Introduction to Dynamical Models in Biology
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Lecture 10
Stability of Steady States

Hello. Welcome to module 4 of second week's lectures on introduction to dynamical models in biology. Earlier we have discussed about steady states. Situation where the dependent variable is not changing with time. In this module, we will learn about stability of steady states. Before we try to understand what do you mean by a stability of a steady state, let us recapitulate about steady states.

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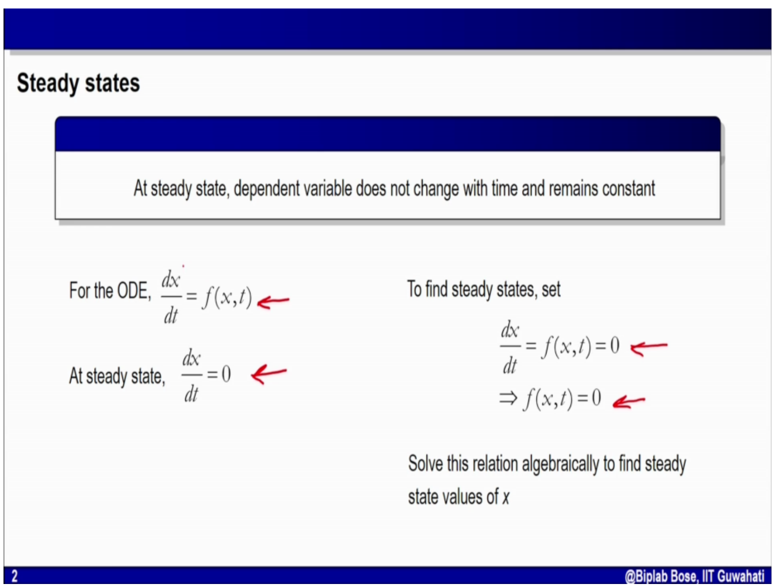

The basic concept of steady is like this. You have a dependant variable say X, that is changing with time so the situation when the dependant variable does not change with time is the steady state of the dependant variable or the (sa) steady state of the system. So to get that steady state, what we can do, mathematically we can write, suppose the system is represented by a ODE as given in the slide, $\frac{dx}{dt}=f(X,t)$ so by definition at a steady state, $\frac{dx}{dt}$ will be 0. As $\frac{dx}{dt}$ is 0, that means X is not changing with time. To find the value of X for which we have steady state in this system, what do you do? As we have discussed earlier, we set $\frac{dx}{dt}=0$, that means $f(X,t)=0$. This is coming from the ODE that has been defined here so once I have $f(X,t)=0$ then I can solve this one algebraically and find out the value of X for which we will have $\frac{dx}{dt}=0$ so that is our steady state.

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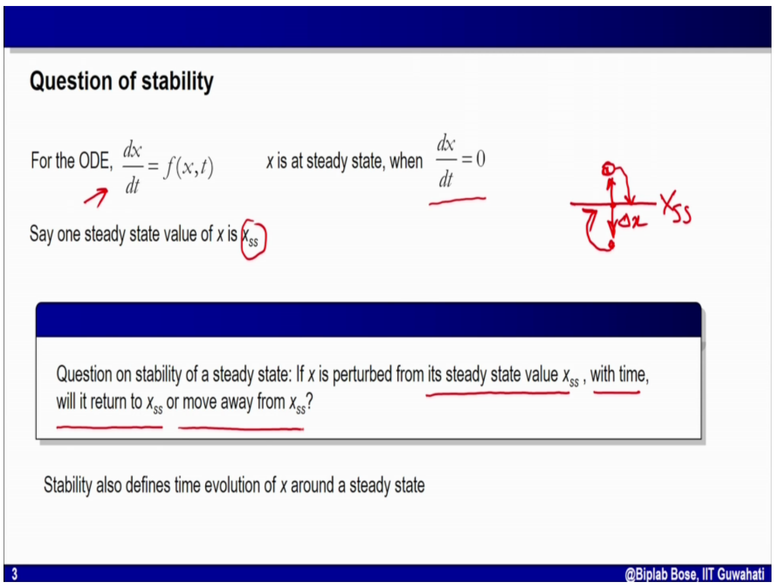

Now let us try to understand the concept of stability of steady state. So suppose I have a ODE as given here, $\frac{dx}{dt}=f(X,t)$. As I discussed just now, at steady state, $\frac{dx}{dt}$ will be 0 so let us call the value of X for which $\frac{dx}{dt}=0$ in this ODE as X_{SS}, the steady state value of X, X_{SS}. So, now suppose the system is at X_{SS}, that means right now the value of X is at X_{SS} so the system is in steady state or $\frac{dx}{dt}$ is 0. Now suppose I slightly pull X away from X_{SS} that means I am here at X_{SS} and suppose I pull it up. The value of X, I pull it up somewhere here. Then the question come what will happen to the system as time progress will X come back to its steady state value or if I do the opposite thing, I just pull it down, the value of X, I reduce the value of X slightly from X_{SS}. So suppose I reduce the value of X slightly to here, this is a small change in X, ΔX .

And then I ask the question, what will happen as time progress? Will the value of X again go back to X_{SS}? So if I have a situation where I disturb the system from its steady state with slight disturbance we call it perturbation, so if I perturbed the system slightly and then allow it to evolve with time, if the system come backs again to the (sa) same steady state, we call that steady state as a stable steady state so as I have written here, If X is perturbed from its steady state value X_{SS} with time, will it return to X_{SS} or move away from X_{SS} so this is the question that we ask for stability analysis of a steady state.

