Introduction to Dynamical Models in Biology

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Lecture 11

Phase Plane Analysis - 1

Hello. Welcome to module 5 of second week of our course on introduction to dynamical models in biology. In last couple of lectures, we have discussed about direction field, steady state, stability of those steady states. If you remember, all the example that we discuss there are involving just one single ODE. That means we have only one dependent variable which is (cha) changing with time. But for most of the real life problems in biology and others, actually the number of dependent variable is more than 1. So if you remember the initial discussions on writing ODE based model, if I have more than 1 ODE or more than 1 dependent variable, for each dependent variable, I have to write a ODE representing rate of change of that dependent variable with time.

That means if I have 2 dependent variable, I will get 2 ODE. If I have 4 dependent variable, I will get 4 ODEs. These ODEs together will be called system of ODEs so in this module today, we will try to understand the issue of stability for a system of ODE and for a single ODE, we have done some direction field analysis, we will have an equivalent thing here which we will call phase plane analysis. So let us start.

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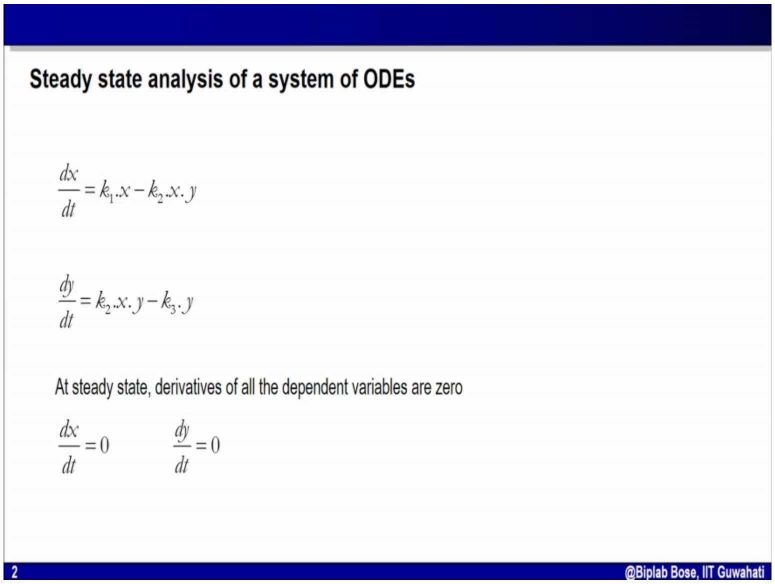

So suppose let us take a simple example. We are taking the example of predator prey model so in a (pre) predator prey models, there are 2 dependent variables. The population size of the prey and the population size of the predator, one represented by X and the other one represented by Y.

So I have some generalized equation written here. $\frac{dx}{dt} = ¿$, this is the ODE is $¿K1*X - K2*X*Y$. For a predator, the (ra) the rate of change is $\frac{dx}{dt} = K2*X*Y - K3*Y$. X and Y here are dependent variable changing with time. K1, K2 and K3 are the parameters. Let us don't bother about the parameters right now.

Now how do we define the steady state for this system? For a single ODE, we have defined steady state (at) like this. Steady state is the value of X, the dependent variable for which its derivative is 0 so if you give me a ODE, $\frac{dx}{dt}$ is a function of X and T then I make $\frac{dx}{dt} = 0$ and identify the value of X algebraically for which $\frac{dx}{dt}$ will be the 0. For a system (re) represented by more than 1 dependent variable, for example this predator prey system, the steady state is the situation where both the derivatives, derivative of both the dependent variable are 0. So if I go back to this predator prey model, I have 2 dependent variables, X and Y so I have 2 derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$, so steady state is possible or achieved when $\frac{dx}{dt}$ is also 0 and $\frac{dy}{dt}$ is also 0. Both the derivatives are 0 so at steady state, derivatives of all the dependent variables are simultaneously 0.

