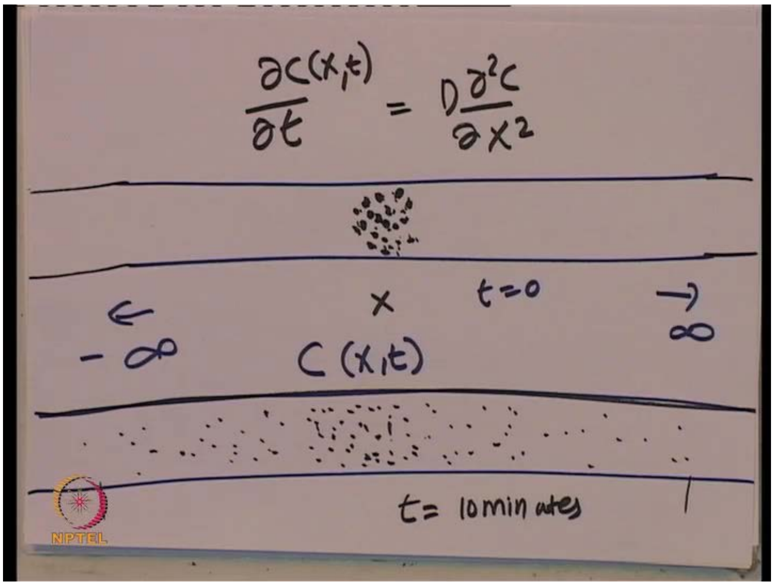

So, think about a very long pipe. So, very long pipe– this goes to plus infinity and this goes to minus infinity. So, very long pipe. Now in this pipe, at time is equal to 0, I am putting a few drops of some molecules– could be either ink molecules or it could be protein molecules, but a small amount of protein molecules I am putting here at this particular position x. So, at this…. Only at this particular point, you have protein molecules. So, if you plot the concentration as a function of space, it is only at x there is concentration. So, no other all… here all the concentration of this is 0, but there is water here.

There is water molecules everywhere. So, pipe, and this will go to infinity; now what happens after sometime. So, this is t equal to 0. Now, if you look after 10 minutes or few minutes, so let us say after 5 minutes, if you look at it, if you look after 5 minutes, so we have this pipe here and this particular point x, and what you will see is that this would have diffused. So, you would see still there will be lot of molecules here, but some of the some would have diffused away.

Most of it will be here, but some, slowly, would have diffused. So, the concentration would have decreased here, but concentration here increased. So, this is t equal to, let us say, ten minutes. So, it diffused from up to this. So, if you look at a particular molecule, that molecule might have reached somewhere here, or some other molecule might have reached somewhere here. So, the diffusion equation that we said tells you how does the concentration changes with time, so del C by del t. How does a concentration at any point x at any time t changes with time? So, if you look at the concentration here, it was 0 at t equal to 0; after 10 minutes, it became non zero. So, how does this concentration change?

That is what this diffusion equation tells you. If you solve this diffusion equation– if you solve this diffusion equation, what you get is essentially C as a function of x and t. How does the concentration change with space and time? At any… if you ask this particular position, what will be the concentration after 10 minutes? What will be the concentration after 20 minutes? What will be the concentration here after 30 minutes? What will be the concentration here after 40 minutes?

We can ask these questions, and this… by solving this diffusion equation we can get answer to these questions. This is a… these questions… the solution of this equation tells you what is the… what is the concentration at any point– at any time. Now, we can ask, knowing C of (x,t); we can ask another question, which is something may be interest to us: let say if we put a drop of molecule protein; let us say we put drop of ink or a some protein, let us say actin or any other protein that you like. If you put certain amount of protein into this particular place, you can ask the question– after 10 minutes, how far this would have diffused? That is also even, and that is also basically given by this the solution of this equation.

One should be able to calculate how much this would have diffused on an average. So, if you do many experiments, things would have diffused, and how much on a average, how long… how sorry how far this protein molecules would have diffused in, let us say, 10 minutes. If you wait 10 minutes on an average, how far they would have diffused? So, this is a question that can be asked from this particular equation. So, how do… how do we know C of this? Let us say we have this equation, which is diffusion equation, and we solve we can get this C of (x,t). How do we know what is the distance these above things traveled? So, how far this would have traveled? So, you can calculate few quantities.

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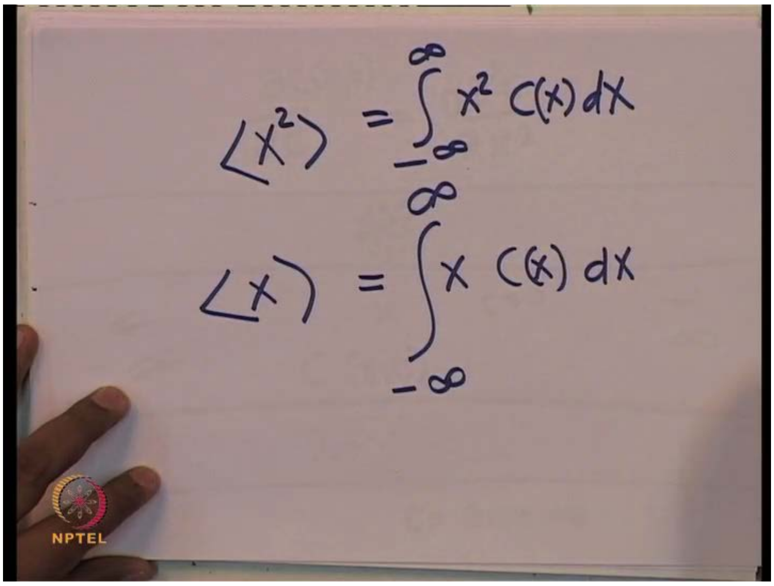

