Introduction to Dynamical Models in Biology
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Lecture 6
Numerical solution of ODE - 2

Hello, welcome to module 6 week 1 of our course on introduction to Dynamical models in biology. In the last module we have learned the basic concept of Numerical solutions of ordinary differential equation. We have learned how to use Euler's method we have learned the basic concept of Euler's method. Now we will learn in this module better algorithm to implement numerical solutions of ordinary differential equations.

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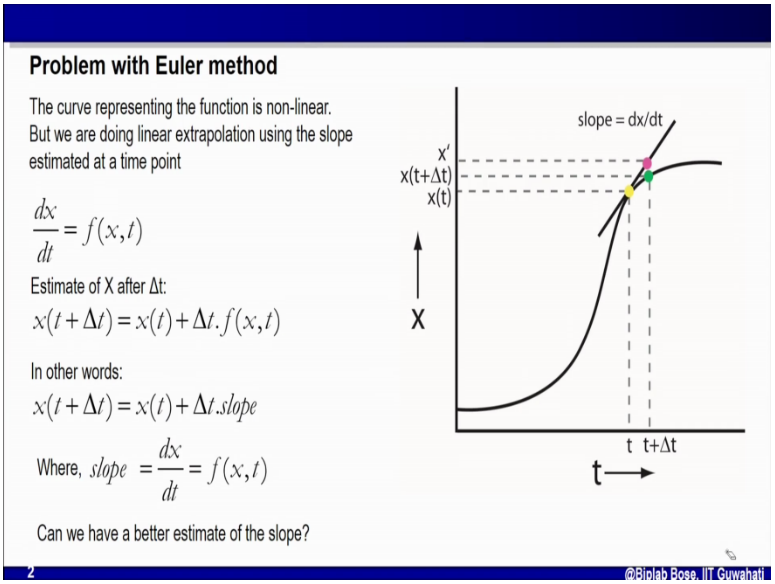

Now let us look back to Euler's method to know why we want better algorithms. In Euler's method what you were doing? Suppose your function for X is this one, the solid line shown here in the graph, t is varying in this horizontal axis and X is varying on the vertical axis. At time t, value of X is this the yellow point at time $t+\Delta t$ if you increase (Δ) time by Δt, the real value the exact value along this function for X will be this green one that is $X(t+\Delta t)$.

In Euler's method what you are doing you are estimating the value of $X(t+\Delta t)$ using this relationship. $X(t+\Delta t)$ is equal to $X(t)+\Delta t$ into function of X and t . Where you got this function? We got this function from the differential equation. Now in other words what you are doing is that $X(t+\Delta t)$ is equal to $X(t)+\Delta t¿$ and slope. Remember the derivative of a function at a particular point is nothing but slope at that point. So at time t, X is this yellow part and if I draw a slope or a tangent along this line that is the curve line which is the function for X or if I draw a tangent here the slope of that tangent will be dx/dt . So what you are doing? We are getting the value of X at the next time point that is our $t+\Delta t$ by doing linear extra correlation along this tangent, along this tangent you are doing linear extra correlation and you are getting the value of X at this new time point as shown in by this pink dot.

Now as you can see ,obviously, as you are following this tangent, you are following this slope we are getting a value slightly higher than the real value of X which is shown by green. So you can see there is a difference between the pink dot and the green dot, this error is coming because from the yellow point we are doing linear extrapolation along the tangent having the slope dx/dt . We can rectify this one if I correct my estimate of slope slightly and generate another line here having a different slope and if I follow this line most probably I will reach at the right value at the green point. So today in this module we will study an algorithm which replaces this estimate of slope by a better estimate of slope rather an average estimate of slope which is much better in extrapolation and getting the value new value of $\frac{t+¿}{X¿}$ $\Delta t¿$

This method in general know as Runge Kutta method and there are multiple algorithms belonging to this family. What we will discuss here 4th order Runge Kutta method. So we will discuss about this algorithm and have to remember that there are many modern versions of these algorithms are available having different advantages and most of the time we will use Various versions of this algorithms in different forms for solving ordinary differential equation.

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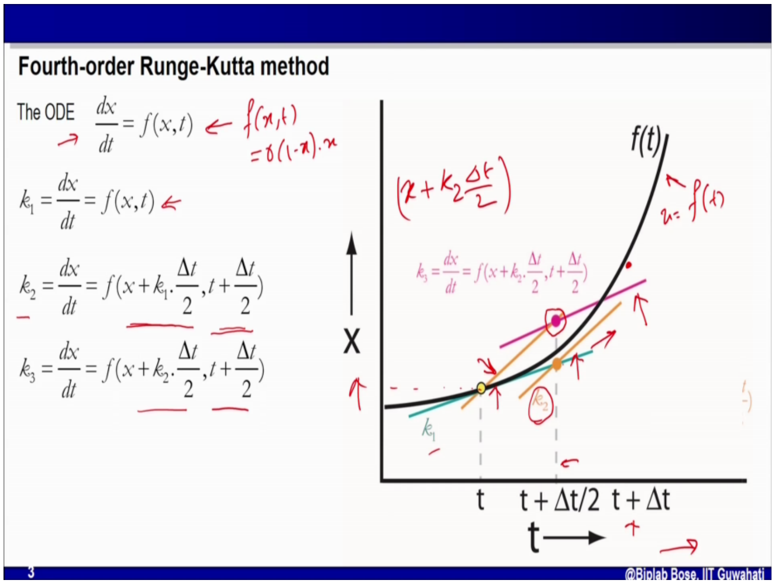

So let us start I will give a function ODE dx/dt which is equal to $f(X,t)$ in my earlier example actually $f(X,t)$ when I was discussing about the spread of a disease, $f(X,t)$ is equal to $r(1-X)X$. So I am writing this generalize form here. Because we are trying to understand this method by generalized fashion so dx/dt is equal to something which is equal to X at time t. So K1 as in all our method K1 the slope is equal to the slope dx/dt which is equal to this function $f(X,t)$. So if I look graphically what I have a horizontal axis, I have on time this vertical axis we have X and this curve and this is nothing but time function of representing, X is equal to function of time. So I am at time t.