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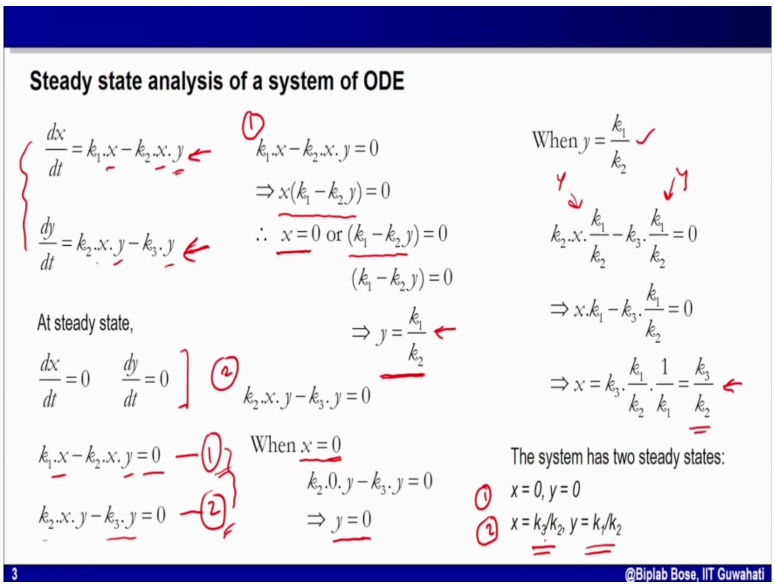

Now how can we calculate that algebraically? Let us try to see it. So we will take the example. I have 2 ODE, $\frac{dx}{dt} = K1*X - K2*X*Y$. $\frac{dy}{dt} = K2*X*Y - K3*Y$ as we have defined at steady state, both the derivatives has to be equal to 0, that means $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ so that means I can make this one equal to 0 and this one equal to 0. So let us make. So that means $K1*X - K2*X*Y$ would be equal to 0 and from the second ODE, using this concept, idea of the steady, we can say $K2*X*Y - K3*Y = 0$ this is my first equation. This is my second equation and I have to solve them simultaneously to get the value of X and Y for which these 2 equations are true. So let us try how to do that. So let us take the first equation. Equation 1. $K1*X - K2*X*Y$ =0. I can take X common so I will get $X*K1 - K2*Y = 0$. That means either $X = 0$ or $K1 - K2*Y = 0$.

Now if $K1-K2*Y=0$, I can separate out Y on the left hand side and the rest on the other side and I get this relation, $Y=\frac{K1}{K2}$. That means from first equation, I am getting either $X=0$ or $X=Y=\frac{K1}{K2}$. Now we will put this value of X and Y in the next equation to solve. So the next (equa) second equation is $K2*X*Y-K3*Y=0$ so let us consider when $X=0$ so if $X=0$, the second equation becomes $K2*0*Y-K3*Y=0$. That means $Y=0$. That means when $X=0$, Y have to be 0 to simultaneously make these 2 equations 1 and 2 correct. Let us take the other one. We have got from the first equation, when $Y=\frac{K1}{K2}$.

So let us take the second equation and plug the value of Y in that so we get $K\frac{2*X*K1}{K2}$.

This is nothing but $Y-K3*Y$ that is $\frac{K1}{K2}=0$. We have only X so we separate out X, take X on the left hand side of the equation and rest of the things, the constant, the parameter on the other side and we get $X=\frac{K3}{K2}$. So what we have got, we have 2 equations, 1 and 2 which we obtained by making the derivative equal to 0 for both the dependent variable and we have solved them simultaneously so we got 2 solutions. One is $X=0$, Y = 0. The (oth) other one is $X=\frac{K3}{K2}$ and $Y=\frac{K1}{K2}$ so that means the system has 2 steady states.

The first steady state is $X=0$, $Y=0$. You can easily see if you put this value of $X=0$ and $Y=0$ in this original ODE, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ both will be 0 and the other steady state that we have in this system is $X=\frac{K3}{K2}$ and $Y=\frac{K1}{K2}$. You can also check it, take the value of X here and put here and here in the first ODE and in the second ODE, take the value of Y and put the value of Y here in the first ODE and in this to second ODE, you can use simple algebra and calculate and see that for this value of X and Y both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ is equal to 0 so our system of this predator prey has 2 steady states.

