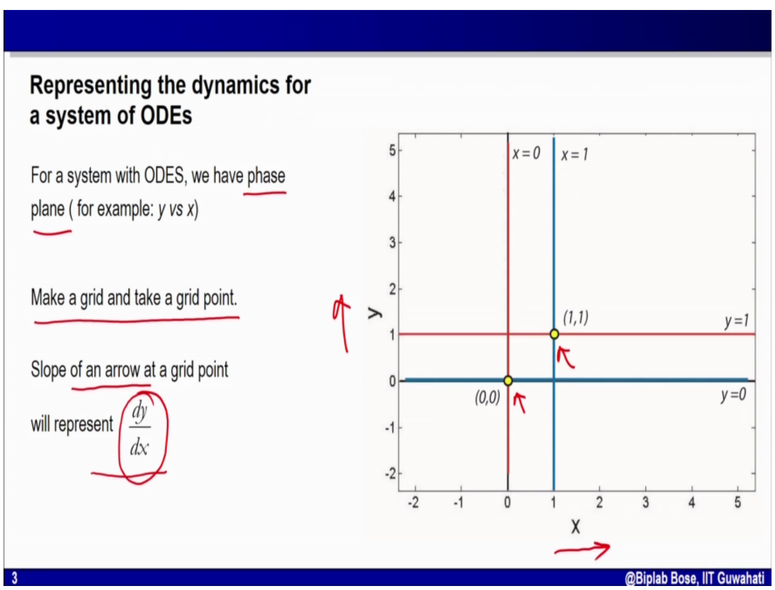
Introduction to Dynamical Models in Biology
Dr. Biplab Bose
Associate Professor
Department of Biosciences & Bioengineering
Indian Institute of Technology, Guwahati
Lecture 12
Phase Plane Analysis - 2

Hello, welcome to module 6 of second week of our course on Introduction to dynamical models in Biology. In the last module, we have started discussion on phase plane analysis and we have discussed how for a system of ODE we can use nullclines to identify the steady states and also how can we show those steady states in phase plane. Now remember when we built dynamical models we are interested in multiple things of the system. One obvious issue is that we want to understand the existence of steady states and the behavior of those steady states as well as we want to understand the dynamics of the system around the steady states.

(Refer Slide Time: 1:26)



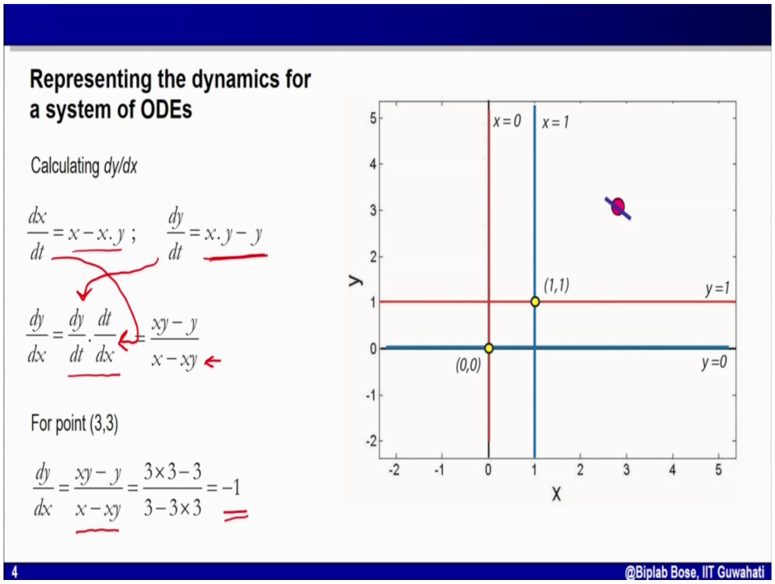
So, to understand the dynamics in case of one dimensional system for example when you have one single ODE what we use is direction field. I have shown the direction field that we have discussed earlier here. So, here I have only one ODE representing the rate of change of x with respect to time so, in direction field I have plotted time in the horizontal axis and the dependent variable x in the vertical axis and then I divide this whole x versus t plane equal grid points equi distance and each of these grid point what I do draw a arrow and the slope of that arrow is equal to $\frac{dx}{dt}$ and you can calculate $\frac{dx}{dt}$ at each of this point based on the value of t and x and the ODE given to you.

Now if I have a system of ODE that means I have more than one ODE because I have more than one dependent variable then how can I represent the dynamics of the system, dynamics of the dependent variable with respect to time in two dimensional space. That's what we will discuss today. And what we will use is actually use a phase plane so, if you remember for a phase plane, we suppose I have two variables dependent variable x and y that means I have two differential equation one giving $\frac{dx}{dt}$ and another one is $\frac{dy}{dt}$ so, if I have two dimensional system two variables only... dependent variable. I can draw a phase plane with x in the horizontal axis y in the vertical axis.

