Introduction to Dynamical Models in Biology
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Lecture 4
Modeling population growth

Hello. Welcome to Module 4 week 1 of our course on Dynamical models in Biology. In the last module we have discussed about using ordinary differential equation to create mathematical model for spread of infectious diseases. In this module we'll continue in along that, we'll create some new model using ordinary differential equations. So today we'll, in this module we will try to model growth of a population. Population growth is a very common dynamical process in biology. Population of a city increases with time, population of animals in a forest increases with time, you may be growing bacteria in a fermenter, number of bacteria will increase with time, in a tumour number of cancer cells increase with time and we want to model that change, that growth in population. Population growth can be complicated. For example human population growth is a bit complicated with respect to growth of bacteria or growth of tumour cells. May have different processes involved in that.

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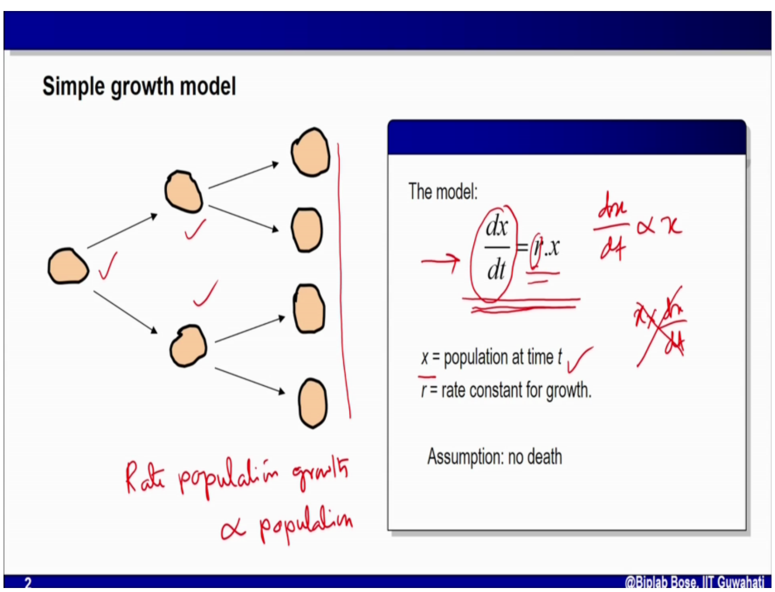

In this module we'll try to get a simplified model to understand how ordinary differential equation can be used to model population growth when the growth of population is very simple, like growth of bacteria in your fermenter, which happen by binary fission, from one cell you get two, from two you get four. With some assumption you can make the model for growth of cancer cells in a tumour like that because from one cancer cell two cells will be produced, from two cells four will be produced, like that. So these are just binary processes one to two, two to four. And I want to create a mathematical model for that. Remember we will use ordinary differential equation and in a dynamical model, ordinary differential equation represent rate of change.

So here we'll write ordinary differential equation to represent change, rate of change, rate of change in the population size. So if I can use an logic that, that rate of that population growth I want to measure actually and obviously this will be proportional to the current population. So if you have more bacteria obviously the rate will be effective rate of growth of the bacteria will be higher. So if you represent this proportionality in terms of a ordinary differential equation I get this equation. Here X represent population of bacteria at time t that is right now and r represent rate constant of growth, so dx/dt that is the time derivative of X that is rate of change of X. Rate of change in population size equal to $r*X$

Essentially, dx/dt is proportional to X and we have used r as a proportionality constant, and r is a constant. It is a parameter of this ODE. And X is the dependent variable, t is the independent variable. So if you look into the equation we have $\frac{dx}{dt}=r.X$. Here the power of X is one, the power of the derivative is also one. There is no product between X and its derivative. So that means we don't have this product, that means this equation, this ordinary differential equation is a linear equation. So now, once I have this model this equa.. this ODE which is my mathematical model we want to ask certain question to this model.