These types of analysis, these types of questions that if I move away or if I perturb the system slightly from a steady state, will it collapse back to its steady state or it will move away from its steady state is also important if I want to understand the dynamics or time evolution of X around the steady states so another purpose of understanding the stability of analyzing the stability of a steady state is to understand the time evolution of X around a steady state. So let us try to understand the stability issues of a ODE based systems with using this concept.

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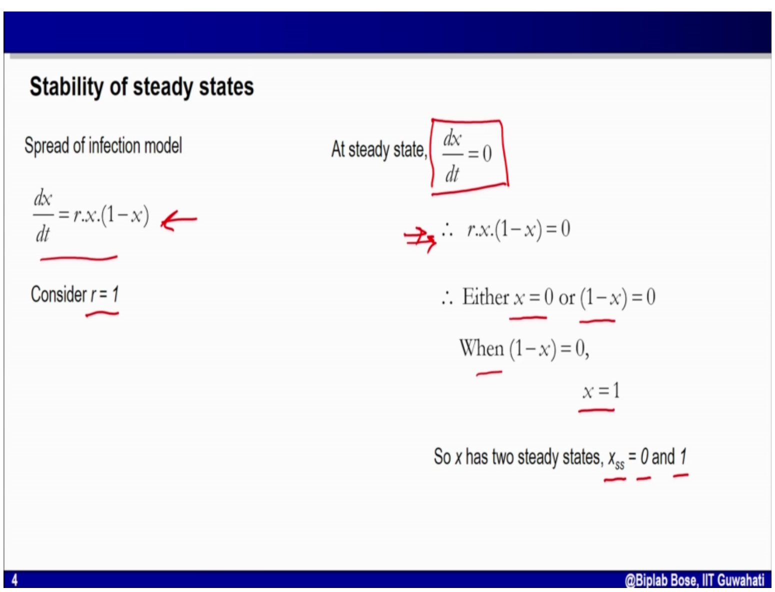

So let us specific example. We will go back to our example of spread of infectious disease so the system is represented by a ODE, $\frac{dx}{dt} = r * X(1-X)$. If you remember, X is the fraction of population that is infected with the disease. Let us consider *r=1*. It will be easy for our calculation. And now we will find out first the steady states of the system and then we will try to understand the issues of stability of the steady state for this system. Let us see at steady state by definition $\frac{dx}{dt}$ =0, so at steady state $\frac{dx}{dt} = 0$ that means from the ODE given here, I get $r.X(1-X)=0$.

So now $r=1$ that means using this relationship, this equation, $r.X(1-X)=0$, it is true if and only if your $X=0 \vee 1-X=0$ so both of them are possible either or. So if I consider $(1-X)=0$ then I can use simple algebra to get $(1-X)=0$, I get $X=1$ so this system has 2 steady state. X_{SS}, that steady state value of X is either 0 or and 1. For both the values, the system will be in the steady state so let us try to understand the concept of stability of these 2 steady states using direction field.

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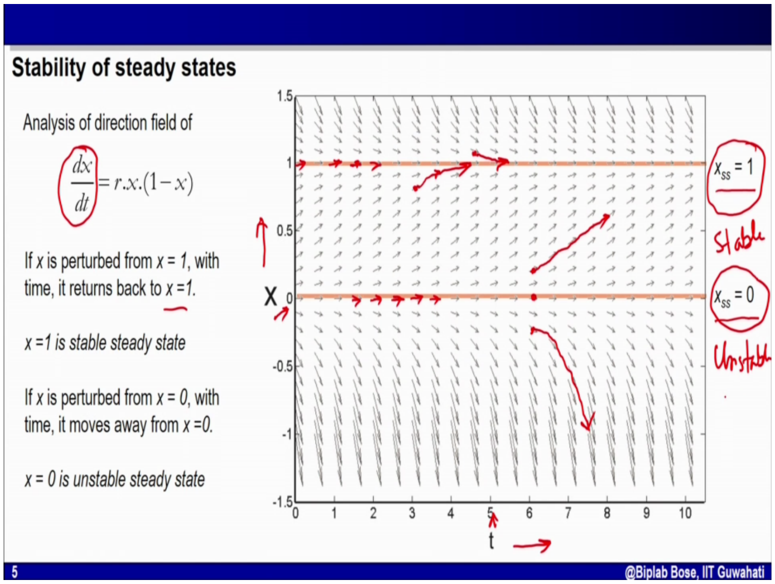

If you remember direction field is nothing but a graphical representation of how the dependent variable changes with time. So to draw a direction field, what we require in the horizontal axis, I have plotted time T, in the vertical axis, I have plotted X and now, I have divided the whole plane, X versus T plane in grid points, equidistance grid point and at each grid point, I have drawn a arrow representing the differential derivative $\frac{dx}{dt}$ and I have filled this whole space, whole plane of X versus T with those arrows so I have a direction field. Now just now we have calculated at $X=1$ and at $X=0$ we will have steady state so let us look into the direction field. At $X=0$, if you move along the time axis, you will see all arrows, all arrows are horizontal that means here along $X=0$, $\frac{dx}{dt}$ =0.

Similarly at $X=1$, all arrows are actually horizontal. That's why I have drawn 2 reddish lines to represent that we have steady state at $X=1$ and $X=0$ the direction arrows which represent $\frac{dx}{dt}$ are all 0s. Now let us try to understand the concept of stability. Suppose I am at time 0.5 here and I am at steady state of $X=1$ so I am at time 0.5 and $X=1$ so the system is at steady state. I pull X a bit higher and drag it here, slightly higher. The question is if I leave X there and allow to progress with time, evolve with time, what will happen to X? As I see the arrows, as I have pulled the X here, following these arrows, X will fall back on $X=1$.

