Introduction to Dynamical Models in Biology

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Lecture 9

Understanding Steady State

Hello! Welcome to module 3, week 2 of our course on interaction on dynamic models in biology. In this module I will introduce you new concept that is steady state and how to analyse the steady state. Analysing of steady state could be crucial in understanding dynamical systems and there are oiled into mathematical formulations, we primarily understand the concept of steady state using intuitive techniques.

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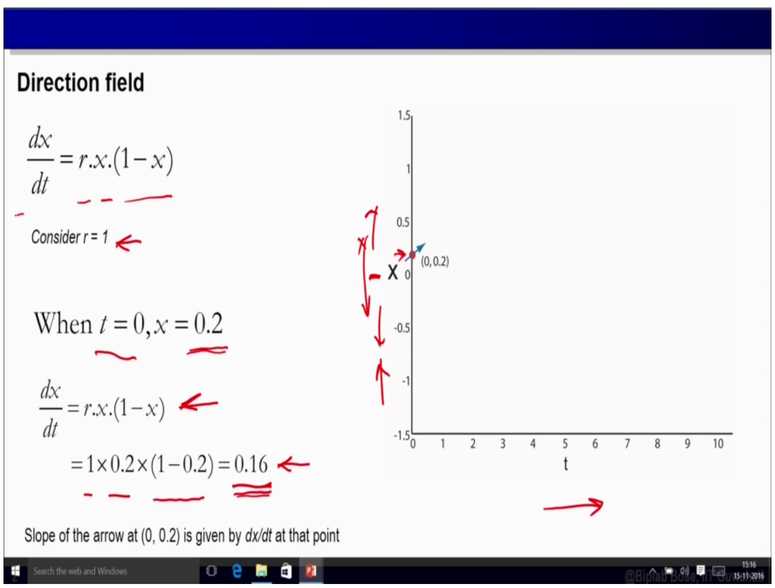

Let us start with some basic idea and I will introduce a concept of direct field in the beginning. Let us take the old model of spread of infectious disease. So, what was the ODE? X is the fraction which is infected with the disease, $\frac{dx}{dt}$ is the rate of change of population infected with the disease is equal to r.x (1-x). So, we are going to analyse in using the direction field. Let us see how to do that. The first thing I do I plot t, time in the horizontal axis and x in the vertical axis. So, let us take it time point for example t = 0 and at that time let us consider x = 0.2.

Notice one thing in my plotting region although x can never be negative, I have also shown negative values of x in this vertical axis. So, 0 is here, positive values are in this direction, negative values are in this direction. I have drawn this so that when I will draw direction field some interesting phenomenon we can see here. So, if I take a point t = 0 x = 0.2 and show it in this plotting region then that will be this red dot. t = 0 x = 0.2. Now let us calculate $\frac{dx}{dt}$ at that time that time point and at that value of x. So, $\frac{dx}{dt}$ = r.x (1-x). I have already written r = 1 so put

$\frac{dx}{dt}=1\,X\,0.2\,X\,(1-0.2)$ that is 0.8 and the multiplication gives you 0.16.

Now at that red dot in my plot let us draw a blue arrow that I have shown here a arrow having a slope which is equal to $\frac{dx}{dt}$. So, here at that point $\frac{dx}{dt}$ is 0.16 that comes around 9 to 10 degree angle so, put a arrow having a angle of 9 or 10 degree based on this $\frac{dx}{dt}$ and notice the arrow here is pointing up because with time $\frac{dx}{dt}$ is changing in the positive direction because I have a positive value here. So, what I have shown here in the graph I have taken a point at 0, 0.2 in this space of x versus t and then I have drawn a arrow there having a slope equal to $\frac{dx}{dt}$ at that point.