You can swipe them there is no issues that y x should always in the horizontal axis. So, I have drawn input x on the horizontal axis, y in the vertical axis. Now, we have discussed that I can draw nullclines on this plane and the intersection point of the nullclines for example this yellow dots here are actually the steady state. Now, in case of direction field for one single dependent variable we divide the plane in equi distance grid point and at each grid point I draw a arrow with a slope equal to derivative but here we have x and y axis so, I cannot draw a arrow for with slope $\frac{dx}{dt}$ or $\frac{dy}{dt}$ so, how can I draw a arrow here or I can put something to represent the dynamics of y and x with respect to time.

Remember my intention is to visualize the change in y and x with respect to time in this y and x plane. What we will draw here is rather I will divide this whole y versus x plane which is actually the phase plane. This phase plane into equi distance grid and at each grid point I will draw an arrow and the slope of the arrow will be equal to $\frac{dy}{dx}$. Derivative of y with respect to x that's what we will draw. So, that will show how y is changing as x is changing and as x will change with time so, y will also change with time. So, that's how I can visualize the change in y and x in a xy plane. But the issue is how can I calculate $\frac{dy}{dx}$.

(Refer Slide Time: 4:53)

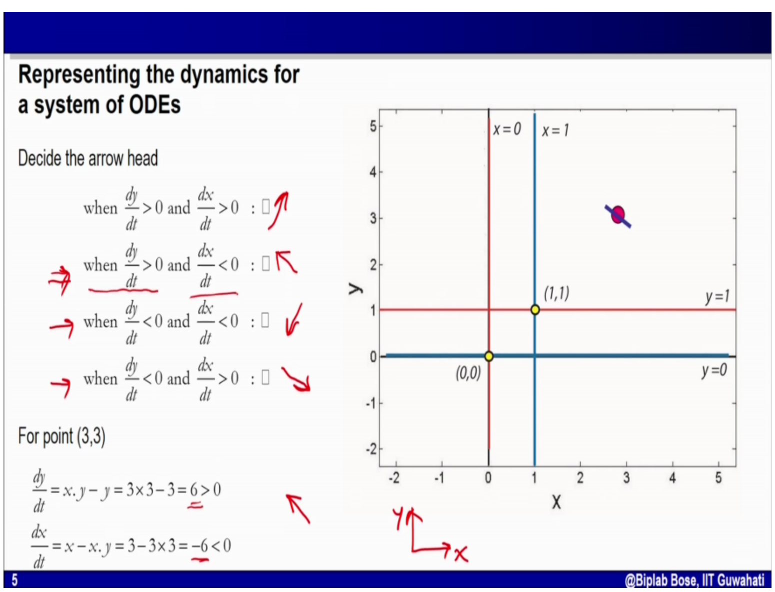


Let us see with a particular example. We will take old example of predator prey so, I have two ODE's here because I have two dependent variable one representing the prey the other one representing the predator. So, prey is x and predator is y so, I have $\frac{dx}{dt}=x-xy$ this is first ODE second ODE is $\frac{dy}{dt}=xy-y$ is the common system of ODE that we have discussed earlier for predator prey models. For this system I want to draw the phase plane and I want to divide that phase plane in equi distance grid point and at each of these grid points I want to draw an arrow representing $\frac{dy}{dx}$.

So, how can I calculate $\frac{dy}{dx}$. here? Let's see so, $\frac{dy}{dx}$. I can write is equal to $\frac{dy}{dt}$. into $\frac{dt}{dx}$. Now $\frac{dy}{dt}$ you have already given from original ODE and I can inverse this $\frac{dx}{dt}$. to get $\frac{dt}{dx}$ so, what I get? I get $xy-y$ this is this one $\frac{dy}{dt}$ into inverse of this one. So, inverse of this one is here. So, I have $\frac{dy}{dx}$ which I calculate from $\frac{dx}{dt}$ and $\frac{dy}{dt}$ because it is the combination of both the derivative. Now let us take a particular point and calculate the value of $\frac{dy}{dx}$ at that point in this xy phase plane. So, let us take a point 3,3 I have shown this by red dot here 3 versus 3, y is 3, x is 3. Now use this value of 0.33 to calculate $\frac{dy}{dx}$.

How can I calculate? $\frac{dy}{dx}$ equal to as I have calculate $\frac{X*Y-Y}{¿}$)/(X-X*Y) so, proved the value f x and y here. Both x and y are 3 so, on the numerator I will have ($3*3-3¿/¿$ (3-3*3) and that gives me -1 . That means I have to draw a arrow at this rate point which you will have a slope of $-¿$ 1 so, essentially I want to draw a arrow there with angle 135 degree. So let us draw arrow there. So, I have drawn a small arrow remember without a arrow head with a slope which is equal to $\frac{dy}{dx}$ at that point and the value of $\frac{dy}{dx}$ at that point was $-¿$ 1. So, I have drawn the line representing the arrow now I have to decide the arrow head. Will the arrow move towards the left upper corner or the arrow head will be on the other side. So, how to decide the arrow head position.