This yellow dot is the value of X here at time t so this $X(t)$ now if I draw a tangent here on slope which is equal to dx/dt as given there I have this bluish line and if I take a time step at $t+\Delta t$, Δt time step and draw a vertical line as shown in this dotted line, this dotted line intersect with this blue line at this red point. So this is my estimate for Euler's method for the value of $X(t+\Delta t)$ and obviously you can see this is quite different from the real value here. Now what if I do some modification? What I do I extend this tangent at this yellow point along with a slope K1 up to the midpoint that is up to Δt by 2 so this is 1/2 way $\Delta/2$. So if I draw a vertical line by $t+(\frac{\Delta t}{2})$ there is a dotted line shown here that intersects this blue line and this orange dot, so this position it intersects.

What is the value of X here? The value of X at this position is $X+K1$ into the interval $\Delta t/2$. So the value of X here in this orange is actually $x+K1*\Delta t/2$. So now what I will do, I will estimate the slope if I draw a tangent there and if I estimate the slope of that I will get the slope K2, so using this differential equation $\frac{dx}{dt}$ I calculate the slope of this orange line drawn at this orange dot that will be equal to K2 equal to $\frac{dx}{dt}$ equal to function of $X+k1(\frac{\Delta t}{2})$ and $\frac{t+\Delta t}{2}$ because these positions coordinates are $\frac{t+\Delta t}{2}$ and $X+k1(\frac{\Delta t}{2})$.

So I get the slope at this orange point, once I have got the slope here at this orange point I draw a line at this original yellow point using the same slope. So I get a orange line here, if we will clean this part so things will become clear for you. So I am taking this orange line slope K2 and drawing a orange line at the original position of X at this yellow point, so I get this new orange line. And I extend that new orange line till it intersects the vertical line drawn at $\frac{t+\Delta t}{2}$. That is this vertical line. So I get a pink dot here. This is the point of intersection. What is the value of X there? The value of X there would be value of X at that pink point will be $X+k2(\frac{\Delta t}{2})$ that will be the value of X. So now if I draw a tangent having a slope or a line having a slope $\frac{dx}{dt}$ the slope of that line will be K3 equal to $\frac{dx}{dt}$ equal to ¿ f ¿ $X+k1(\frac{\Delta t}{2})$¿ is the value of at that position and the value of time at that pink position that is $\frac{t+\Delta t}{2}$. So in this way what I have got? I have got the slope of this pink line which is k 3.

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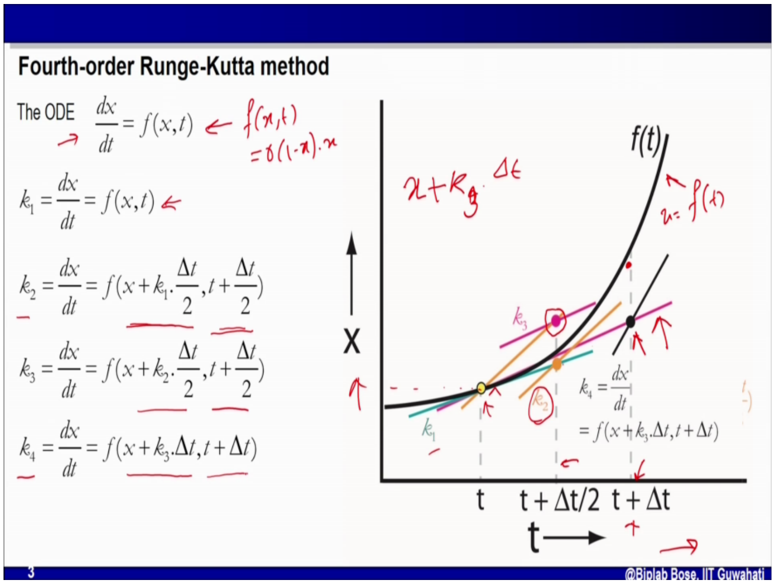

So now let us clean it bit and see what I will do next, now take that slope K3 and draw a line at this yellow original point where X was having the same slope as K3. So I get this new pink line, so I have a new pink line drawn at the original value of point at X and extend it till it intersect the vertical line drawn at $t+\Delta t$ that is the time step you are taking so that intersection point is this black dot. What will be the value of X there? The value of X in that black dot will be X the original value of $X+K3*\Delta t$. So you have taken the slope of K3 and extended up to by an increment Δt . So the new value of X at this black point is $X+K3+\Delta t$. Now at this black point I draw a black line having a slope $\frac{dx}{dt}$. So the slope of that line will be K4 which is equal to $\frac{dx}{dt}$ at that point. At that point the $\frac{dx}{dt}$ will be function of $x+K3+\Delta t$. and $t+\Delta t.$ So the coordinates here the t is $t+\Delta t$ and X is $X+K3*\Delta t$ sop I can calculate the slope at that point. So what I have now is 3 slopes original K1, K2, K3 and K4.