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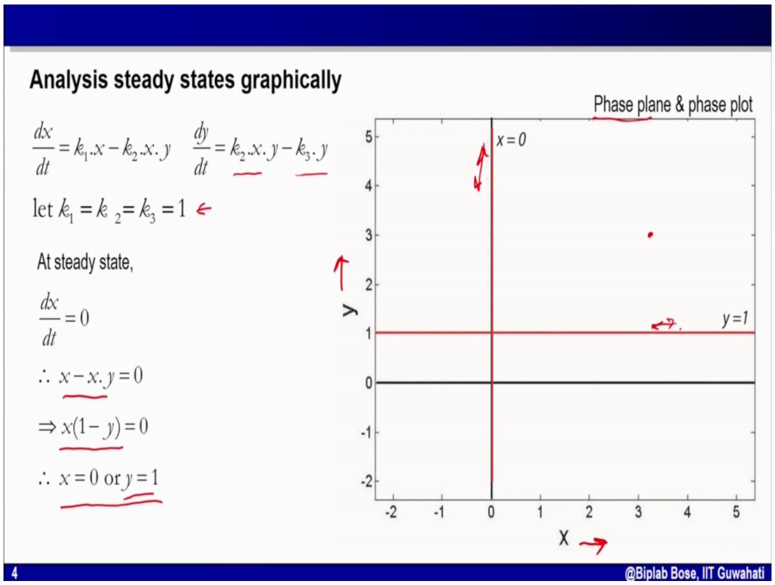

Now let us try to understand the steady state or identify the steady state graphically. Remember to understand the behavior of X a dependent variable with respect to time when we have only 1 dependent variable and 1 ODE, we have used direction field. Now, in this case, we have 2 variables, X and Y so I cannot use the concept of direction field directly if I want to see the evolution of X and Y both together. So we will use use a different trick, a different method. So to learn that, let us simplify our problem. We keep the same ODE, the predator, prey model. $\frac{dx}{dt} = K1 * X - K2 * X * Y$ and $\frac{dy}{dt} = K2 * X * Y - K3 * Y$. To make our life simple, I have considered all the parameters, K1, K2, K3 is equal to 1. That will make our calculation easy, you could have (take) you can take any other value. So now at steady state $\frac{dx}{dt} = 0$ so that means $X - X * Y = 0$ so that means I will get, if I get X (equ) common so I get relation $X * (1 - Y) = 0$.

So either $X = 0$ or $Y = 1$ so I have 2 solutions to this equation. Now I want to represent it graphically, how should I do it? Remember in case of 1 variable, I can always plot the (vari) dependent variable versus the independent variable, the time. In this case, we will do something else. We will plot Y and X both, both the dependent variables in these 2 dimensional space so that's why we have plotted X in this horizontal axis and Y in this vertical axis, so any point on this space, XY (place) plane is the value of X and Y at that moment. As time will change, value of X and value of Y will change as per the given ODEs so we will get multiple points and these points will keep on moving.

So what I have done here, I am trying to plot X and Y both together as the change with time. We will try to see how they change with time and the how the plot looks like but let us come back to the initial basic issues of this plot so this type of plot is called phase plane and phase plot. What is phase plane? I have 2 dependent variable. X and Y. 1 is the horizontal axis, the other is vertical axis and these planes, X and Y plane is called phase plane. And the plot that we will make here will be called phase plot so look back to the steady state relation. At steady state, $\frac{dx}{dt}=0$, from that we have got, $X=0$ and $Y=1$, so let us plot this on this phase plane of X versus Y plane. How would they look like? $X=0$ is this vertical line. At all these value, $X=0$ and these red horizontal line here is $Y=1$.