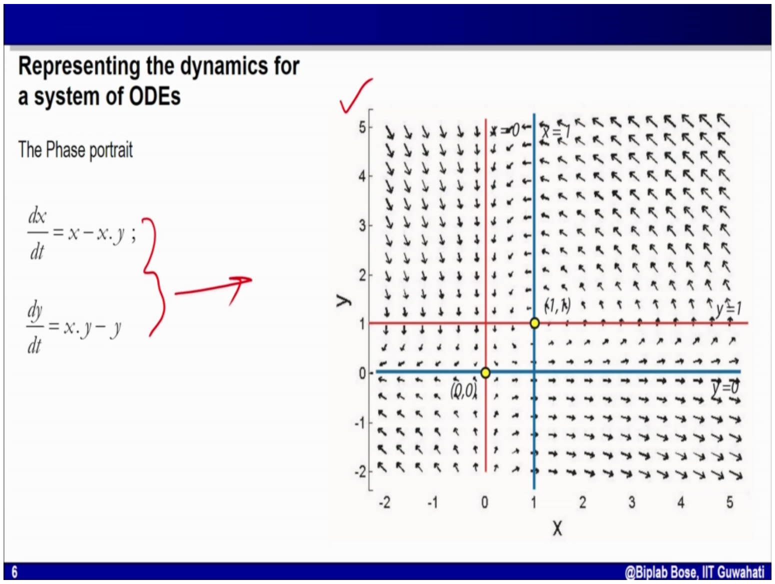
(Refer Slide Time: 7:56)



Let us use some thumb rule to do that what we can do? We can do like this. Suppose when $\frac{dy}{dt}$ >than 0 and $\frac{dx}{dt}$ > 0 that means with time both x and y will increase and I will put a arrow like this pointing up right hand side. When $\frac{dy}{dt}$ > 0 that means y is increasing with time and $\frac{dx}{dt}$ is decreasing with time that means $\frac{dx}{dt}$ < 0, then I can have a arrow like this. So, y is increasing, x is decreasing. Remember my plane I have x in the horizontal axis, y in the vertical axis. When $\frac{dy}{dt}$ < 0, and $\frac{dx}{dt}$ is also less than 0 that means with time both y and x will decrease so, I can have arrow in this direction. When the forth case comes when $\frac{dy}{dt}$ < 0 but $\frac{dx}{dt}$ > 0 then I can have arrow in this direction that means x will increase in horizontal axis, y will decrease in vertical axis. So, this is the thumb rule that we can use and it is very easy thumb rule.

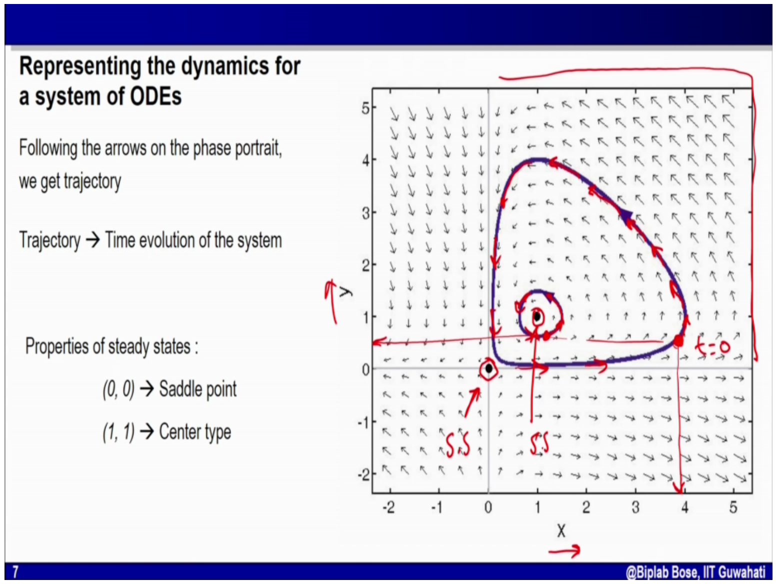
Let us go back to our point 3,3 and calculate what will be the arrow head direction so, at point 3,3 $\frac{dy}{dt}$ is given by the OD $xy-y$ that is $3*3-3$ that is 6 so, 6 > 0. $\frac{dx}{dt}$ is x minus x into y that is $3-3*3$ so, I get $-¿$ 6 so, $-6<0$ so, I have y derivative positive, x derivative negative that means my arrow will be in this direction. Following this particular rule when $\frac{dy}{dt}$ > 0, $\frac{dx}{dt}$ < 0, arrow will be on the left upper corner direction so, let us put the arrow there. So I have got the arrow here. So this is how I can take a point on the phase plane x versus y plane and calculate the slope of a there from $\frac{dy}{dx}$ and then I can decide the direction of the arrow head based on the values of $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

(Refer Slide Time: 10:45)



So what I can do? I have whole phase plane of x versus y and I can divide them in equi distance grid point and at each grid point I can draw the arrow using the rules that I have discussed just now so that's what I have done here for this predator prey model I have x versus y I have shown this colored lines are the nullclines that we have discussed the intersection point of the nullclines are the yellow dots those are the steady states and you can see the arrows here at equi distance grid point each of the arrow is representing $\frac{dy}{dx}$ that is how y and x are changing with respect to time as y is changing with respect to x that gives an x change with respect to y and the direction is based on whether y is decreasing or x is increasing like that as we have discussed just now the rules. So in this way I have filled the whole space with arrows.