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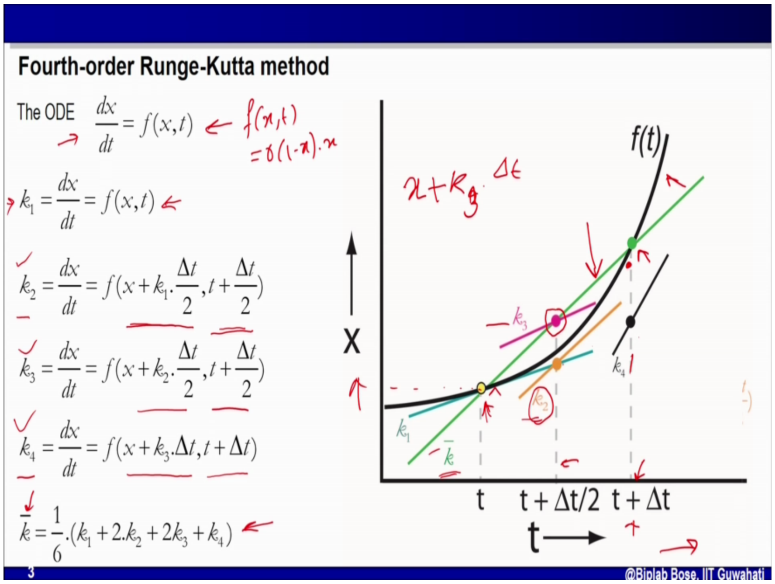

So what I do I take a average of these 4 slopes K1, K2, K3 and K4 and I take a weighted average that means I give different weight to each of this slopes. For K1 and K4 we will give equal weightage where for K2 and K3 I give twice the weightage. So I get the mean value of K is $\frac{K1+2*K2+2*K3+K4}{6}$, because remember $1+2+2+1=6$, so if I now draw a line having mean K slope along with this initial position of X that is this yellow point then I get this green line. So what I have done if I clean a bit, so what I have done I have three slopes K1, K2, K3 and K4 I have taken the weighted average of all these four slopes and got a new slope K mean and I have drawn a new line the green line shown in the graph which has a slope of mean K and goes at the initial position of X at time t. As you can see if I extrapolate this line at $t+\Delta t$ time it will reach the green point and the green point is on this original black curve which represent function of time or represent the behavior of X with respect to time. So in this way we have extrapolated the average value of the slope and got the new value of $X(t+\Delta t)$ time after the increment of time by Δt.

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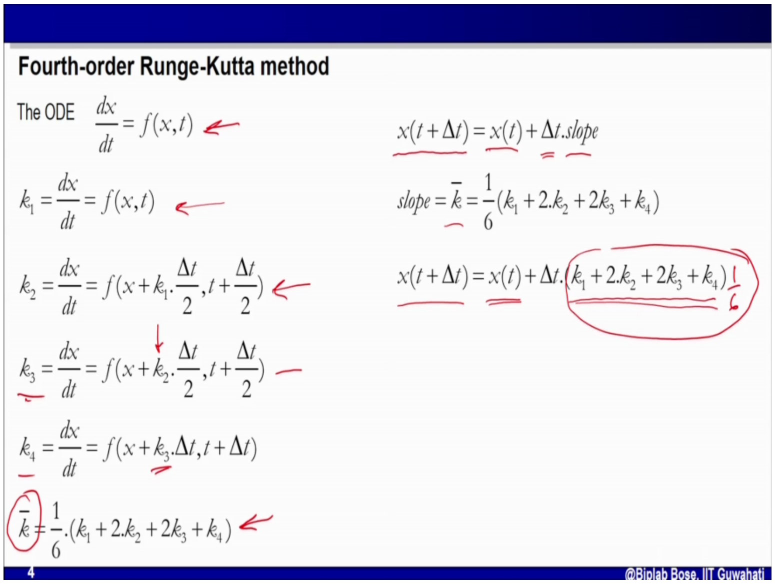

So if I brief what we are doing here you have a ordinary deferential equation given first which is in generalized form have written, $\frac{dx}{dt}$ is a function of X and t. I calculate first the first slope K1 using the same Euler's method which is nothing which you are using Euler's method K1 is $\frac{dx}{dt}$ which is a function of X and t, then using that slope I calculate a new slope K2 which is nothing but function of $X + K\frac{1*\Delta t}{2}$ that is a 1 / 2 way of your increment and $\frac{t+\Delta t}{2}$, that way I use the K2 to calculate the K3 and then I use K3 to calculate K4 four slopes. Then I make weighted average of these 4 slopes K1, K2, K3 and K4 to get a mean value of K that is $\overline{K}$.

In general what we have know as we have lean in Euler's method X at a new time point that is $X(t+\Delta t) = X$ at the older time point as X at time t plus the incrementing time Δt into slope. Here we will use the average slope $\overline{K}$. So if X at time t is Xt this one then $X(t+\Delta t)$ would be $X(t) + \Delta t$ time that is the incrementing time into the average time, average slope here we should write 1/ 6, it is missing initially, so this way using this average slope I can calculate the new value of X.