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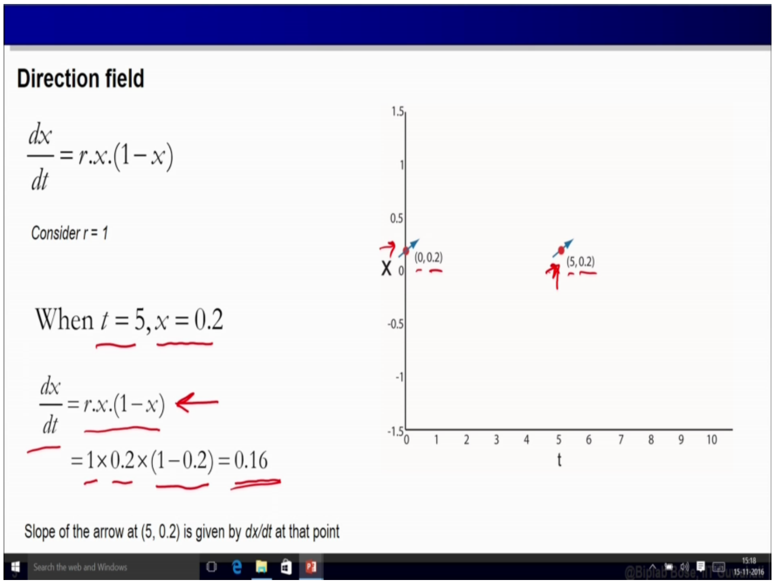

Let us take another point for example t = 5 and x = 0.2 let us calculate what is the value of $\frac{dx}{dt}$ at that point $1X0.2X(1-0.2)$ that is 0.8 you don't have any t term in this equation that means this is not effected by t so obviously I get the same value $\frac{dx}{dt}=0.16$. Let us put that point in this space and put a arrow having a slope 0.16 so that I have done here you have at (point) 5.2 time is 5, x is 0.2 and there I have drawn a arrow having slope 0.16 that is equivalent to almost 9 degree. If you notice these arrow and these arrow are actually parallel and their slope are same, that is because the ODE does not have any t term on the right hand side and only the difference between these two point is that time has changed. In case of the first point t is 0, in the next point t is 5 but x has remained same at 0.2 so, as the right hand side only has x term does not have t term, the slopes of both the arrows are same.

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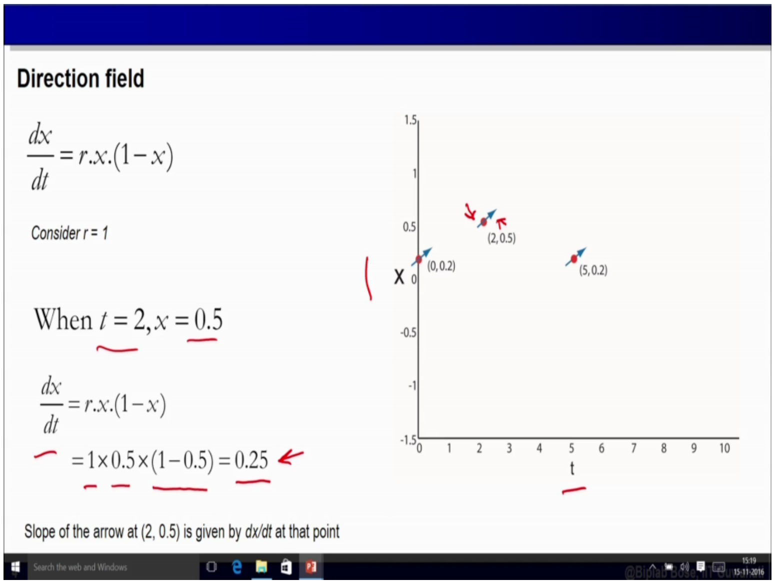

Let us take another point somewhere else. For example at 2.5 that I have shown here 2.5 and calculate $\frac{dx}{dt}$ there. It would be $1\,X\,0.5\,X(1-0.5)$, so, if you multiply it would be 0.25. Let us put a arrow there. Obviously arrow head would be pointing up because the 0.25 has plus sign there in the $\frac{dx}{dt}$, so the slope of that line will be 0.25 as I have calculates, it is equal to almost 14 or 15 degree. So, I have drawn a arrow here pointing up having a slope 2.5. So, what I have done I have a t v/s x space in which I have taken some point and at those points I have calculated $\frac{dx}{dt}$, the slope as per the ODE once I have calculated the slopes, I have drawn arrows with arrow head appropriate arrow head on those points having appropriate slopes as calculated from the derivative.