Let us take another point. Suppose I am somewhere here at time 3, I am here. Now, I pull down the value of X slightly here in this point. So now if I leave X with time, X will evolve along along this direction field, along direction field and ultimately collapse at $X=1$ so from both the side of $X=1$, if I perturb the system, that means if I change the value of X slightly higher or slightly lower with time, X will again come back to value of 1. That's why this $X=1$ steady state is called stable steady state. What is happening, if you disturb a system which is at a stable steady state slightly then in time again it will come back to that steady stable state.

Now let us into steady state $X=0$. Suppose right now X is here, at time 6 at $X=0$. I change the value of X slightly higher here. Now if I leave X to evolve with time now, X will follows these arrows and move away from $X=0$. Suppose I take a value slightly below the original value of $X=0$ and now if I leave it leave it and allow X to evolve with time or change with time then following this direction field, X will move in this direction.

As you can see in this direction field, the arrows are telling me that if I change value of X slightly away from 0, they will move away from $X=0$. That means if the system is at steady state with $X=0$ and now you slightly disturb or perturb the system, the system will not stay at $X=0$ anymore, it will not come back to $X=0$ anymore rather it will go away from it so that's why $X=0$ is a unstable steady state. So we have defined 2 things, stable steady state, unstable steady state. In stable steady state, if the system or the dependent variable, if it is at stable steady state, if I perturb it slightly with time, it will come back to that steady state, just the opposite for the unstable steady state. If the system or the dependent variable is at a unstable steady state, if you do nothing to it, it will stay at that steady state but if you perturb it (sli) slightly with time it will never come back to that steady state anymore. It will diverge away from it. And that's why it is called unstable steady state.

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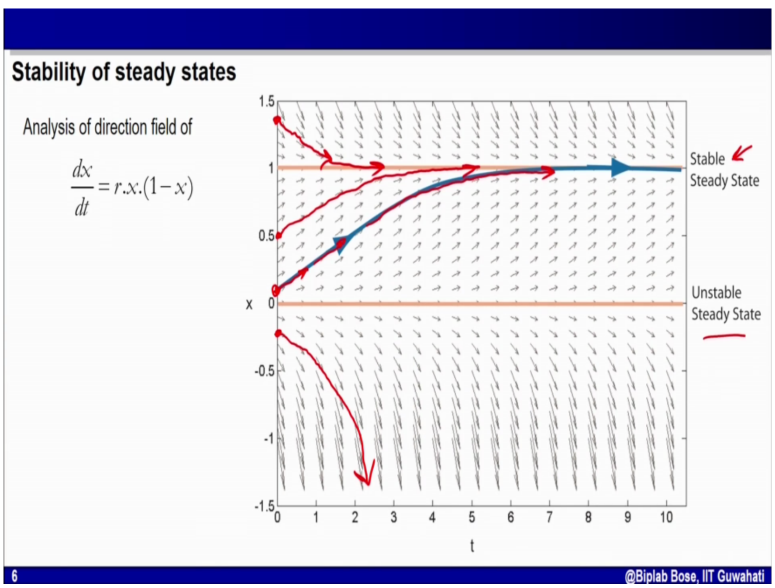

Let us explore the direction field further to understand how the dynamics happens between a stable and unstable steady state. It's the same problem. As I have shown the direction field here in the graph, I have 2 lines reddish color representing 1 unstable steady state, the other one is a stable steady state. So suppose at time equal to 0, I am here. X is here. Then it is slightly away from unstable steady state. As the arrows in the direction field shows with time, X will change and increase and it will move and ultimately will collapse at the steady state which is stable and its value is $X=1$.

If I take another point, suppose at $T=0$, X is 0.5 then following these arrows, I will again, the system will collapse at the stable steady state $X=1$. In this particular problem, X can vary 0 to 1, so in a negative value of X does not make any biological sense, any value of X bigger than 1 also does not make any sense because we are measuring the fraction of people were infected were infected with the diseases but mathematically let us try to analyze further. Suppose I start with the initial value of X somewhere here. Slightly below the unstable steady state in this direction field.

So with time, X will evolve along these arrows and diverge away from the unstable steady state and obviously it does not have any other stable steady state on this lower direction so it will uh keep on decreasing whereas if I start a $T=0$ at a value higher than 1 then along this direction field arrows, the system will move and eventually collapse at the nearest $X=1$ steady state, it's the stable steady state. So this graphical analysis shows that by drawing and marking a (steady) steady state on the direction field, I can identify based on the arrows of the direction field, I can identify whether the steady state is a stable one or unstable one and I can make a prediction about a time evolution of the dependent variable along this direction field from a unstable to a stable steady state.

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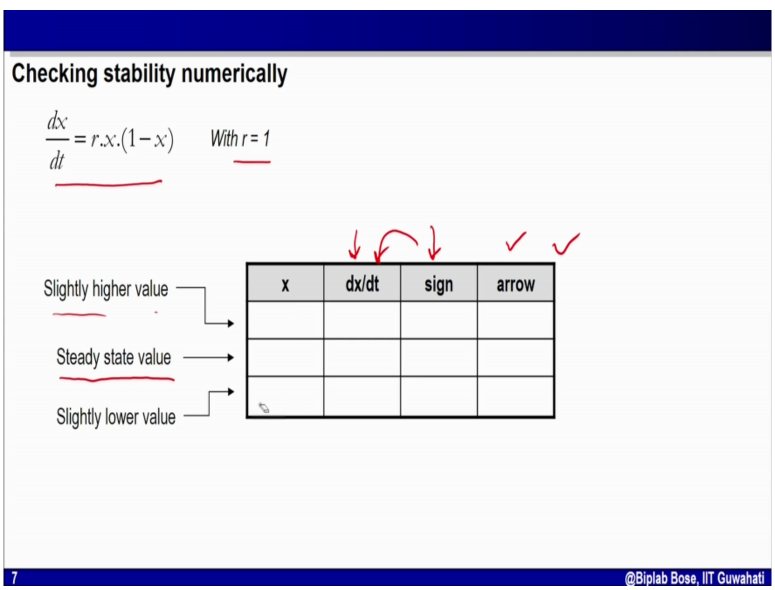