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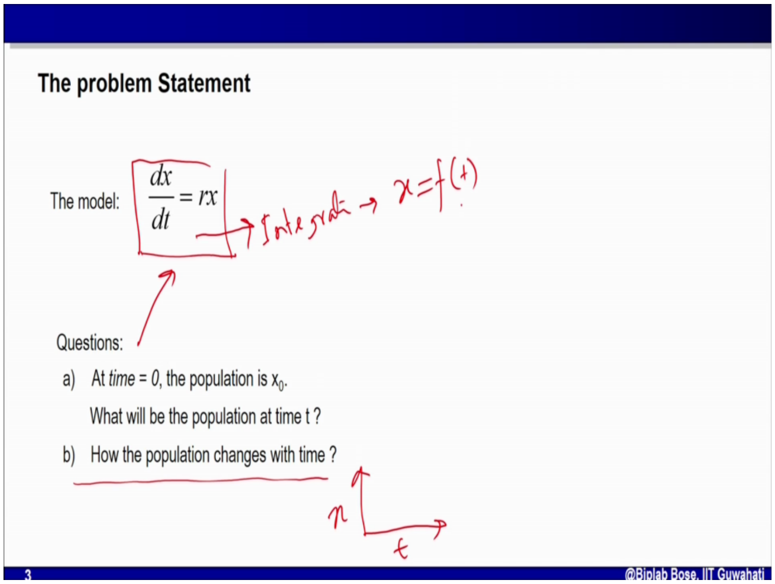

So what type of question I can ask, I can ask like this. That I have started with hundred bacteria I know the rate constant of growth how many bacteria's will be there suppose after four hour? Or I can have a generalized question that I want to know the dynamics of population change. That is I want to know how with time the X, the population size of the bacteria will change. So I have this model $\frac{dx}{dt} = r.X$. This is my model and I am asking this question. If you remember what we did in the last module we essentially integrated this equation, this ordinary differential equation to generate the function... X as a function of time. And once we got that function we can answer these two questions.

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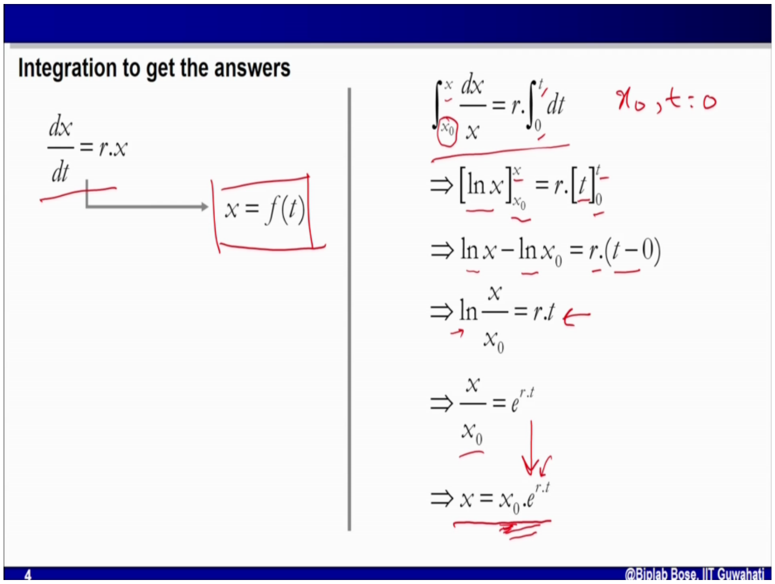

So let us here, we will try to integrate this ordinary differential equation to get this function. How we'll start, very simple. We'll separate X and time t, so that's what I have done I have separated them. So I have to integrate $\frac{dx}{x}$ from X_0 ¿ X and I have to integrate time from 0 to t. So, X_0 is the population at $t=0$. X_0 is the population when time equal to 0. So using the general formula, simple formula of integration, $\frac{dx}{x}=\ln X$, while as dT by integration give me t. I have integrating from X_0 to X and t from 0 to T, so if I simplify I get $\ln X$ minus lnX_0 equal to $r(t-0)$ that is equivalent to t. So if I simplify further and rearrange the term I get, $\ln\left(\frac{X}{X0}\right)=r.t$