(Refer Slide Time: 11:41)



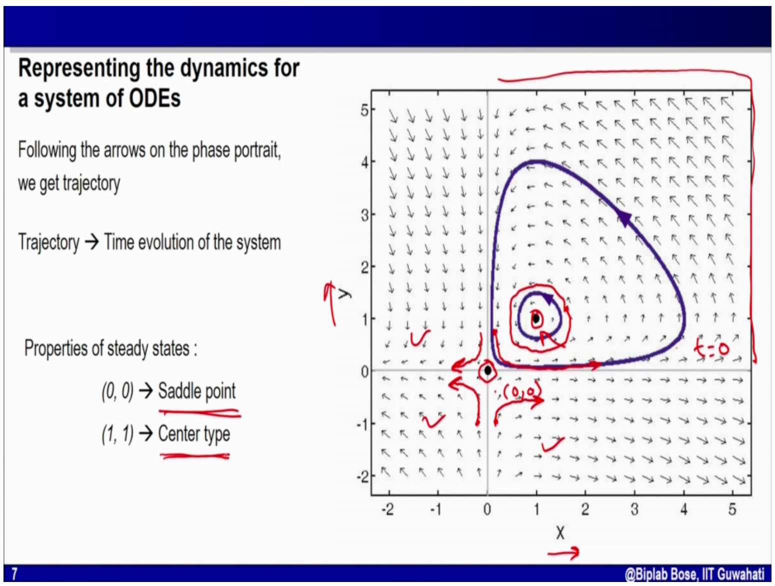
Now what these arrows are telling me? These arrows are telling me the direction of evolution of the system. So let us see how I can understand the direction of evolution of system from a phase plane analysis. So suppose the same plot I have drawn here just the color has been changed so, I have x in the horizontal axis, y in the vertical axis same predator prey model ODE and this is my phase plane. So suppose I start at this position, this is the initial t = 0 position so the value of x at that point is this one near 4 and the value of y at this point is just around 0.5 or something.

So the arrows are telling me that as the time will progress the system the x and y value will change along with these arrows. Just like the direction field I can connect the arrows one after another and I can get the trajectory that is the geometric representation of time evolution of both the variables in this phase plane. So if I start at this particular point t = 0 I will follow this closed loop path and come back to the same position. You can start at another point here and you will follow the arrows and the arrow are telling me that I will follow a closed path like this.

So, the phase plane with these arrows are telling me that if I start from a particular position where both x and y values are positive this is this quadrant this is this quadrant I will follow a closed path where x will increase and then y will also change and essentially I will have x and y oscillating values right because x is increasing and then x is decreasing so I will have oscillation of x similarly y is also increasing and y is again decreasing with time so I will have oscillation of y and that's what we have expected for predator prey model as we have discussed earlier.

Now apart from this trajectory there is another issue here if you remember these black dots shown here are nothing but intersection points of the nullclines so, these are steady states so, this is one steady state, this is one steady state, this is another steady state. So let us look into the stability issue of steady states these steady states the first one this one is 0,0 this one is 0,0 so if you look into this steady state let us clean a bit so that I can understand how the steady states are behaving here.

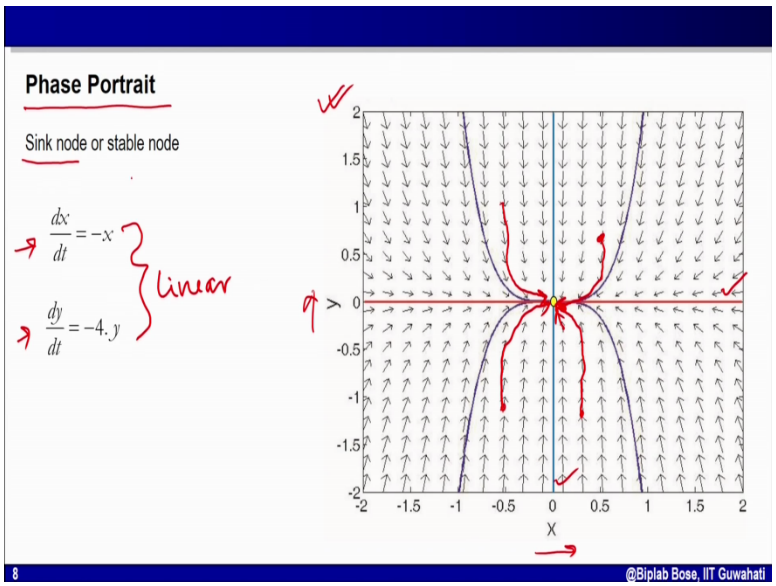
(Refer Slide Time: 14:37)



So, if you look into this particular steady state at 0,0, if I start here at a close point of 0,0 my arrows on the direct phase plane is telling I will move away from 0,0. If I start from here I will again follow this trajectory and move away from 0,0. If I start from here obviously I will move away from 0,0 and if I start here I will move away from 0,0 and follow this closed path. So, that means this 0,0 steady state is such that if you are there you will stay there but if you are slightly away from there that means you have perturbed from that 0,0 you will never come back to 0,0. in three quadrants this one, this one and this one you will move away from 0,0 that means 0,0 steady state is obviously unstable and this type of steady state having this type of trajectory around it are called saddle point I will discuss them separately what we understand by saddle point?