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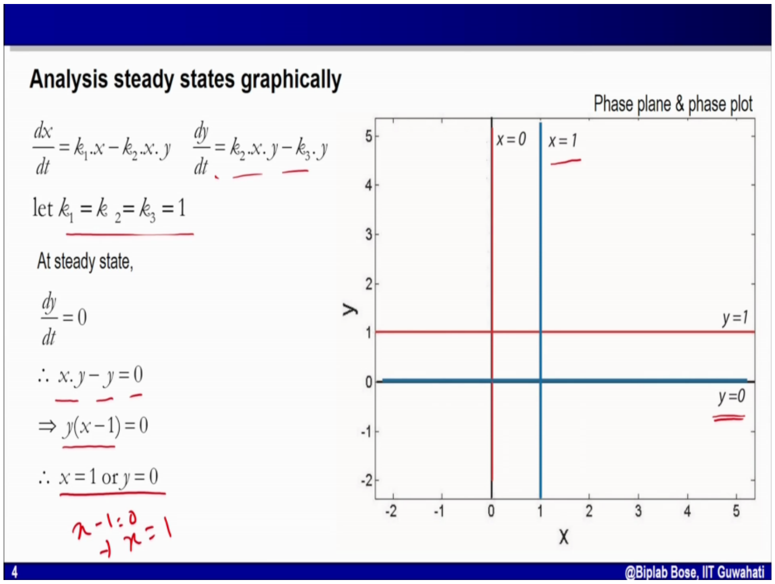

Now, let us do the same thing for the second ODE, as you remember, in steady state, both the derivative has to be 0 so I put $\frac{dy}{dt}=0$, that means from the second ODE, I get $X*Y-Y=0$. The parameters K1, K2, K3 are 1 so my life is easy so I get the relation, $X*Y-Y=0$. Now I can solve this. Obviously I can take Y common so I get into $X-1=0$ so either $Y=0$ or $X-1=0$ so X that means $X-1=0$ that means $X=1$ so I have 2 solutions, $X=1$ and $Y=0$. Just like the previous case, let me plot this $X=1$ and $Y=0$ .in this phase plane. The blue line I have drawn to represent these 2 lines so the vertical blue line is $X=1$ and the horizontal blue line along the axis is $Y=0$. So I have 4 lines 2 red which we have got from $\frac{dx}{dt}=0\wedge 2$ blue which are from uh $\frac{dy}{dt}=0$

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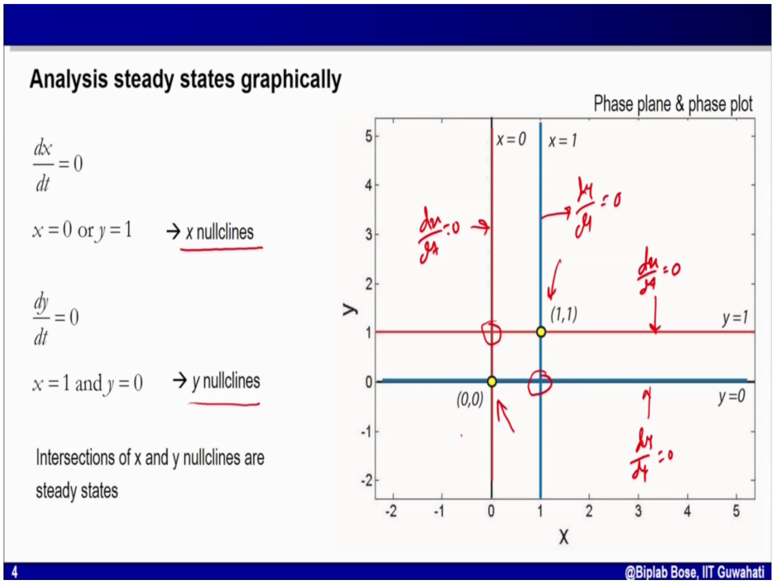

So as I have 2 lines, for red lines, both the red lines, $\frac{dx}{dt}=0$. That means along this line, in this plane, X does not change with time so for this red line and for this red line $\frac{dx}{dt}=0$. Here also, $\frac{dx}{dt}=0$ so these 2 lines are called X nullclines. Null means 0 and clines means slope. So slope of X along these lines are 0, that's why this is called X nullclines whereas for the blue lines, we have got this blue line using $\frac{dy}{dt}=0$. That means Y is not changing with time so for this blue line, $\frac{dy}{dt}=0$ and for this blue line also, $\frac{dy}{dt}=0$, so I have 2 Y nullclines. Again these are nullclines because null means none, not and clines means the slope so obviously it is called Y nullcline because slope of Y with respect to time 0 along this line.