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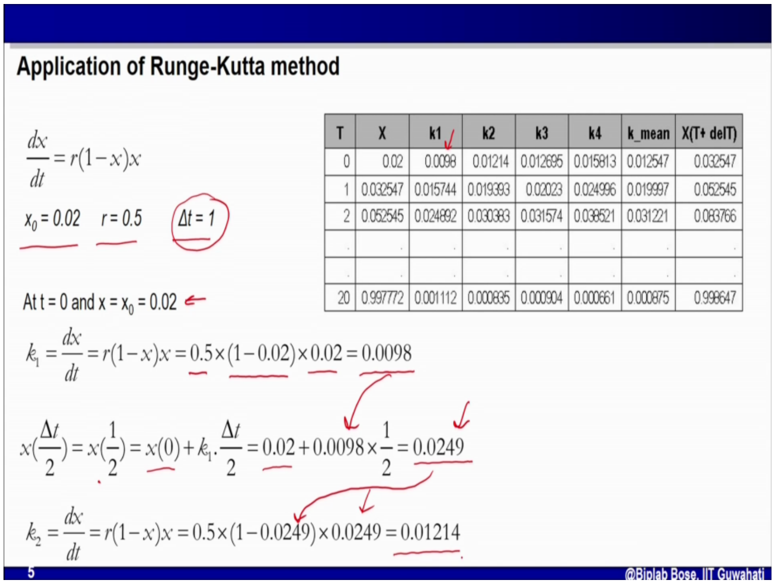

T	X	k1	k2	k3	k4	k_mean	X(T+ delT)
0	0.02	0.0098	0.01214	0.012695	0.015813	0.012547	0.032547
1	0.032547	0.015744	0.019393	0.02023	0.024996	0.019997	0.052545
2	0.052545	0.024892	0.030383	0.031574	0.038521	0.031221	0.083766
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
20	0.997772	0.001112	0.000835	0.000904	0.000661	0.000875	0.998647

Let us try this method. This is the 4th ordered Runge Kutta method on our problem of spread of infected diseases, so if you remember the ordinary differential equation for modeling this spread of infectious disease was $\frac{dx}{dt}=r(1-X)X$. r is given as 0. 5. $X(t)=$¿ 0.02 and I am taking here Δt as 1, so I increment time by one. So how will proceed first I will calculate K1 so consider I m at time point zero two, $t=0$ so the value of X will be 0.02 to as given, this is a initial condition. So at this point the slope K1 will be equal to $\frac{dx}{dt}*r(1-X)X=0.5$ (1-.02) (0.02)=0.0098, so in this table I have written it here point 0.0098. Now I have to calculate, I have to draw at this value of $X(t)$ having a slope K1 and that I will extend up to $\frac{\Delta t}{2}$ times up.

So the new value of X there will be $X(\frac{\Delta t}{2})$ that is X remember Δt is 1 so X $\left(\frac{1}{2}\right)=X(0)+K1(\frac{\Delta t}{2})$, so that is at X0, X at 0 time point is 0.02 it is already given here, plus the slope I have calculated so these slope comes here into 1 / 2, one is Δt that is by 2, that's give me a value 0.0249. So I got a point from $\frac{\Delta t}{2}$ away from time equal to 0 and the value of X I have calculated at that time that is this one. I want to draw a line there having a slope equal dx ¿ dt and the slope there will be K2, $\frac{dx}{dt}=r(1-X)X$. So let us put the numerical value r is $0.5.1-X$ and X is the new value of X into this one this equal to 0.01214 so I write down it here in this table.

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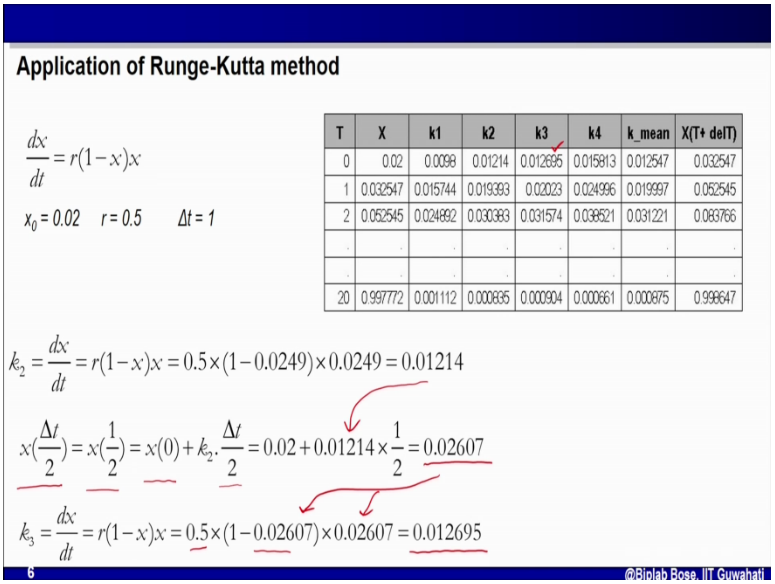

In this way I again try to calculate K3 so to calculate K3 what I will do, using the slope K2 I will draw a line from the position of $X(t)=0$ and extend it so that it reaches up to $\frac{\Delta t}{2}$. So the value of X at that position would be $X\left(\frac{\Delta t}{2}\right)=X\left(\frac{1}{2}\right)$, one is the Δt will be value of X at 0 time point + K2 because now I will use the slope $K2*\Delta t/2$ and if you put this value K2 is this one that I have calculated in the last stage I get the new value of X as 0.02607. So by using this value I will calculate the slope of a line drawn having a slope $\frac{dx}{dt}$, so numerical value of slope will be K3 is the new slope is equal to $\frac{dx}{dt}=r(1-X)X$. r is $0.5.1-X$, I have calculated just now, these 2 values are for X 0.02607 so if you multiply that you will get 0.012695 so I put that value here.