Now let us look into another way of analyzing and identifying stability of steady states so let us take the same example. $\frac{dx}{dt} = r.X.(1-X)$, we have considered $r=1$ for simplicity. What we will do here, we will draw a table with 4 columns. First column will represent the value of X, the second column is the derivative of X as given by the ODE. Then the third column is for the sign, sign of the derivative and then we will have a column for arrow. I will explain what type of arrow we will put here. One important issue of doing this type of analysis is that I have 3 rows. The middle row represent a particular steady state value. On the upper row, just above the steady state value row, you take a value slightly bigger than the steady state value that you have taken in the middle row and in the lower one here, you can see take a value slightly smaller than the steady state value.

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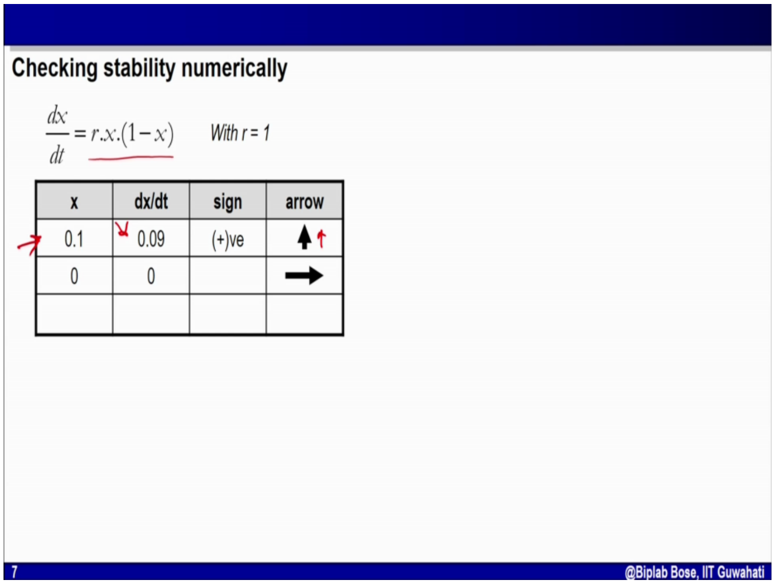

Let us see with examples. So if you remember just now in few slides back, we have calculated that we have 2 steady states for this system, $X=0$ and $X=1$ so let us start analyzing $X=0$ first so I have a table here with 4 columns, 3 rows, in the middle row, I put the steady state value of X, this is my X steady state. That is 0. Obviously the $\frac{dx}{dt}$ for this $X=0$ is 0 and it doesn't have make any sense of taking talking of sign of 0 but I put a arrow which is horizontal here.

Horizontal because we know that from direction field concept, the arrows in direction field when the derivative is 0 is horizontal so you have your horizontal arrow here. Now let us take a slightly higher value of X which is slightly higher than the steady state value. The steady state value is 0. I have taken a slightly higher value that is 0.1. So I put this 0.1 in place of X in this differential equation, I get $\frac{dx}{dt}=0.09$. The sign of this derivative is positive so I give put a arrow pointing up that means $\frac{dx}{dt}$ is saying that X will increase so I put a arrow up as shown here.

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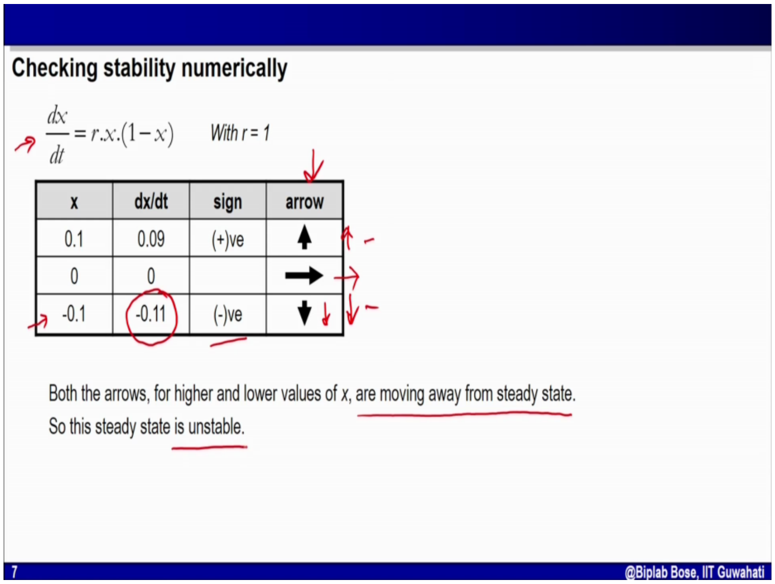

Let us now take a slightly lower value of X from slightly lower than the steady state value of 0 so slightly lesser than 0 is here -0.1 . I again use this ODE to calculate $\frac{dx}{dt}$, $r=1$, $r*(-0.1)*(1-(-0.1))$, if you do that calculation, you will see $\frac{dx}{dt}=-0.11$. So if $X=-0.1$ using the ODE given above I get the derivative is equal to -0.11. The sign of the derivative is negative and as the sign is negative, that means it is saying X is decreasing with time, that's why I put a arrow down. So now once you have made this table focus on this row, row this column of arrows, this is your steady state arrow and you have 2 arrows, 1 is pointing down and 1 is pointing up.