Let us look into the other steady state this one here if I start somewhere close to this point this steady state I will follow a close loop. I will never reach the steady state or I will never move away from the steady state essentially I am oscillating moving in closed loop giving rise to oscillation for x and y so this steady state is called center type steady state. Being this type of steady state if you are there at the steady state you will stay at that steady state for infinite period of time where as if you slightly perturb from it you will start moving in a closed path around that steady state. So, this is called center type steady state we will discuss that also.

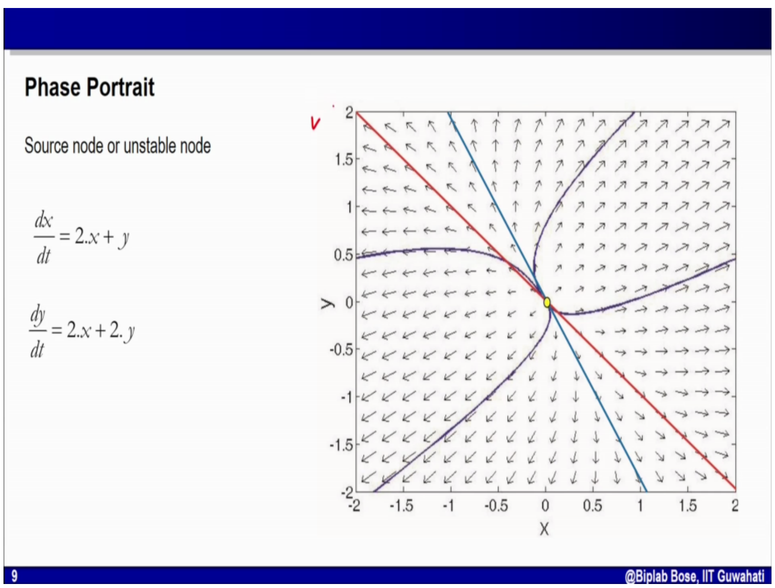
(Refer Slide Time: 16:32)



So, let us look into different types of trajectories in phase plane and the steady state correspondingly. Remember when we draw these arrows on a phase plane and show the dynamics of the system by trajectories around the steady state we call it phase portrait. So, let us start different examples of phase portraits the one shown here is represented by these two ODE's $\frac{dx}{dt} = -x$ and $\frac{dy}{dt} = -4y$ I have just given an example. If you use the same rule of plotting x in the horizontal axis, y in the vertical axis and then divide this xy plane in equi distance grid point and draw arrows there using the rules that we have discussed you will get this type of picture.

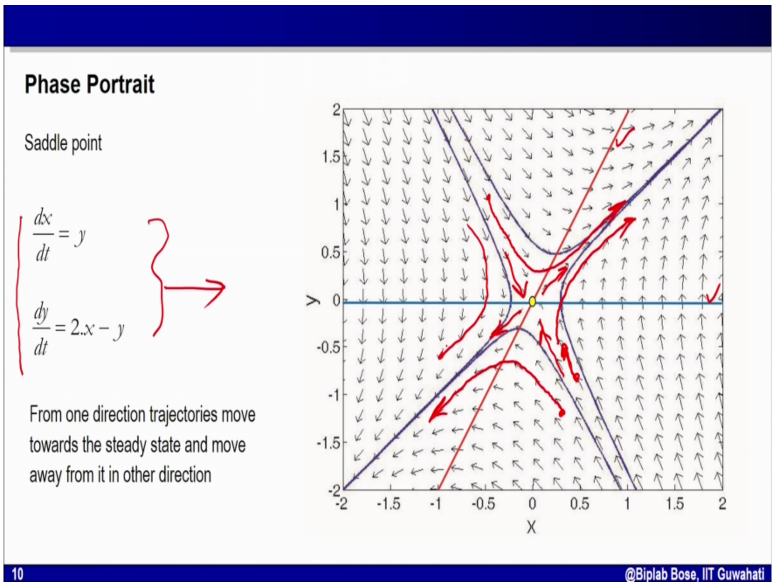
Let us see the nullclines I have one nullcline here I have another nullcline here so they are intersected at this yellow dot so that is my steady state. Obviously this OD is the linear system so, I will have only one intersection point between the nullclines and I have only one solution and this is the homogeneous linear system as you don't have any constant term on the right hand side of the equation so, steady state solution is 0,0. If you look at the trajectory you can see easily if I start suppose from this point I will follow this arrow and collapse at 0,0. If I start from this point I will move towards the 0,0 and collapse at 0,0. If I start from this point I will move and collapse at 0,0 so, from all direction if you follow... if you start from any quadrant position if you start you will eventually reach the steady state at 0,0 asymptotically so that means this is the stable point this is the stable steady state and all the trajectories are converging there so this steady state is sometime called sink. This is called sink node or stable node. I can have the opposite one also that will be unstable.