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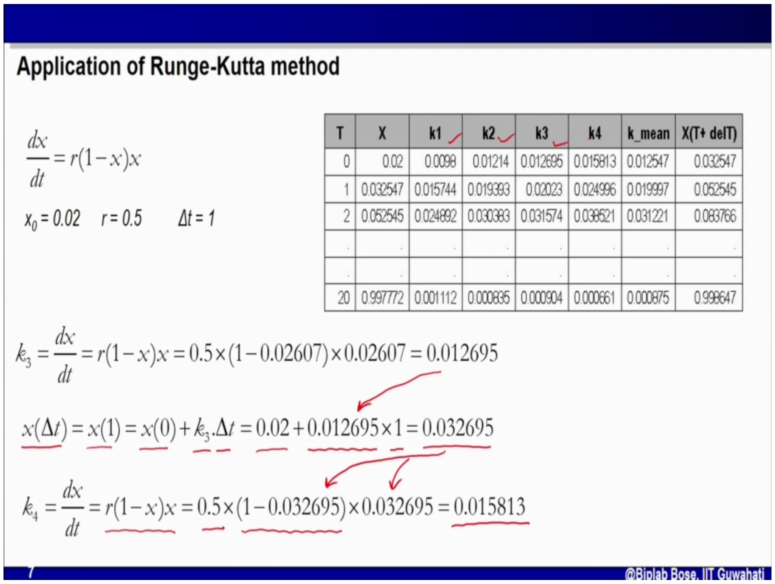

So I got the value of K1, K2 and K3 now I have to estimate the value of K4 using the value of K3 so using the value of K3 I draw a line at $X(t)=0$ and extrapolated up to Δt time step. So the value of X will be $X(\Delta t)$ that is $X(1)=X(0)+K3*\Delta t$. So if the numerical value $X(0)=¿$ 0.02 is initial condition plus the value of slope that I have calculated in the last step into one is the one is the time step is equal to 0.032695 so that is the value of X wherein extrapolate a straight line having slope X = 0 so its $X(0)$. Now let's calculate K4 so if I draw a line $\frac{dx}{dt}$ at that point the slope of that line will be K4 equal to $\frac{dx}{dt}=r(1-X)X$, r is 0.5, $X=1-0.032695$ that I have calculated just now. So now these 2 are here if you multiply you get the value of K4 as 0.015813. So what you have got I have got K1, K2, k3 and k4.

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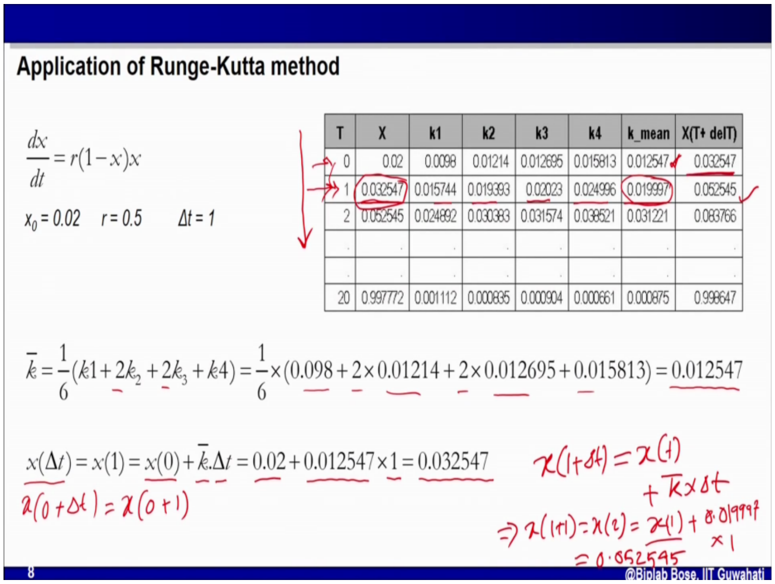

So now I will take weighted average so remember we were giving different weighted to different slopes so for K2 and K3 the weightage is 2 2 for rest K1 and K4 the weightage is 1 so the total is 6 so we have some $\frac{K1 + K2 + 2 * K3 + K4}{6}$. We have already calculated the value of K1, K2, K3 and K4 then put this numerical values here, don't forget to multiply by 2 for K2 and K3 and you get a mean value of slope as 0.012547. So write it down in this table. So if I know increment time by Δt using this mean slope I will get a new value of $X(t+\Delta t)$ so t was 0, initial time was 0 so the value of X at this is essentially this $X(0+\Delta t)$ that is $X(\Delta t) = X(0+1)$ x, remember Δt is one that will be equal to X(0) time point plus mean slope into $\Delta t\, X$ at 0 time point is 0.02. The mean slope is 0.012547 I have calculated here into the time set one so I get a new value of X that is point 0.032547 so in the table I write time at time one the value of X is 0.023547 I have written it here the same thing.