So, we can calculate the average distance this would travel. We can calculate X square average and X average– the… this angular brackets means average. So, this is the average amount of distance the protein would have diffused. So, the…. So, if you… if you want the RMS– root mean square distance, we have to calculate X square average and find square root of this. So, we what we want to understand is that how much… how far these are. So, what is this? What is X square average and what is the X average?

So, if you know that if you want to calculate average of any quantity, you have to know how much average amount of protein in a… in a… in a solution. You have to know at every point how much is the protein, that is C of X, and if you multiply with X and integrate, we get average. So, X average has to be X C of X d X minus infinity to infinity. So, if we do this, so, at every point, there is C of x is a concentration, and if we say that X is the place special, area will X average is given by this particular integral? You can see… we can see this is just like finding the average mark or average any quantity; we will describe we will discuss this more carefully– little more carefully– when we discuss statistics, average, etcetera, but from the common sense, you know that average is nothing but this, and we can also calculate X square average in this particular way.

So, if we find these two integrals and if we know this C of X, we can calculate these integrals and get these two average values. So, for the movements, even if we do not understand why this integral has particular form, sorry, why this average has this particular form, just keep in mind– just realize that this is true; just understand that this is true, and when we discuss statistics, we will… we will show you clearly that X average is nothing but this; X square average is nothing but this, but for the movement, just believe me, and then just know that X average is integral X C of X d X and X square average is integral X square C of X d X.

So, we have to calculate this, that is, we can… knowing C, we can calculate this. So, that is our aim. We can… we can know how far things would diffuse in 10 minutes– that is something useful quantity. If you put a protein and wait for a 10 minutes, we can know how far it would have diffused; this is very useful information, something that our mathematics– using mathematical techniques– we can calculate. So, we understood diffusion equation; we understood continuity equation; and we used some ideas from vector and all that and calculus to learn about this, and we know that it can describe the phenomenon of diffusion. This mathematical equation can describe the phenomenon of diffusion. So, with this, we will stop today's lecture.

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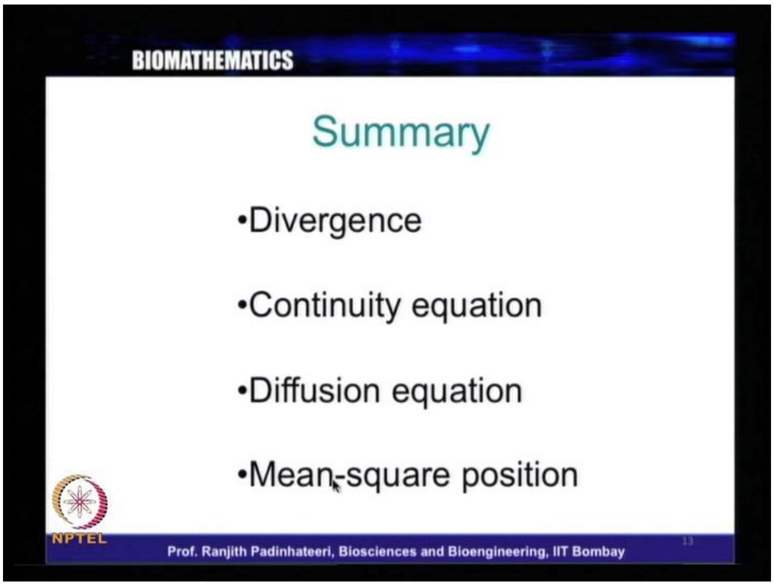

So, let us summarize what we learnt. So, what we have learnt is that we learnt something about divergence, which is del dot J. We said this divergence; we learnt about continuity equation; we learnt about diffusion equation; we learnt about this equation that del C by del t is D del square C by del x square is diffusion equation, and that describes the phenomenon of, say, as molecules diffusing along a pipe or any container.

Anywhere you have a concentration difference, those things will diffuse and described in diffusion equation, and we also average position and mean square position of any particle after a particular time can be calculated by some equation, which is x square average and x average x square. Sorry, it is a x square average and x average will tell you how much things would have moved. So, given….

So, this is… this is what we learnt in this… in this class, and we will we will learn more things about diffusion in the coming classes. We will also learn, when we learn statistics and probability, etcetera, we will learn little more interesting things about diffusion equation in different contexts, but for today's class I want all of you to just understand what is diffusion equation and where is it coming from. It is coming from some common sense of continuity, and that can give you information about how does the concentration vary in space and time. So, with this we will stop today's lecture here. Thank you.