(Refer Slide Time: 18:52)



So, look at this phase portrait here, this phase portrait is coming from the ODE $\frac{dx}{dt} = 2x + y$ and $\frac{dy}{dt} = 2x + 2y$ just for an example I have taken this ODE's. So, for this system of ODE if you draw the phase portrait you get this type of figure. So, again the nullclines are shown by these two colored line and their point of intersection is this steady state shown by yellow dot. If you observe carefully if I start near this steady state I will move away. If I start somewhere here near steady state I will move away. If I start somewhere here I eventually move away. So, you can easily see by observing this trajectories here that if you are at this steady state at the intersection point of two nullclines you will stay there but if you perturb the system slightly the system will diverge away. It will move away asymptotically from this steady state so this type of stead state is unstable and it is like a source is from where all the trajectories are originating that's why this type of steady states are called source node or unstable node.

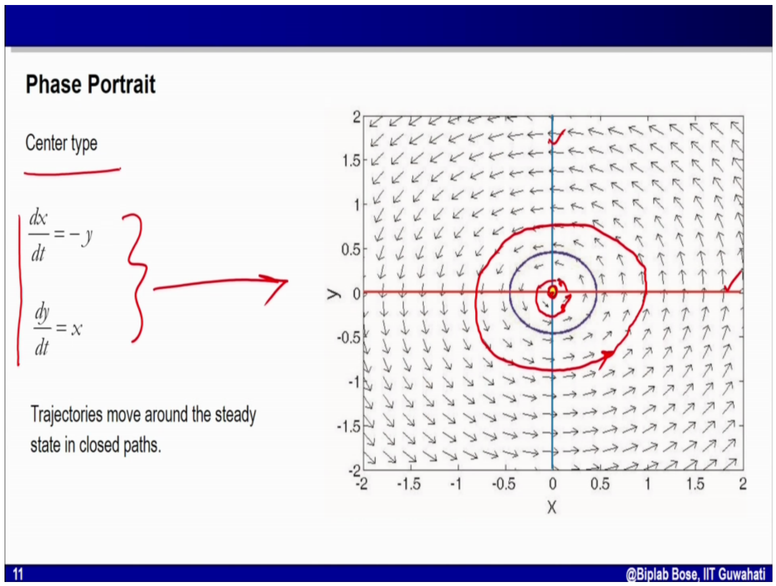
(Refer Slide Time: 20:16)



Let us look into few more example her comes the saddle point this system of OD that is $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = 2x - y$ gives rise to this phase portrait. Again these two color lines are nullclines and this arrows here represents the slope $\frac{dy}{dx}$ and as you can see there are two path through which you can actually reach towards the 0,0 steady state position whereas there are another path which diverge away from it and in between these two path all paths will take you if you start from here for example it will take you towards the steady state and then it will take you away from the steady state. If you start suppose somewhere here you will go close to steady state and again you will move away from the steady state.

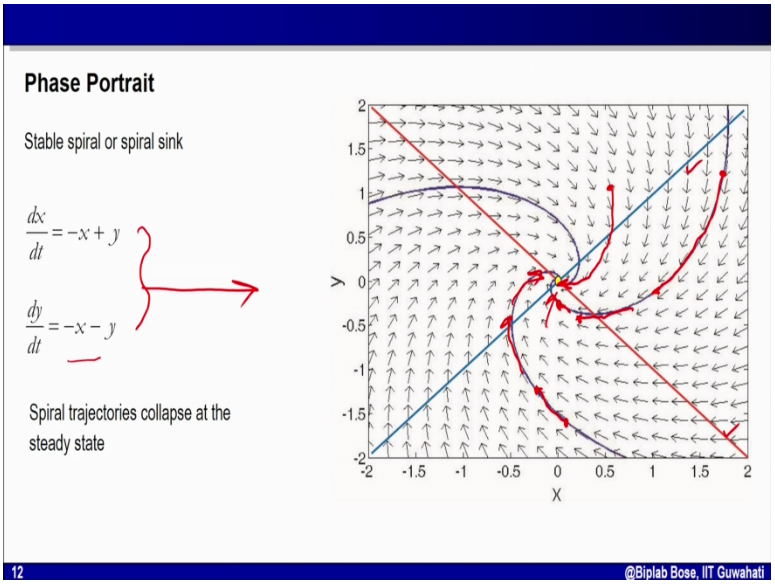
What I am doing I am taking initial point and then following the trajectory the arrows to draw the trajectories so, if I start from this position the arrows are telling me I will go towards the steady state and then diverge away so the same thing is here. So, in this phase portrait we have a steady state and there are only one path from both the direction from which as we following this path you can reach the steady state but for others either you will move away directly from steady state or initially you will move close to the steady state and then diverge so this type of steady states are called saddle point and obviously they are not stable.

(Refer Slide Time: 21:56)



Let us look into the center type or closed path type that we have faced earlier in case of predator prey model. Here I have taken a system of ODE simple one $\frac{dx}{dt} = -Y$ and $\frac{dy}{dt} = X$. This system of ODE give rise to this phase portrait again this colored lines are the nullclines and the arrows represents $\frac{dy}{dx}$ as you can see if you stay at this yellow steady state position you will stay there but if I slightly disturb the, perturb the system then there is a closed path through which the x and y value will keep on changing. That's true from here also if I start from here I will follow this path and something like this I will keep on moving. So, I will have a closed path here this closed path looks like ellipse it may have a different shape also but I have a closed path. So, in this case the steady state is obviously unstable. If you move away from it you don't have any steady state. It does not... you does not... you do not go back to that but if you are slightly away from it you are in a closed path so that's why it is called center type.