That means with time, if you disturb X then with time, X will move away from 0 so as both the arrows, both these arrows are moving away from steady state, as both the arrows are moving away from steady state, this steady state is unstable. It is a simple way of understanding the stability or calculating the stability of a steady state without drawing the complete direction field as you understand to draw the complete direction field, you have to draw lots of arrows. Here we are only calculating the arrows and their direction will be near the steady state.

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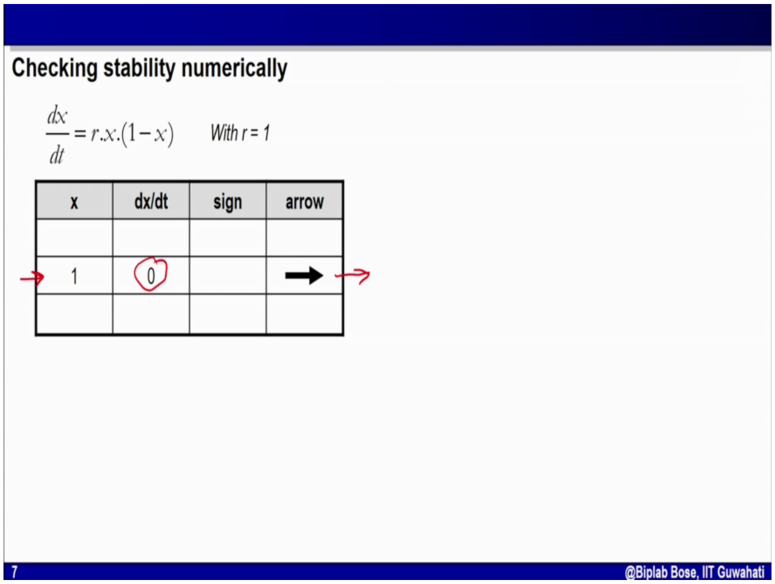

Let us try for the next steady state if you remember, another steady state for this ODE is $X=1$. I put in the middle row. Middle row $X=1$, $\frac{dx}{dt}$, the derivative of that will be obviously equal to 0 and I put a arrow which is horizontal here because $\frac{dx}{dt}=0$.

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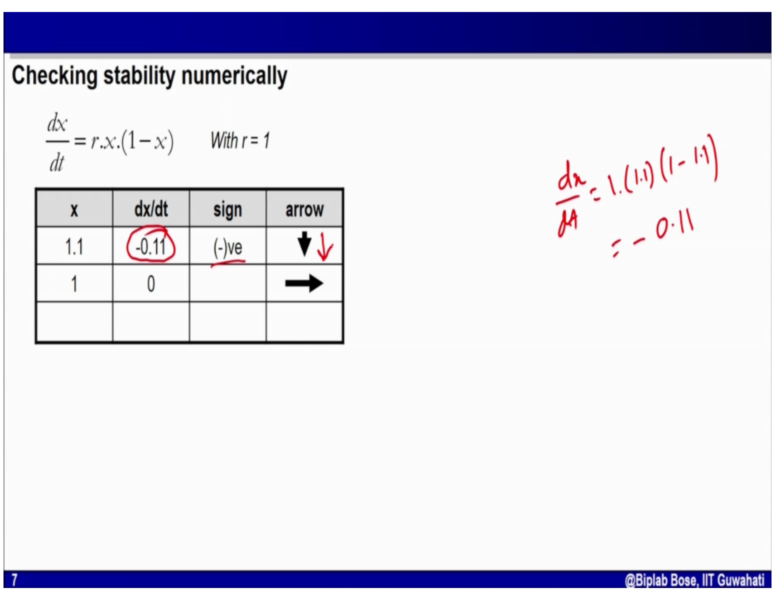

Now I will take a value slightly higher than 1 so what I can take, I have taken $X=1.1$, that means $\frac{dx}{dt}=1*1.1*(1-1.1)$. so this give me a value of -0.11 . So that value is written here. The sign of the derivative is negative so with time, X is decreasing so I put a arrow getting downwards.

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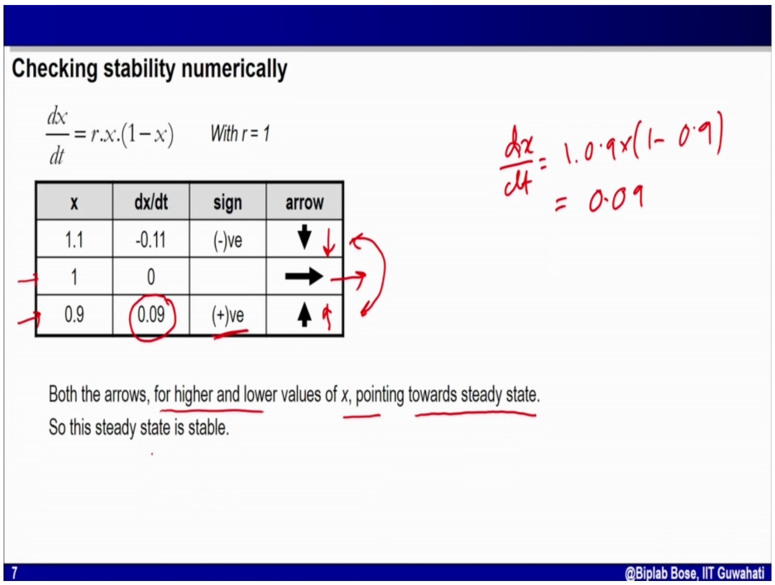

Let us take a slightly smaller value then the steady state value, steady state is 1 I take a smaller value 0.9 so then in this case $\frac{dx}{dt}=1$ that is $r.X=0.9*(1-0.9)=0.09.$ so you have 0.09 here in this column. As a sign of this derivative is positive, that is written here. As the sign is positive, that means with time, X will increase so our arrow in the arrow column is a up one so what we have, I have a horizontal arrow for the steady state. One arrow, slightly lower value of X, I have a arrow pointing towards this one and I have another arrow pointing downwards to this one so that means in this case both these arrows are pointing towards that steady state so for both the arrows, for higher and lower value of X are pointing towards the steady state.