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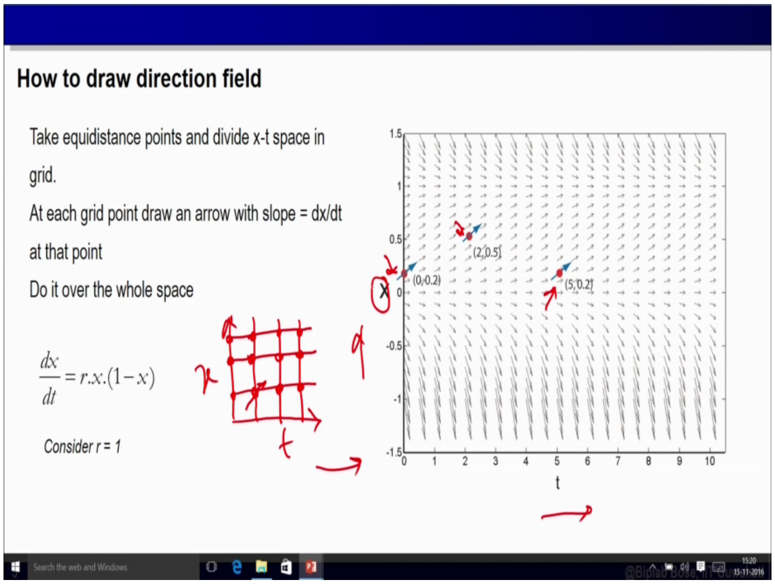

I can extend this further what I can do, I can take (the) divide the whole space of x and y, suppose this is x and t, this is my space x and t and I can divide it in equal grids like this. So, I can divide in equi-distance grid and each of this grid point, these are the grid point, I can calculate $\frac{dx}{dt}$ as I have done earlier and then I can draw arrows, that's what I have done here and shown in this plot this is t, this is x, we have to divide it in equi-distance grid, I have not shown the grid line for clarity and at each of this grid point I have calculated $\frac{dx}{dt}$ and drawn on a arrow. So, for example this one you have seen this one is a gridline with 0.2. This is another gridpoint with 2.5, this is another gridpoint with 5.2 and each of this gridline there is a arrow, the slope of the arrow is equal to the derivative of the dependent variable that is x at that point.

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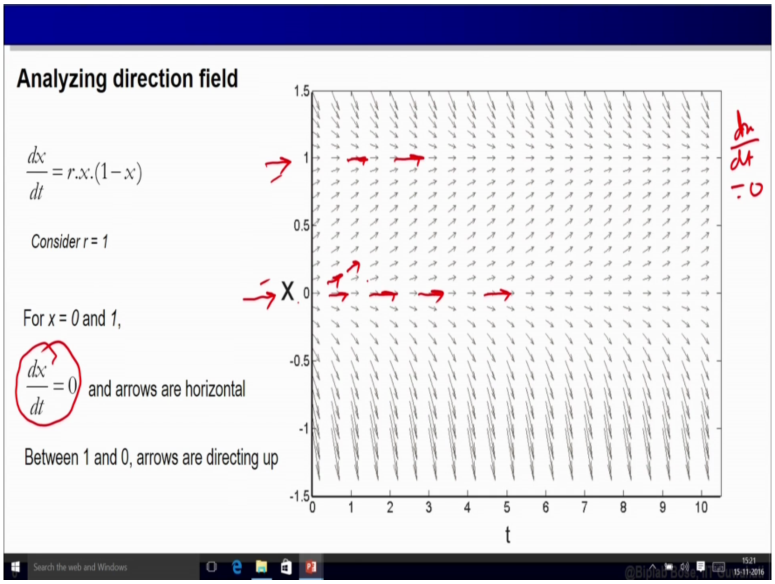

This whole thing, this diagram with arrows with this dependent variable and time space x and t space is called direction field. For any given OD you can draw this direction field. Let us explore this direction field with more. Notice that when x = 0 your arrows are all horizontal. Horizontal arrows are coming because the slope at those positions is equal to 0 that meant $\frac{dx}{dt} = 0$ at those positions. You have another value of x, x equal to 1 where all arrow at any time point you can see are horizontal, here also $\frac{dx}{dt} = 0$. In between 0 and 1 the arrows are pointing up and they are reaching toward from 0 to 1 they are moving towards 1. What are they telling, let us see.