So if I see this method I have moved from time 0 to time one by increment Δt is equal to one using the mean K, now I am at the X(t)=1 , I will again calculate K1, K2, K3 and K4 by using the same method and can calculate the weighted average method and I have done this and shown in the table that is 0.1997. Now using the same addition that this mean into Δt that is $X(1+\Delta t) = X(1 + Kmean * \Delta t)$ so this is nothing but X one plus one that is X at time point 2 will be X as 1+K is 0.01997 into Δt is one, X at one is this value 0.023547 so if you sum up you will get the value of x at 2 that is this value 0.052545 so in this way you can keep calculating these 4 slopes at each of this X as on increasing time by Δt and can calculate the value of X at the next time, that's what I have shown briefly in this table. And you stop at the final time point for example the time point may be 20 and you get the value of X at time point 20.

So this is in brief the 4th order Runge Kutta method there is exponential mathematics method behind this we have avoided that and tried to intuitively and graphically understand how this algorithms works. Now let us look that by doing this complicated calculation of average K are we gaining something. What I have done in this slide I have compared the Euler's method in this 4th order Runge Kutta method both are numerical method and try to understand what is the different result obtain for the same system by using this algorithms.

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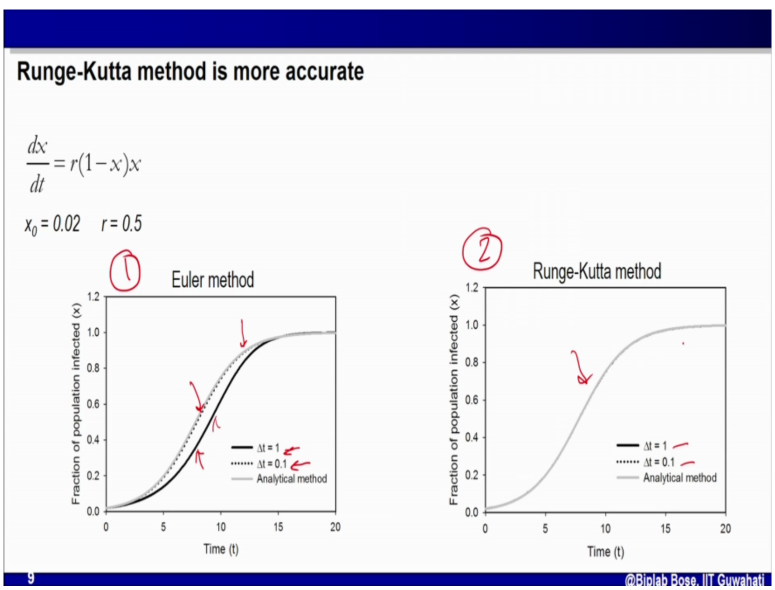

ODE is the same one that we use for modeling for spread of infected disease $\frac{dx}{dt} = r(1-X)X$. $X(0)$ that is X_0 that is equal to 0.02 and r is 0.5 as we kept it earlier, so the first graph here I have shown is the result from Euler's method. We have seen this graph earlier the grey line is the exact analytical method, result from the exact analytical method the dotted line here shown is the result of Euler's method when you use $\Delta t = 0.1$. It matches almost with the analytical method with the symbolic method but the problem start when you start taking bigger Δt, Δt is equal to 1 then the result is this black line, as you see and you notice earlier also that deviating from the real original result. The result obtain by analytical method that means this approximate method is an error when I increase $\Delta t =$¿ 1.

Now let's see what face when we use Runge Kutta method that we have discussed last. So again this plot the plot 2 is the same system again I have plotted the exact same solution like grey line and I have done 2 simulations using Runge Kutta algorithms with Δt =1 and Δt = 0.1. You can see clearly in this one that all the three have the same data and they have got collage in the same line and you cannot see them because they are overlapped. So it shows that in the Runge Kutta method we have used the average slope, this average slope technique is giving me more better more accurate result than the Euler's method. As I said earlier most of the recent versions of algorithms that wide utilize solving ordinary differential equation in biology and other fields are actually derivatives of the 4th order Runge Kutta method and in general we can say it belongs to the same family.

Now what I have discussed with you now that if I take a average slope at a particular time and then estimate X at the new time point we are getting better result, but still the issue of Δt lingers here. If you remember in our previous module we have discussed that if your Δt is bigger your error will increase and even if you are using Runge Kutta method there will be situation where Δt , if you will take bigger Δt you will have huge errors in your calculations. So how to decide what that will be the value that you take. If you remember if I use very large Δt I will have erroneous result, if I use very small Δt using the algorithm or Runge Kutta method I will get good result and the error will be less but the problem is your computation time will increase.

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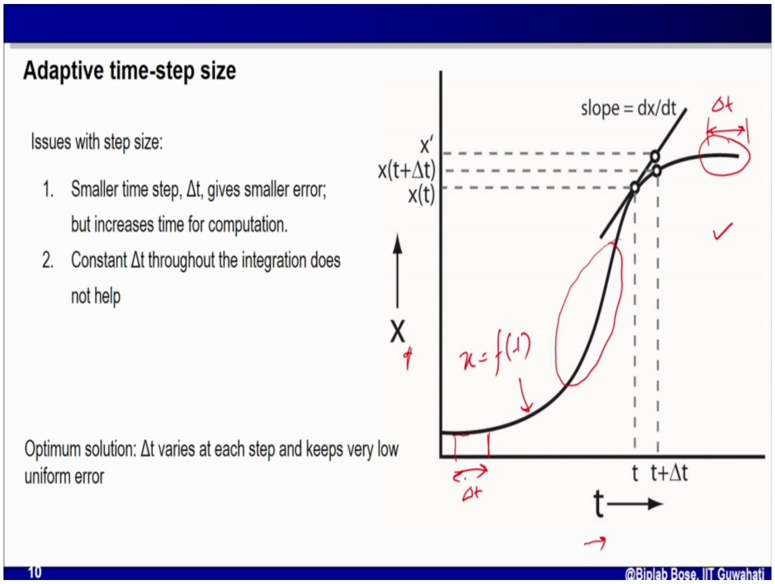