That means this steady state is a stable steady state. Again I have not drawn the complete direction field. Rather I have calculated the derivative and the direction of the arrows just near a particular steady state and have tried to decipher the stability of that steady state. This is much simpler than drawing the complete direction field and analyzing. In this way, you can analyze the stability of any steady state. The trick is first identify the (possi) all possible steady states and then make this table for each steady state and see the sign of the arrows along the horizontal one that is the steady state one. If both the arrow point towards the steady state then it is a stable one. If both the arrow point away from the steady state then it is a unstable one.

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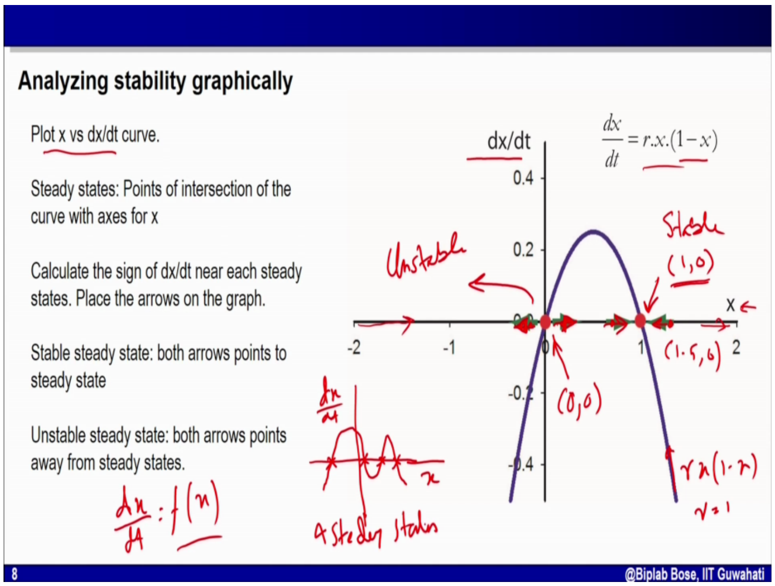

There is another way of looking at and analyzing the stability and representing it graphically. For example here what I have shown plot X versus $\frac{dx}{dt}$ that means in the vertical axis, you put $\frac{dx}{dt}$, the derivative of X and in the horizontal axis, you put X. So now in this plane, I can draw the ODE that is $\frac{r*dx}{dt} = r*X*(1-X)$. so this is the function of X so I can plot it, plot, I have curve, I have shown this one, this blue line is this curve so this is nothing but $r*X*(1-X)$. where $r=1$, I have taken $r=1$ to draw this one.

Now what is steady state in this curve, see we know $\frac{dx}{dt}=0$ at steady state. $\frac{dx}{dt}=0$ along this horizontal line. In all this line, horizontal (li) axis, $\frac{dx}{dt}=0$ so that mean at all the point where this blue curve intersect the horizontal line or either the horizontal axis are the steady state so the 1 intersection point is this one, the red dot I have shown. The other intersection point is this one and you can see easily, this first intersection point this one is nothing but (1,1), 1 is the value of X and 0 is the value of $\frac{dx}{dt}$, the other intersection point is at (0,0) when $X=0$, $\frac{dx}{dt}$ is also 0.

So just looking at this graph, I can identify the number of intersection points where the ODE, the curve representing the ODE, is intersecting the axis for X and those are my steady states. Then I can do the similar calculation the way I did in in the previous slides that I take a value slightly bigger than a steady state. For example, let us take this, this one (1,1) so I take a small value of suppose 1.50 and then I calculate what is the derivative at that value of derivative at that time, $\frac{dx}{dt}$ at that point.

So obviously that will be negative as you can see 1.5 > 1 so $1-1.5$ will be negative so it will be pointing towards the steady state as X will decrease with time. Similarly I can take a points slightly before the steady state that is the lesser value than and I can calculate $\frac{dx}{dt}$ there and based on the sign of that, I can easily see the sign will be positive. That means X will increase with time so I put a arrow towards the steady state so as both the arrows are pointing towards the steady state (1,0) I call this as a stable steady state.

You can do the equivalent thing for (0,0) point, this is also a steady state so I take a value slightly bigger than $X=0$ and calculate $\frac{dx}{dt}$ there and the sign of that $\frac{dx}{dt}$, you can see say for example, $X = 0.1$, you can check it. $X = 0.1$ will give a $\frac{dx}{dt}$ positive that means my arrow will point like this. X will increase with time whereas if I take a smaller value of X say -0.2 something like this and if I, we can calculate $\frac{dx}{dt}$ using the ODE, r $\frac{1-X}{¿X*¿}$), I will get a negative sign so the arrow points to lower value of X.