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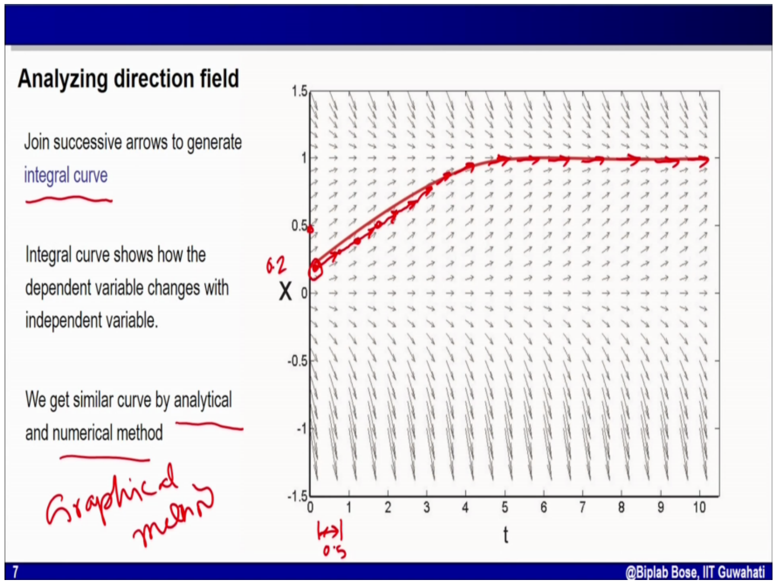

Let us take a point for example at time 0, let us take a point 0.2 here. So, that is 0.2 for x at t = 0. The interval between these two grid point is 0.5, so, if I start at this 0.2 position when t equal to 0 after 0.5 time I reach this point following this arrow then if I increment time further by 0.5 following this arrow head I reach here then incrementing time further using the same arrow head I will (in) reach there then if I keep on following these arrows and I keep changing time with increment of 0.5, I will keep on following these arrows and eventually I will reach 1 when the arrow heads will become horizontal and I will keep moving along this as the slope becomes equal to 0. So what I have done, I have started with a initial value with t = 0 I was at 0.2 and then I have followed successive arrows and their direction to move through time and in this x v/s time space.

I have drawn it cleanly you can see this is the line, this line is called integral curve. In fact we have got this integral curve by integrating and numerical method, the method I have used here is based on direction field and is sometime called graphical method. Graphical methods are very useful when you cannot integrate a particular ODE just looking at the graphical representation of direction field and the integral curve you can get a quality behaviour of the system. Let us take another initial point t = 0 x is 0.5 and then if I keep on following the successive arrows then I will get another integral curve as shown here by the green one.

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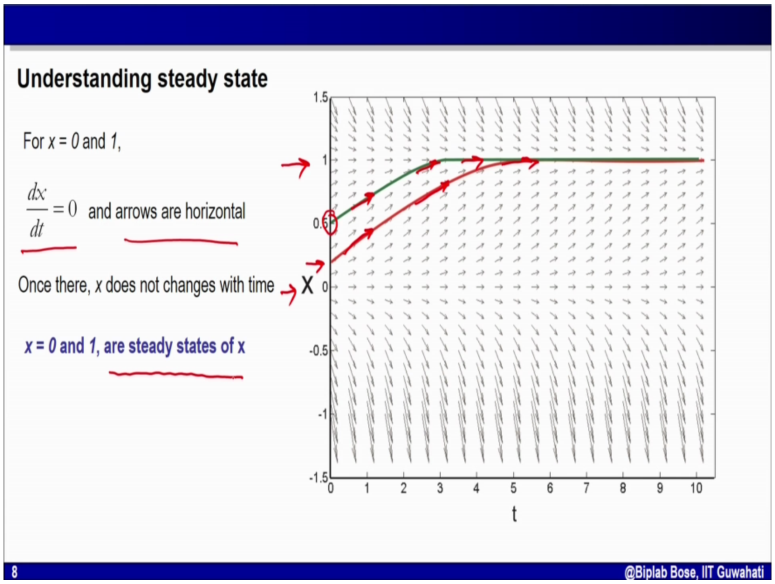

So, why I am discussing about integral curve and direction field? If you look into this direction field I have two places x = 1 and x = 0 where the direction field's arrow are horizontal that means $\frac{dx}{dt}=0$. When I start at a initial value of x 0.2 I move along this integral curve and eventually land up at this $\frac{dx}{dt}=0$ at x = 1. Similarly, when I start at 0.5 I follow this green line and eventually reach x = 1 where $\frac{dx}{dt}=0$. So, I have two value of x where $\frac{dx}{dt}=0$ and the arrows are horizontal. These two values are called steady state of x.