Let us do further simplification then we get, $\frac{X}{Xo}=e^{rt}$. And remember I want to get X as a function of time so I want to keep X on one side of the equal to side and everything else on the other side. So that's what I have done in the next step. I've, I have rearranged the term algebraically and I have got $X=Xo.e^{rt}$ If you look into the right hand side of this equation X_0 is a constant because you have decided that, that is the initial number of population, initial size of the population, r is a rate constant there is a rate constant of growth and t is the time. So this is my function.

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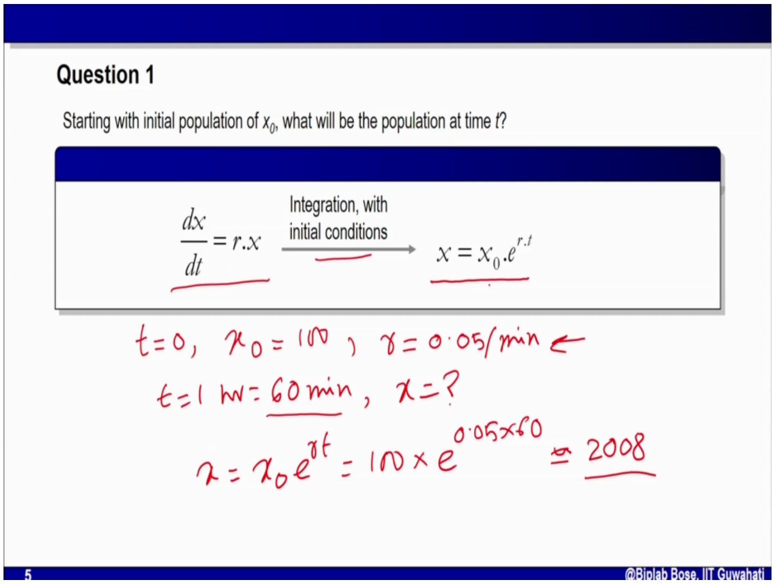

Now once I have this function I want to go back to my first question. So suppose at $t=0$, suppose $t-0$ my initial population of bacteria is 100. That means I have seeded 100 bacteria in my fermenter and I know the rate or the rate constant for growth is 0.05 per minute. Remember bacteria's grows very fast, so I want to know after one hour T equal to one hour that is 60 minute what will be the size of the bacterial population. So how can I answer this question.

My model is $\frac{dx}{dt} = r.X$ by integrating that with initial condition I have got $X = Xo.e^{rt}$ So I put the values $X = Xo.e^{rt}$.

This is X_0 for me in this example is 100 into $e^{r*0.05}$ into remember the time is 60 minute, because your r is in minute scale I have converted one hour into 60 minute this one I have already done the calculation. This is almost approximately you can calculate it is equivalent to 2008 cells. So that means my mathematical model is saying that if I inoculate 100 bacteria and if the growth rate constant is .05 per minute after one hour I will get 2008 bacteria. So this was a specific question and specific answer.