So as I can see for (0,0) point, both the arrows are moving away from (0,0), this is my unstable steady state. This method of graphical representation is good because by the (inter) number of intersection points, I can easily identify how many possible steady states are there. If I have ordinary differential equation something like this, suppose I have an ordinary differential equation $\frac{dx}{dt} = f(X)$ and if I plot $\frac{dx}{dt}$ in the vertical axis and X in the horizontal axis and then if I plot this function and suppose I have the function is like this, that means I have 1, 2, 3, 4 intersection points, that means this system has 4 steady states.

So this type of representation in $\frac{dx}{dt}$ versus X plane is very good to identify number of possible steady states but you have to remember that you have to be able to plot the function properly in this plane. Let us now move into another type of stability. We have discussed stable steady state, unstable steady state. What if we have a mixture of both? Can there be?

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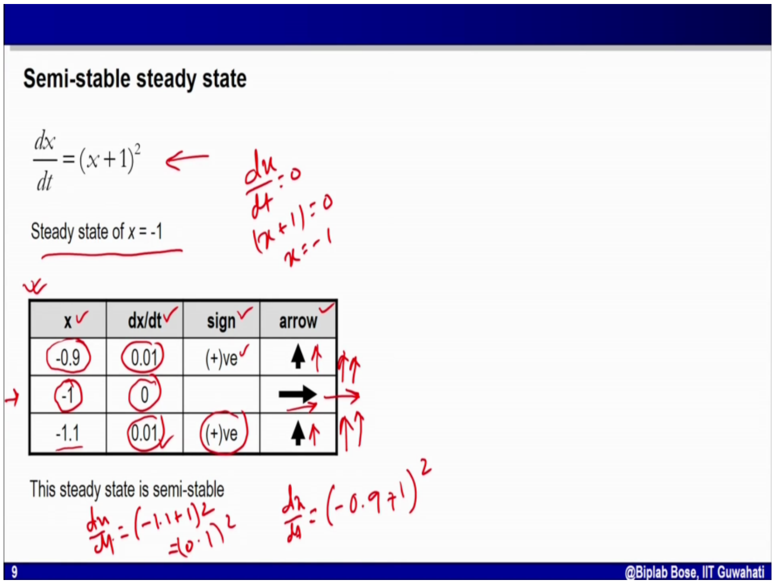

Let us this example given here, the ODE is $\frac{dx}{dt}=(X+1)^2$, I can easily calculate uh the steady state, let us put $\frac{dx}{dt}=0$, that mean $(X+1)=0$ so $X=-1,$ so the steady state is -1 . I want to check its stability. I will take the numerical route rather than plotting the direction field. What you need? You have to make a table, the way I have shown here, 4 columns, 3 rows, first columns is for X, second column is for the derivative, third one is for the time and the fourth one is the arrow.

In the middle row, I put the value of the steady state, that is -1 . Obviously if $X=-1$, $\frac{dx}{dt}$ will be 0 so I put a horizontal arrow. Let us now take slightly higher value. Remember the higher value, the value greater slightly greater than the steady state will be in the first row so I have taken -0.9 and you can easily calculate so $\frac{dx}{dt}=(-0.9+1)^2$ so the value is 0.01 and obviously, the sign will be positive because you have a square term so you have positive sign.

So I can put the arrow representing up whereas if I have, if I take 1.1 (slig) $-¿$ 1.1, slightly lower value than $X=1$, again although value of $X=-1.1$, as we have a square term $X+1$ square term, what we will get, we will get a positive value. We can calculate it easily for example, $\frac{dx}{dt} = (-1.1+1)^2$ so that will be uh 0.1^2 and you will get this value so obvious that this positive value and the (sam) the arrow will be pointing up. So in this way, you can see here, this is the horizontal arrow representing the steady state. 2 arrows, slightly lower of X, slightly higher higher value of X are 1 is pointing towards the steady state. Another one is going away from the steady state. So in this case, if you approach from the one side, you will reach towards the steady state, if you start from another side, you will go away from this steady state so this type of system is semi stable.

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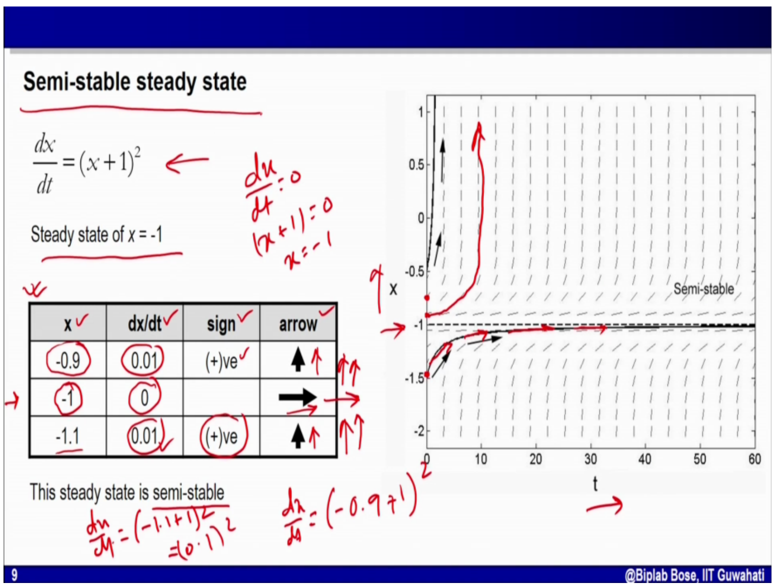

Let us look into the direction field of this. I have T in this axis, X in this axis. What we have plot, we have plotted the arrows, the arrow head has not been given to keep it clear. This has only one stable, one steady state that is −¿ 1 so I have drawn a dotted line here. If you start from a lower value than the steady state, suppose this −¿ 1.5, this arrow says that you will move along this and collapse at this steady state but if you start at a slightly higher value, say suppose somewhere here then you will move along this and go away. So if I start from the lower value, I collapse at the steady state and if I start from a higher value, I move away from it, so from one side it has stable type behavior, from the other side, it has a unstable behavior. That's why this is called semi stable steady state.