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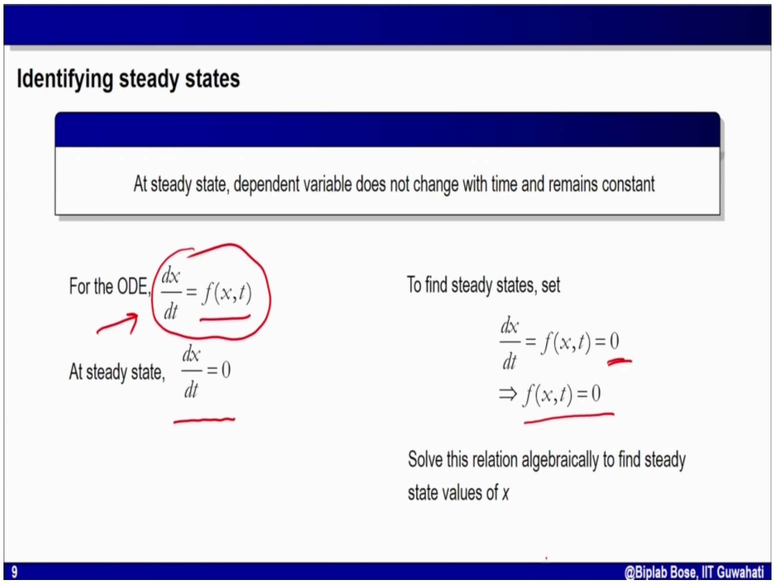

These are steady state because the dependent variable x is not changing with time and constant for these values. If use, r at those values the system will not change with time, it will stay there. So, that's why these two values are called steady state. So, by mathematical definition if I have a ODE which is $\frac{dx}{dt} = f(x,t)$ generalised we have written then the steady state is where $\frac{dx}{dt} = 0$. Now, how should I find the steady state so suppose I have been given with a function $\frac{dx}{dt} = f(x,t)$ so, I put that equal to 0. Then I will simply do a algebra that $f(x,t) = 0$, I will separate out x and t so, I get the value of x for which $\frac{dx}{dt} = 0$.

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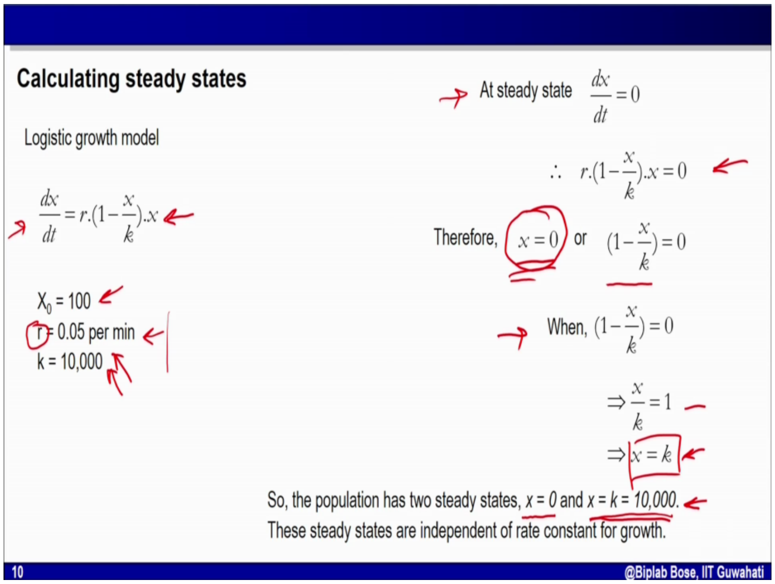

Let us see example, we will take the example of logistic group so, the model OD is $\frac{dx}{dt} = r.\left(1 - \frac{x}{k}\right)$, x is the size of the population. So, initial value of x is x_0 = 100, the rate constant for both of the population is 0.5 per minute and K the carrying capacity is 10,000. I want to find out the steady state for this system that means I want to find out the values of x, for which $\frac{dx}{dt} = 0$. How I use algebra to that? By definition a steady state $\frac{dx}{dt} = 0$ that means $r.\left(1 - \frac{x}{k}\right) = 0$ because that's what is the ODE telling me.