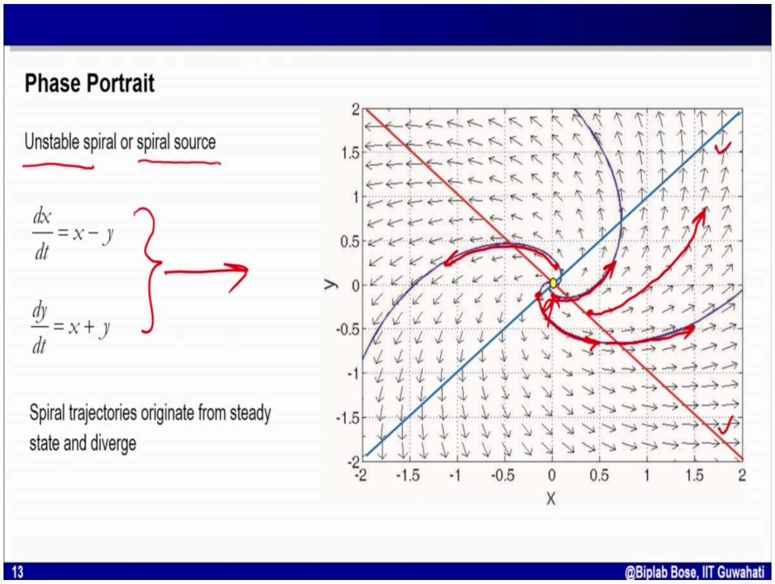
(Refer Slide Time: 23:15)



Now comes another type of phase portrait. I have taken two ODE's $\frac{dx}{dt} = -x + y$ and $\frac{dy}{dt} = -x - y$ these two ODE's is represented by this phase portrait. Again I have two nullclines here by the colored line and the arrows represent $\frac{dy}{dx}$ and the intersection point of nullclines is the yellow dot which is showing the steady state. If you start from anywhere suppose here on this phase plot and then the arrow tells you that you will follow this one this path and collapse at steady state. If you start from here you will follow this path and collapse at steady state. If you start from here you will follow this path and collapse at steady state.

And notice the path this path are spiral type path so, if you start from a particular position on the phase plane you spiral and move in spiral and then the system collapse at the steady state, so obviously this steady state is stable because it is sucking everything towards it. If you are at that steady state you will stay at that steady state but if you are disturb from the steady state use the following spiral trajectory with time will again collapse to that steady state so this is the stable spiral or spiral sink. What happen if you have opposite one?

(Refer Slide Time: 24:44)

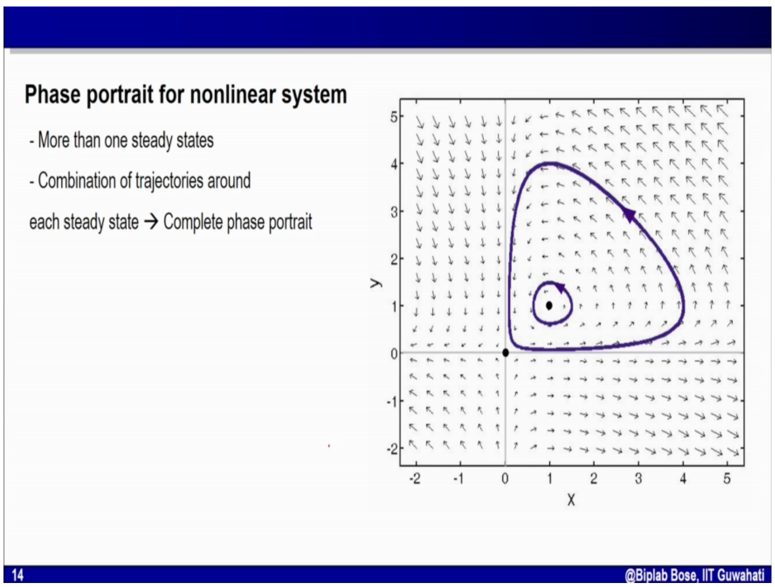


Let us take the example so, I have OD $\frac{dx}{dt} = x - y$ and $\frac{dy}{dt} = x + y$ these two OD give rise to this phase portrait. Again these two colored lines are nullclines and the intersection point is steady state. In the earlier case the path trajectories were collapsing at the steady state but look into this plot here the arrows are moving away from the steady state. So, if I am at steady state and if this get disturb, suppose it disturb you to here then the system will not go back to the steady state but it will follow a spiral curved path and diverge away. Similarly if we distance the system and then release it the system will move away in this direction. If I disturb, perturb the system from steady state and bring it here and then diverge.