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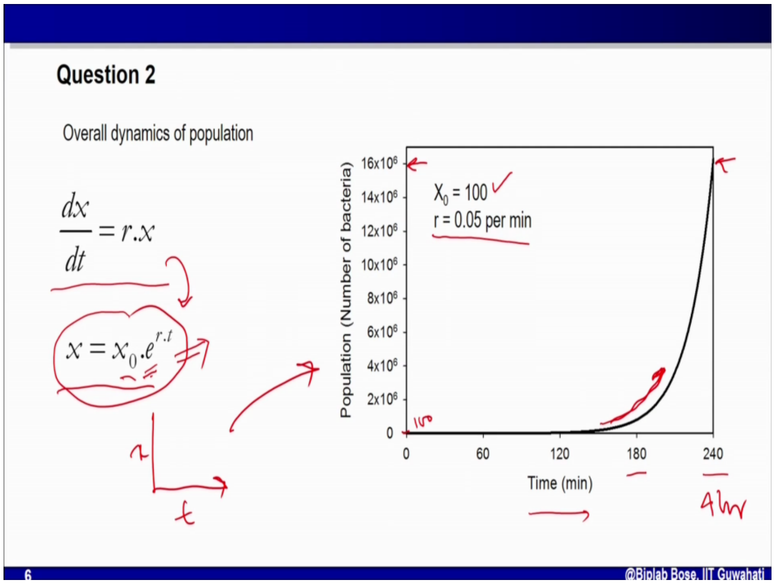

But suppose I want to get a generalized dynamics, understand the generalized dynamics of the bacteria how will I approach it. So my model is $\frac{dx}{dt} = r.X$ by integrated integrating I have got the function, $X = Xo.e^{rt}$. And I want to plot this one so that I want to plot X in the vertical axis and T on the horizontal axis, and that's what I have done here. What I have done, I have taken different values of time from 0 to 240 minutes that is mean 4 hour and I have taken a fixed value of X_0 that is 100, already I know r equal to .05 per minute. So using this function I have got this curve. So see we have exponential term here. $X = Xo.e^{rt}$. That means the growth of the population is exponential and that's what you can see here. We started with 100 bacteria here, just close to 0 and within 3 hour it has started increasing and it has stated increasing exponentially. By four hour we have reached almost 16 x 10 to the power 6 cells. This is a huge exponential growth.

So what we have done till now. We have fitted a ODE based model for simple growth of bacteria. You may, make some assumption and you can use this type of model for growth of maybe tumor cells also. We ask two cons questions. One is a specific question. That if I start with a particular number of cell we know the rate constant for growth what will be the size of the population after one hour or two hour and the second one is I want to answer, understand the generalized dynamics of population growth which I have shown here by this plot. Now, isn't there something wrong with this plot. You see the bacteria's are growing exponentially.

As we are not considering any death of bacteria with time exponentially these thing will keep on increasing towards infinity. That never happens. If you are growing bacteria in a fermenter or on a agar plate they don't keep on growing. They after sometime saturates. They hit a ceiling after which they don't grow further. That is true for a tumor growth also, that is true for growth of population in a city also. That is true for growth of population of animals in a forest also. The population size cannot keep on increase infinitely. There must be certain limit.

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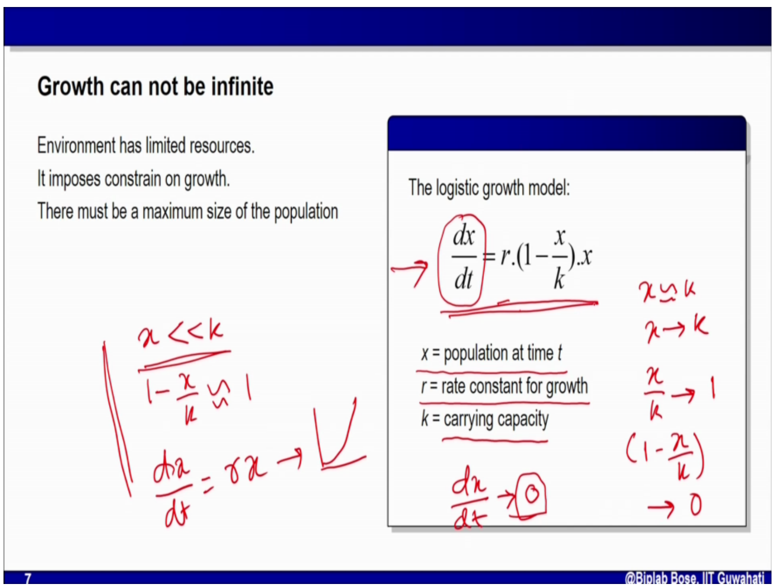