Now once I have this relation and r is a non zero thing so, r is non minus zero that means either x = 0, either x equal to 0 or 1 minus x by k equal to 0, it is simple algebra so, that means if x = 0 $\frac{dx}{dt} = 0$ so that is my one steady state, simply I got it. Now, the other option is 1 minus x by k can also be 0 then also DXDT will be 0. So, when $\left(1 - \frac{x}{k}\right) = 0$ I can separate out x and everything else that becomes $\frac{x}{k} = 1$ that means x = k. So, what I have got I have two steady states for this logistic group model one is x = 0, on is x = k for both these values $\frac{dx}{dt} = 0$.

Now you have given me that k = 10,000 that means the population has two steady state, one at x = 0, other one as k that is 10,000. So, if your population is already at 0 there x = 0, the population will remain like that, that's true. If you don't have the organism then what you will grow? If your population is at carrying capacity this is 10,000 that is another steady state so, if you have already reached the carrying capacity there will be no further growth so, it remain at steady state. So, this system has two steady state and using simple algebraic separation we can find out the values of x for which we have the steady states. Here notice one interesting thing for both the steady state there is r. So, the steady state is a independent of rate constant for growth. They only depend upon the carrying capacity.

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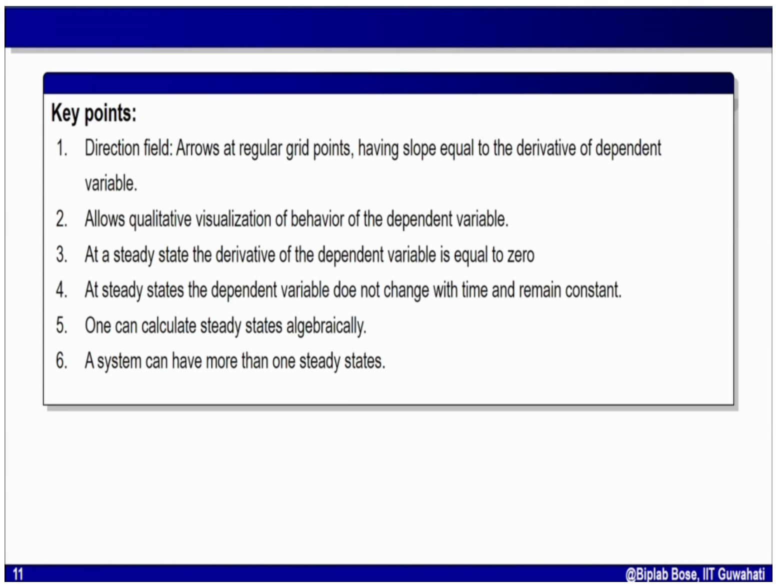

So, if I jot down the key points, what I have here. In this video we started with discussion with direction field. Direction field is a quality way to see to see how the ODE behave in time versus dependent variable space. So, what you do in case of direction field. Suppose you have a ODE involving dependent variable x and independent variable time. So, you plot x versus (x) t space we divide that space in equal size grid and each grid point we calculate the derivative of x that is DXDT and then there we draw some arrow having the slope which is equal to DXDT. So, if you fill the whole space with these arrows we will have a direction field. And if you look at the direction field image it's almost like the field images that you have seen in physics textbook for magnetic field or electrical field.

So, once you have the direction field, you can decide the initial position that is t = 0 the position the value of x and then you can follow one arrow after another as the arrow point from the initial position and join those dots to give a integral curve. That curve will give the time evolution of x, the way it will keep on changing with time. This is equivalent to what you have got by integrating and then plotting the function of t for x or you get by numerical solution. So, direction field give a visualisation of evolution of x with respect to time, the dependent variable with respect to time.

Now, in this direction field there are places where $\frac{dx}{dt}=0$. The value of x for which the derivative for is it is equal to 0 is called steady state. At steady state the system does not change. For example, the fish model when the system, the population is in steady state or population equal to 0 or 10,000 the carrying capacity the population will not change. So, the derivative of population size at that point with respect to time is equal to 0. So, system can evolve from a non-steady state position to a steady state position and at steady state it will not change with time. We can calculate the steady state algebraically and remember a system can have one or more steady state and you have to find them and understand the behaviour of those steady state. In another module we will discuss different behaviour and properties of steady state. Today we end here. Thanks for watching. See you in the next module.