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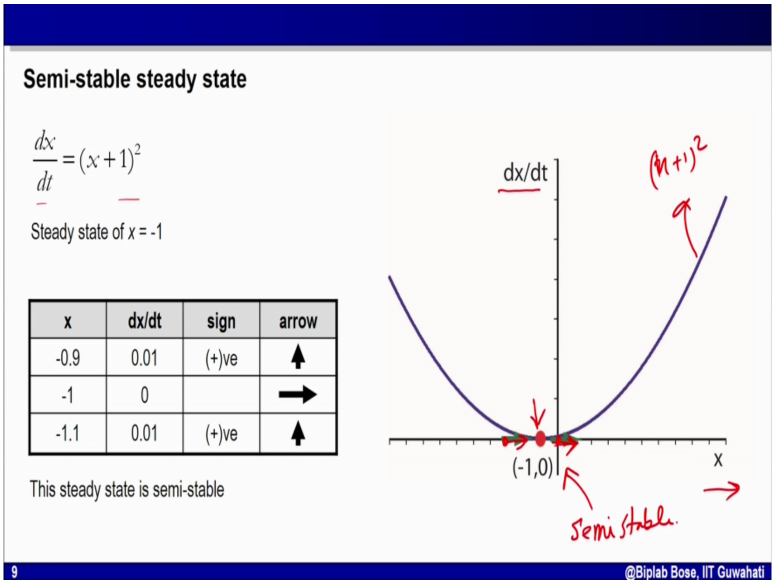

Now look let us look into the same system in $\frac{dx}{dt}$ versus X plot. So my equation is $\frac{dx}{dt} = (X+1)^2$. I have plotted X in the horizontal axis, $\frac{dx}{dt}$ in the vertical axis and this curve represent $(X+1)^2$. So you can see this curve is intersecting the horizontal axis, that the axis for X only at 1 point that is (-1,0) that means this system has only 1 steady state. I can take a smaller value than −¿ 1 and then I can calculate the $\frac{dx}{dt}$ as the $\frac{dx}{dt}$ is positive so it is moving towards it. I can take a bigger value of X as and calculate $\frac{dx}{dt}$ there, obviously the $\frac{dx}{dt}$ is positive so X is increasing with time so my arrow is in this direction so 1 arrow is moving towards the steady state, one is moving away from the steady state. So this one is semi stable, so it is semi stable steady state.

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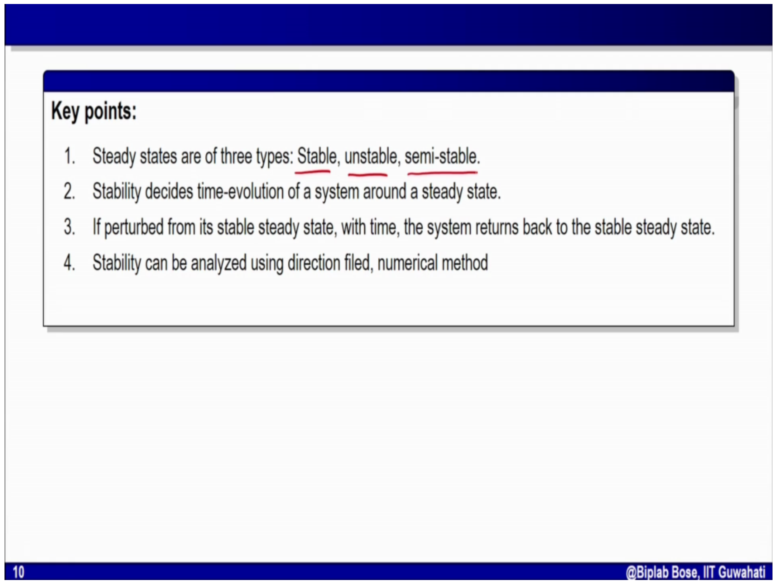

Let us now jot down what we have learned here. We have seen that steady state can be of 3 types. Stable, unstable and semi stable. Stability analysis also helps us to understand how the system or the given variable will evolve with time around the steady state if you have a stable steady state and if you start near that stable steady state, you will collapse at steady stable steady state. If you are at the stable steady state and if you perturb the system slightly, it will again come back and collapse at that stable steady state. If you are at unstable steady state, if I perturb the system, it will move away from that stable steady state. In case of semi stable one, if you are near the steady state, from 1 side, you can reach the steady and if you are on the other side of the steady state, with time you will move away from it.

We have shown 3 methods of understanding and analyzing the steady state stabilities. One method is using the direction field. You draw the direction field, X versus T plot with the arrows showing the direction field and then you see how the arrows are moving towards the steady state. Are they moving towards the steady state or they are moving away. A shortened version of that is numerical method the way I have discussed in a tabular form. That is much easier to do because you have not to draw the whole direction of it so I always advise to do that because it is very easy to execute and from that, you can easily decipher whether a particular steady state for a ODE is stable or unstable or semi stable.

In some cases, you may like to plot, the $\frac{dx}{dt}$ versus X plot and then draw the function representing the ODE there and then curve will intersect the horizontal axis that the axis for X in 1 or less or more time. So depending upon the intersection point, number of intersection point, you have so many steady states and you can similarly analyze the stability of the steady state at each of this position calculating the $\frac{dx}{dt}$ around it, the sign of $\frac{dx}{dt}$ around it. That's all for this module. We will continue discussing about steady state and stability issues further. Thank you.