So this limits comes from the crunch of resources. If you are growing bacteria on a plate after sometime there will be no food, there will be no space further to grow the bacteria. If the population of a city increasing after sometime there will no place for the people to stay. There will resource crunch for food, job, etc. So population will ultimately plateau and will become stationary. So how can I make this initial model of population growth bit more realistic to incorporate this idea that the population cannot keep on growing infinitely. I tried it by getting a very simple model. I tweaked the equation, that ordinary differential equation to create a new ordinary differential.

Here also your X is the population at time T R is the rate constant for growth. So, dx/dt is the derivative which represent, Rate of change of of your population is equal to I have written r into obviously X in between I have $(1-\frac{x}{k})$ What is K? K we call carrying capacity. How much bacteria can grow in my agar plate? How much bacteria can grow in my fermenter of one litre size? How many people k can stay in a city? So K is the carrying capacity of the environment. So if you look into this equation, what will happen?

Initially when X is much smaller than K, that means the population size is much smaller than K then 1 minus X by K is almost equivalent to 1. So then my rate equation will become $\frac{dx}{dt}=r.X$. Just like my previous equation. So this will give me exponential growth. But after sometime, after sometime what will happen, X will become close to K. Or I can write X will start approaching the carrying capacity K. So then X by K will be moving towards 1 and 1 minus X by K will tends towards 0. So my rate of change dx/dt will tends towards 0. That means initially, initially I will have growth, exponential growth but as the population reaches the carrying capacity of the environment the growth rate will start reducing and eventually it will become 0. This type of equation is called logistic growth equation and we'll call this model logistic model.

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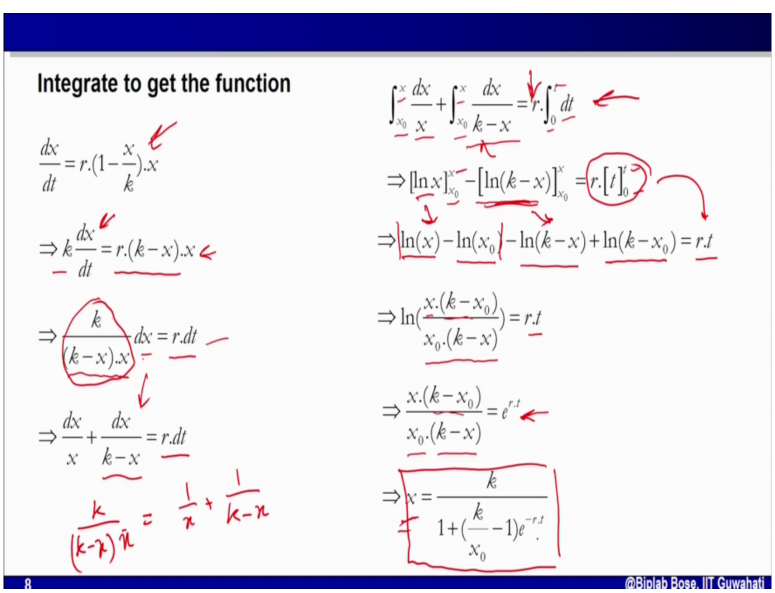

Now let us go back to questioning this model. If you remember we can ask two type of question. We can ask a specific type of question that I have a particular population right now, what will the population in size after two days three days or something sometime after if there is time or we can try to understand the generalized question, that what is the generalized dynamics of the population. Now to get this answer we have to integrate my ordinary differential equation. So this is my ordinary differential equation here. In this case I will try to integrate it. Before integration I want to rearrange term. So I have rearranged this term I have multiplied both side by K. So I get $K\left(\frac{dX}{dt}\right)=r(1-X).X$.