So I have 4 nullclines, 2 X nullclines shown by red colored line, 2 Y nullclines shown by blue color line. Now, let us see where X nullcline and Y nullcline intersect. At 2 points where these blue and red lines are intersecting. 1 is (1,1) shown by this yellow (po) dot. Another one is this (0, 0). Don't confuse with this intersection point of 2 red lines or 2 blue lines. They are intersection of the same type of nullclines. Intersection of red null um lines are intersection of X nullclines and intersection of 2 blue lines are nothing but intersection of 2 Y nullclines. We are not interested in them. We are interested in intersection of X with Y nullcline so I have two such positions where a blue line representing the Y nullcline is intersecting the red line representing the X nullcline and that are 1,1 and 0,0 and why I am interested in that, because at this point, both the nullclines in have slope 0 that means at this point, at this intersection point, $\frac{dx}{dt}$ is also 0, $\frac{dy}{dt}$ is also 0.

As both these derivatives are 0, that means these are steady states for the system as we have defined earlier. So just plotting these nullclines on the phase plane and looking for intersection point, we can identify how many steady states are there. So if I have 2 dependent variables, I can plot them in a phase plane and then I can put each of the derivative is equal to 0 and draw nullclines and I will try to look for the intersection point. It can happen that for the system, there is no steady state so the nullclines does not intersect at all with each other that means you have not steady state there. If these 2 nullclines intersect at only at 1 point that means this system has only 1 steady state. If the nullclines intersect at 2 points as they have intersected in these example that means the system has 2 steady states. Once we have identified the steady states then we can ask the questions of stability of this steady state and also ask how the system, the X and Y value will change with time or evolve with time around the steady state.

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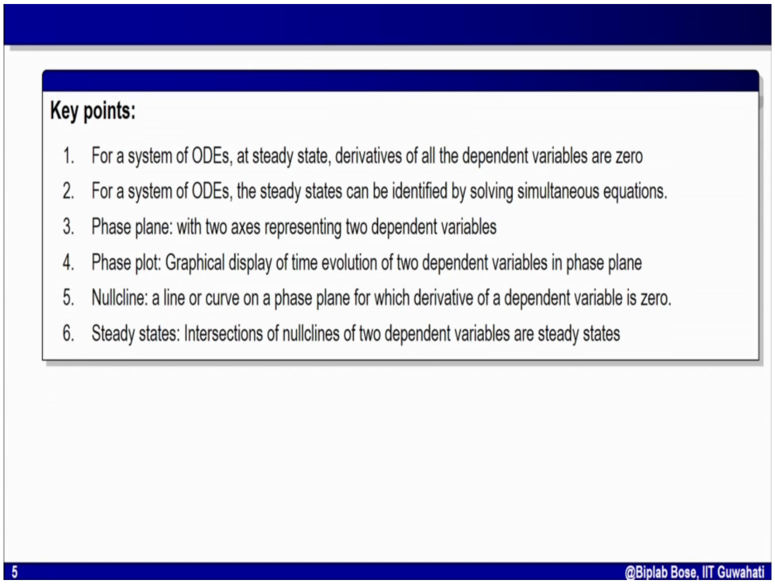

We will discuss those in another module so let us jot down the key points. For a system of ODEs where you have more than 1 dependent variable, a steady state, at steady state, the derivatives of all the dependent variables are 0. We can identify the steady state by putting all the derivatives equal to 0 and then solving those equations simultaneously using simple algebra. We can identify the steady state. We have also discussed about phase plane. In a phase plain, 2 axis represent 2 dependent variable and actually we want to understand the time evolution of these 2 variables in this plane and want to see this in this plane.

If a phase plot is nothing but a graphical display of timely evolution of 2 dependent variables in a phase plane. We have not drawn the phase plot yet, we have just introduced the concept here in this module. Now in a phase plane, if you draw all the lines or curve for which the derivative of a dependent variable is 0, those lines are curves will be called nullclines and as I have discussed, intersections of nullclines of 2 dependent variables are the steady states of the system. That's all for today, thank you for watching. See you in the next module.