Now there is another issue in this who algorithm whether it is Euler's method or Runge Kutta method if you have noticed I have used a fixed value of Δt for my whole calculation, but look at this graph shown in this slide the same plot for the spread of infectious disease module that we have so we have time in horizontal and X in vertical axis and this solid track line is function of time, X is a function of time. As you can see in the lower part where time is less, lower value. If I take you can intuitively understand, if I take a large delta T my error will not be so significant because the line is almost flat, almost linear, similar thing at the end here also the system is almost linear if I almost horizontal. If I take a Δt large I will not have much problem whereas in this middle region X is changing drastically very fast with respect to time I have to take small small interval of Δt so that I get less errors.

That means by just looking at this graph I can say it is not wiser to use the same Δt for the whole calculation, maybe where X is not changing much with time I can use large Δt and reduce time of computation and then we are at the region of the function where X is changing very fast with respect to time I use small small Δt increment so that I get less error. That means I have to adapt delta along my solution as I am solving this ODE I should keep on changing or adapting Δt based on the need or based on the function. That's why we got adaptive time steps size algorithm. So what we will do, in this algorithm we try to estimate the optimum delta T as we keep on solving a particular ordinary differential equation.

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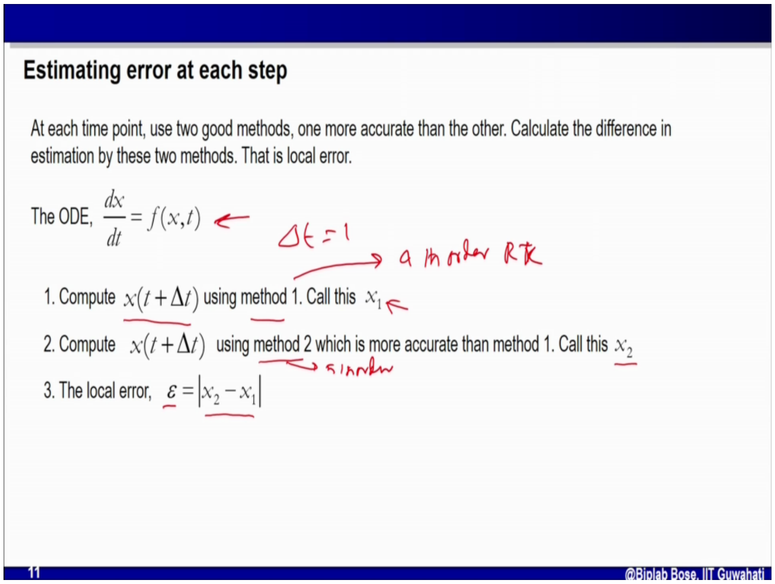

There are many ways of doing this one I will discuss one generic method which has been implemented in many algorithms in various variations. To use the adaptive method what I require is that I have to take a Δt and then I have to estimate how much error is there in my calculation if I take that Δt. For example suppose you are at time equal to zero and you want to move forward and you have taken Δt as one, so you have to estimate there if I take Δt equal to one how much error do I get. Is the error very high? If the error is very high, I will reduce Δt may be into 0.5 and try again. So I want to continuously calculate error in my estimate and change value of Δt as I solve the ODE.

Now if I have to estimate error I should know what is the real value that means I should know what is the real function of X with respect to time. Now the problem is if I know the real function I don't need to solve it and as we have discussed analytical solution of most of the ODE is very difficult in most of the cases, so actually I do not know the real value of X at a particular time point as per the real function as I don't know the real function then how can I estimate error. There is a way out and that way out I will discuss now. Rather than comparing with the best one the real one with your algorithm, you take 2 algorithms, one algorithm more accurate than the other one and you know that. So maybe there are many algorithms in the family of Runge Kutta method, we have discussed there are 4^{th} order, there are 5 orders method of Runge Kutta algorithm also and that one is better than the 4^{th} order one. So we can take 2 different algorithms, one is better than the other one.

So what you do? You calculate the new value of X at the next time point using these 2 methods and calculate the difference between these 2 methods, result between these two methods… so that difference in result is obtained by these 2 methods is the estimate of error. So what I am doing. I have an ODE given to me and I take a Δt time, suppose that is equal 1 and then I compute value of X with $t+\Delta t$ using method 1 so that may be 4^{th} order Runge Kutta Method and I called that result X1, then I do the same calculation to get the value of $X(t+\Delta t)$ using a new method this may be 5^{th} order of Runge Kutta method which is known to give more accurate than the 4^{th} order Runge Kutta method and I calculate the value of $X(t+\Delta t)$ let us call that X2 and now I calculate the difference between X2 and X1 and that is my error and error is absolute value of difference between X2 and X1. So once I have calculated the error now I do some test. As a user of this algorithm I decide before hand how much error I will tolerate. So I have decided a upper limit of tolerance and a lower limit of tolerance. Suppose I can decide the error should not be more than 10^{-6} and it should be in the range of $10^{-8}¿10^{-6}$. So this is my tolerance limit.