So, if I start from a initial value here I will follow this path. So, no path is going back and converging at the steady state, if you are at the steady state the system is at the steady state it will stay there if you disturb the steady state it will diverge away from the steady state in a spiral form so that's why it is called unstable spiral or spiral source from which all the trajectories are originating. So what I have discussed till now is different types of common phase portrait that you can expect. Both some of them are stable some of them are unstable. For example I have discussed stable node, unstable node. I have discussed stable spiral, unstable spiral then we have saddle type steady state of phase portrait and then I have we have center type phase portrait.

Now if you notice all the equations that I have used are actually linear equations so that mean I have only one steady state because for a linear system the intersection of nullclines will be only at one position so, I will have only one steady state. Now we have non linear system of ODE and that will be very common in biology as we have discussed repeatedly. For example predator prey model I have two dependent variable connected with each other using the couple ODE which are non linear in nature because you have $x*y$ terms there also. So if I have this type of non linear ODE what will happen to the phase portrait? So, obviously I can have multiple steady state for a non linear system right.

(Refer Slide Time: 27:24)

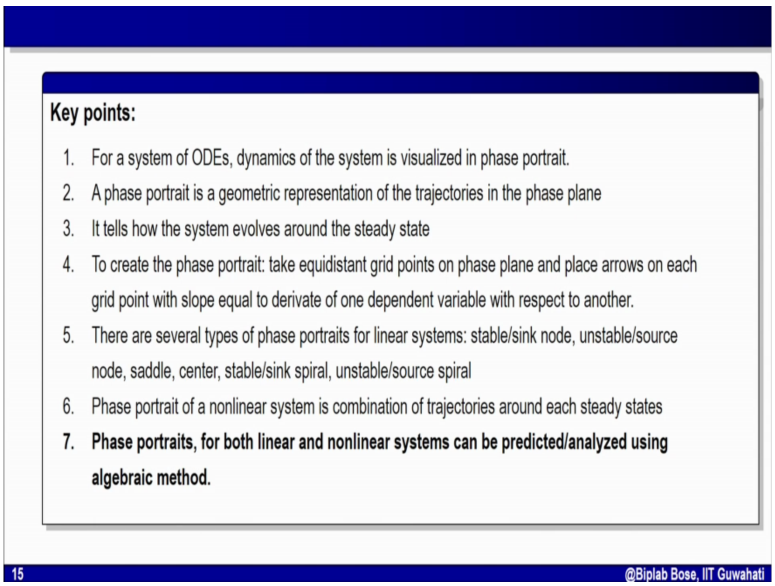


So for example in a phase plane phase portrait drawn here for the predator prey model I have one steady state here, I have one steady state here then total phase portrait of the system is nothing but a combination if you notice combination of two phase portrait for this first steady state here I have a center type steady state, I have center type phase portrait where the arrows are all moving in closed path around the steady state whereas for the other steady state at 0,0 these are saddle type so the arrows are moving away and total phase portrait is a combination of these two phase portrait.

We can sue the same technique for any other non linear system we can identify the steady state by the intersection of the nullclines and then I can put grid points at equi distance and calculate

$\frac{dy}{dx}$ to get the arrows and decide the arrow head based on derivatives of x and y and what I will get is the combination of phase portrait for individual steady state. They will club together to give rise a geometric representation of trajectory as a whole in the phase plane. Now although we can manually draw this type of trajectories and phase portrait by hand using paper and pen but usually it is time consuming. There are many tools to do that if you can use Matlab there are Matlab codes available to draw these type of phase portrait and they are quite easy.

(Refer Slide Time: 29:06)



So let us come down to the key points that we have discussed today for a system of ODE the dynamic of the system is visualize by phase portrait. For a single ODE you can draw direction field but for a system of ODE you have more than one ODE you draw this portrait you choose any two dependent variable and draw phase portrait. If phase portrait is nothing but geometric representation of the trajectories of the phase plane. As you are representing the trajectory the phase portrait tells me how the system evolves around the steady state and remembers, I am getting the steady state through the intersection point of the nullclines. To create the phase portrait what I have to do I have to divide the phase plane in equi distance grid point and then I have to place arrows on each of this grid point and the arrows will have slopes equal to the derivative of one dependent variable with respect to the other one at that point. So if I am drawing phase portrait in y versus x plot then I will have the slope equal to $\frac{dy}{dx}$.

Then we have discussed that there can be different type of phase portrait for linear system I can have stable or sink node. I can have unstable source node. I can have saddle type. I can have center. I can have stable sink or spiral. I can have source spiral. These are essentially phase portrait for individual linear system. I can consider the phase portrait for a non linear system as a combination of multiple phase portrait corresponding to each of the steady state. So phase portrait of a non linear system is nothing but combination of trajectories around each steady states. Now you have to remember that what we have discussed till now is nothing but graphical representation of the time evolution of dependent variable in a system of ODE.

Here you have to draw the figure to understand the dynamics but many a times actually we want to know the dynamics purely based on mathematical analysis or algebra. So there are good method of analysis of dynamics and steady state of linear and non linear system through algebraic method. We have not discussed that because that is beyond the scope of this course. Thank you for watching this module. We will learn about different types of dynamic behaviors in the further videos.