Then remember I have to separate out X and t . So I have done that. So I have rearranged term here to separate out X and t on both on the other side of the equal to sign. So I get $\frac{\frac{k}{k-x}*X}{dX}=r.dt$. I can rearrange this term further. This whole thing, this whole thing $\frac{k}{k-x}*X$ is nothing but $\frac{1}{x}+\frac{1}{(k-x)}$. So from this I get $\frac{dX}{X}+\frac{dX}{k-X}=r.dt$. So now we will integrate both side. So that's what we have here. We integrate, dX/X from X_0 that is the initial population size to the final population X. $\int_{Xo}^{X}\frac{dX}{k-X}$. And we integrate dT from 0 to T.

Remember I have r it is rate constant, it is a parameter, it is not changing with time. And I have K the carrying capacity which is also a parameter and constant is not changing with time. So if I use the general formula of integration, from integrating $\frac{dX}{X}$ I get lnX, integrating $\frac{dX}{k-X}$ I get - lnK -X. So I have to integrate from X_0 to X. And this one also to be integrated from X_0 to X and time from 0 to t . So by simplification what I get? I get ln of $X-lnXo$ this whole thing is coming from this one. From this one, from this one I get $lnK-X+lnK-X0=rt$. I am getting this from this one. So if simplify further coupling all those, taking all those logarithmic term together I get $(lnX*1-X0.k-X0)/X0*K-X=rt$.

I will do further rearrangement. I'm taking the log out, so I get $\frac{K-Xo}{X*(¿)} ¿ ¿ ¿$. Exponential term on my right hand side. Remember I want X as a function of time so I will keep X on one side of the equal to sign and keep everything else on the other side. So that's what I have done here. Rearrange the term and I get $X=K/1+\left(\frac{k}{X0}-1\right)*e^{rt}$. You can try this rearrangement yourself. It will be clear and you'll see the simple rearrangement. So this is the function, final function that I have got here. $X=K/1+\left(\frac{k}{X0}-1\right)*e^{rt}$. So using this function I can actually understand the dynamics of the process.

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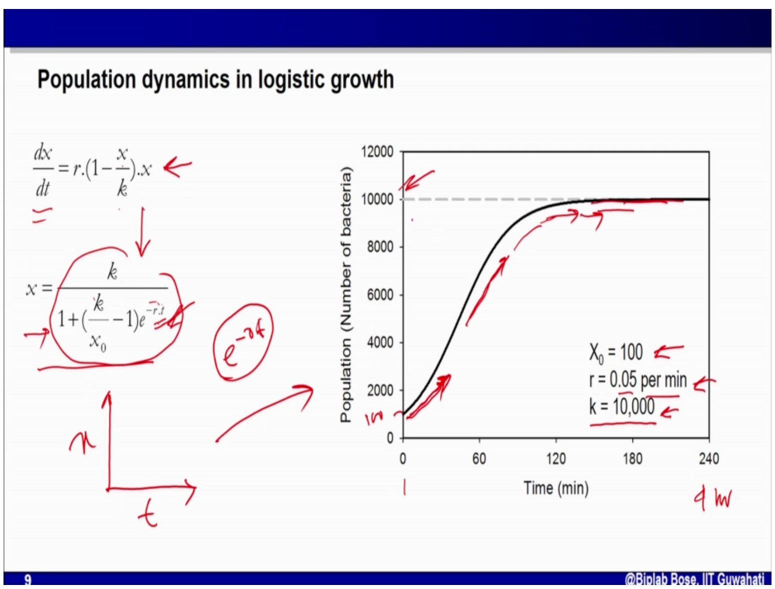

So what I started with. I first made a mathematical model in terms of ordinary differential equation, I got the function of a function to represent X in terms of function of time by integrating that equation and now I want to plot X in the vertical axis and time in the horizontal axis. And that's what I have done here. For X_0 equal to 100, just in the earlier example we started with 100 bacteria, rate of growth r , is a constant for a growth is 0.05 per minute, remember this rate constant not rate. K is the carrying capacity, that is 10,000 I have taken, that mean my plate or my fermenter can carry or accommodate only 10,000 bacteria. So now if I put this value in this function I get a curve like this for different values of t.