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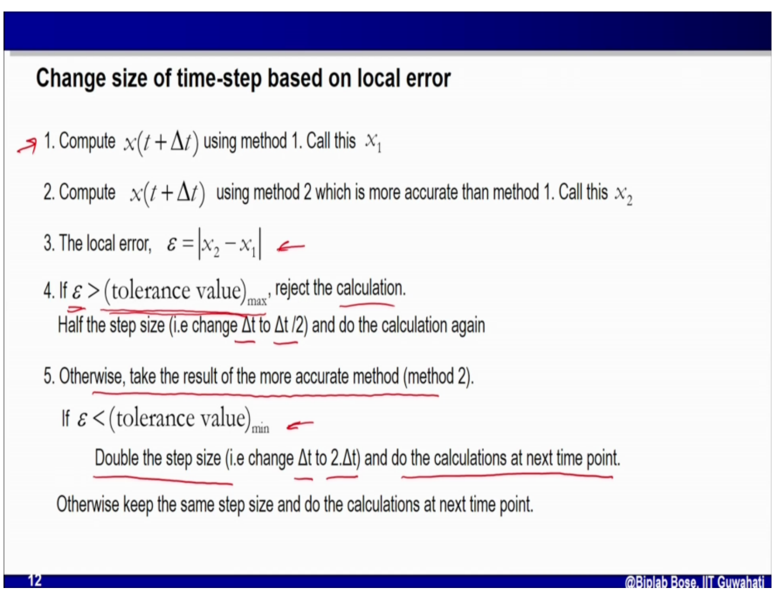

So what I do I have computed the error just I have said and then I use the logic if this error is bigger than the maximum value of tolerance that means the error is bigger than the tolerance I will accept I reject the calculation and I say this Δt is not good enough and I reduce Δt to half of it Δt /2 and I go back to again to step one into the whole computation like method 1 and method 2 and calculate the error again, and again I check the error I am getting is bigger than the maximum tolerance level that I will allow or not. Suppose now the error is not bigger than the tolerance value that I will allow the maximum tolerance value that I can allow other than that is either smaller than that or equal to that so if that is true that it is not bigger than the maximum tolerated value I will allow then what I will do I will accept that result obtained by the accurate method that is the second method, otherwise take result for the more accurate method that is method 2.

Now if the error is less than the tolerance value that is the minimum tolerance value that you can allow what you do you increase the time step because you realize the Δt you are using is giving you very accurate results so what you don't I increase Δt slightly and you will see whether that will still allow me to work in a reduced error way. So what you are doing here is you double the step size from Δt to $t.\Delta t$ and do the calculation for the next time point otherwise you maintain the Δt as the same value. Now I move to the next time point and do the same calculation and use the same logic from 1, 2, 3, 4 and 5 and calculate the value of X. So what we are doing is that depending upon the error, whether the error is big or whether the error is small we are changing the value of Δt .

Whether the value is big or small is decided by the user who decides the tolerance limit. So as I said I can say I want the error to be between $10^{-8}¿10^{-6}$. so my tolerance limits maximum value is 10^{-6} . or whereas the lower value of tolerance is 10^{-8} . In this way I can use the Runje Kutta method same Runje Kutta method with an adaptive time stamp and of the years that will give me better results and faster simulation.

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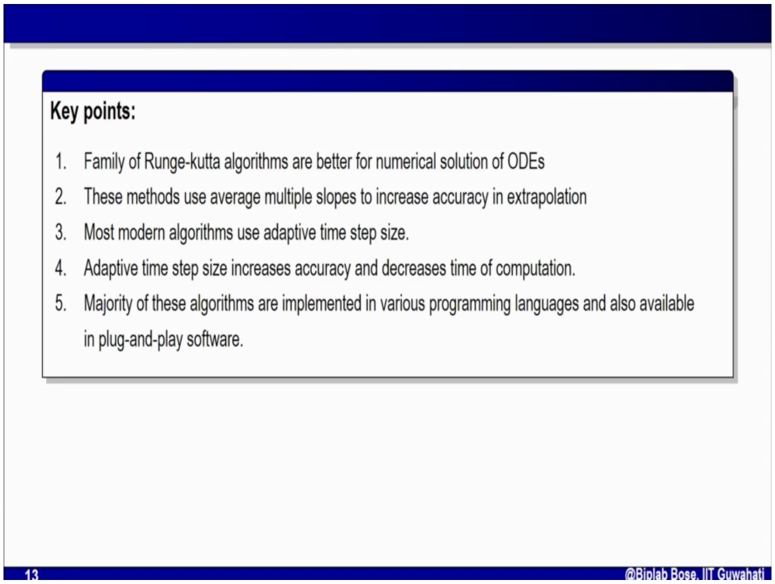

Most of the algorithm that is implemented in the readymade software that people use, biologist use to model ODE based problem actually have these types of adaptive time steps, algorithms and they are easily useable. So let us jot down the key points that we have learnt in this module. We have discussed the basic features of 4th order Runge Kutta method remember there is exhausted mathematical explanation for this algorithm, we avoided that and we have tried to understand that intuitively and graphically and also remember this is not the only algorithm available as a Runge Kutta algorithm. There is a family of algorithm which uses same idea of using average of multiple slopes calculated at a particular position.

In comparison to Euler's method, in Euler's method you only have one slope that can be used for calculating your value of the X at the next time point. This method, this family of algorithm based on Runge Kutta method and others which are usually used really with adaptive time step algorithm so that the Δt , the time step is not fixed and keeps on varying depending upon where you are in the function. That's all for today thank you for listening.