So see here I have plotted from 0 to 4 hour. This is 4 hour. Just like the previous plot. In the previous case we had it was exponentially growing like this that is not happening here. You can see, here in this case the population is growing and then after sometime it is saturating at the carrying capacity 10,000. When X is small, then K your rate is higher, so initially you have growth as T tends to infinity this becomes smaller and smaller. The whole thing become, the lower denominator become one and X reaches towards K that is the carrying capacity. So what is happening here? We start with 100 cells, initially increases exponentially as it reaches towards the carrying capacity K it becomes shallower, shallower and then it becomes flat and hits the ceiling, the saturation point of 10,000. There is no further growth of the bacteria.

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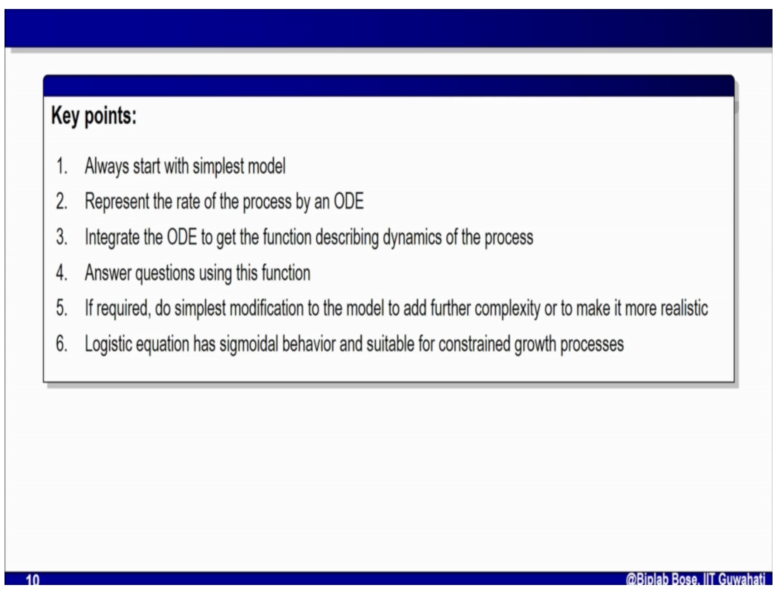

So if I jot down what we learnt from these two exercises remember I started with the simplest model, the model of growth of bacteria without any bound. I have not considered death. I have not considered any constraint from the environment on the population. And we got exponential growth. We have used the ordinary differential equation to represent the rate of change of the population size then integrated that to get the function. The function that represent the population size with respect to time and then we answered questions. Then we moved into making bit complex model. So starting with a simple model, then I moved into logistic model where we have considered that environment has limited resource to accommodate a population.

So we have introduced a concept of carrying capacity and just tweaked our previous equation with small addition here and there to create a new ordinary differential equation. And that ordinary differential equation represents the reality far better than the previous one because now population cannot grow continuously. After sometime it will reach the carrying capacity of the environment and the population growth will stabilize and remain fixed at that value. And we created the ODE, integrated that. After integrating we have plotted $t \, vs \, X$ population in uh, change in population with respect to time and we got a sigmoidal curve. Initially there was increase, faster increase and then the increase become slower and reaches a plateau.

So this type of model is called logistic model. And the equation that we use is called logistic equation. You can use this type of logistic equation in other cases also beyond population growth where the process is constrained by the resources from the environment other. So initially there will be higher rate of the process and then it will saturate. And we will get a sigmoidal dynamics in respect to time. That's all for this module. Thank you for watching. We'll build new modules in future.