Introduction to Biomedical Imaging Systems
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Lecture - 07
LSI Systems

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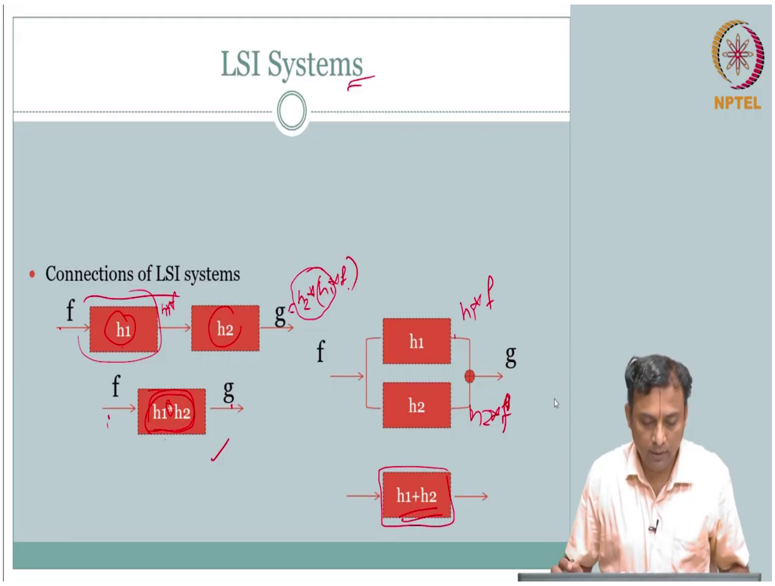

So, having looked at linear shift in variance of system and how to character a system right, we talked about a convolution being an important operator right.

So, let us so, this is fine, we are talking about f of x and g of x which are all in the spatial domain. So, we could use convolution property right and we talked about how the system transfer function concept right, system impulse response point, spread function, how do you use the system impulse response function that connects your input and output through the convolution operator.

So, clearly that is fine, this is very intuitive, this is straightforward, but there are other ways through which other operators through which we could still analyze the system properties. So, if you looked at your 1D, if you recall your 1D systems analysis, you notice that we use to call their time domain analysis, or you know time series.

So, you have signal usually the independent variable being time right, you will analyze time domain, or you could actually capture the signal in the frequency domain and so, there is a important operator that comes in which important transform that comes in which is Fourier transform.

So, now that we have seen how convolution could be used right that connects the system function and your input and output let us see how we can make use of Fourier transform or the other domain or other equivalent domain to analyze the system. So, this is going to be again a very quick review ok.

So, you will appreciate that all of the like we saw for the systems review, the Fourier transform is going to be a direct extension from 1D to 2D of course, there are couple of aspects that you know may look initially little complicated, but once you understand how to read that right, read the equation in English make it intuitive, I think it is not going to be that complicated. So, you need to so, you have to take it at the level of a review material ok.

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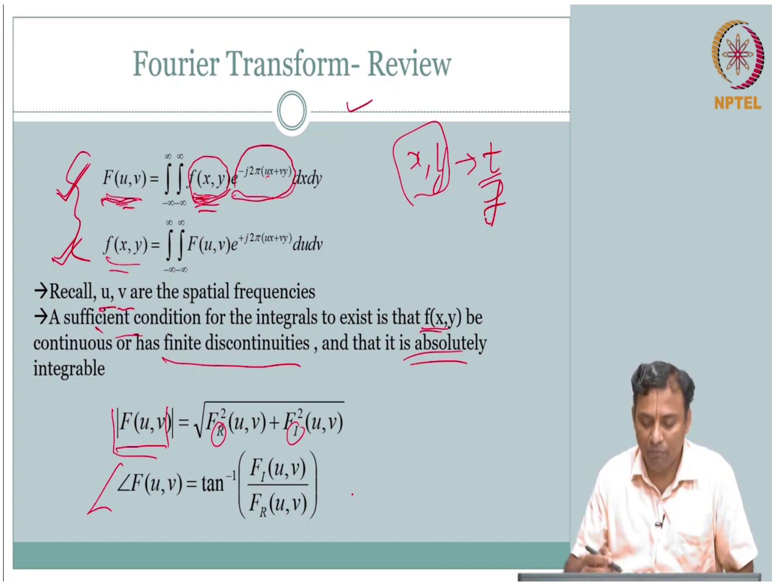

So, let us start with Fourier transform right, Fourier transform a review. So, what do we mean by that? So, there is a f of x comma y which is what we have been dealing which so far right, f of x comma y is your spatial variable in two-dimensions that is what we have been working upon. So, now, this is called as your forward direction, this is your Fourier transform and this is your inverse Fourier transform.

So, if you really have a careful look at what it is of course, u and v are the spatial frequencies. So, if you recall at this point of time before we jump in and interpret, just to just for you to feel convinced that this is a direct extension like I said last concept as well, you can do x of x comma y was your t right; x comma y was your t so, your u and v are your spatial frequencies which was say f or 2 pi f would be your omega right so, whichever way you are familiar with, you can do that.

But if you substitute this, you will quickly realize that f of t e power minus j 2 pi f t dt only one variable right and there will be one integral your right, your LHS would be f of j omega or whatever frequency denotion you do. So, it is a straightforward extension only thing is the time variable instead of time variable, now we are going to deal with spatial variables.

So, therefore, we already interpreted, visualized these frequencies right. Go back to the example where we talked about lines sorry, we talked about the pattern right, it was vertical, parallel bars right black and white alternate and then that was at an angle so, we visualized frequencies.

So, now, what here you see is how do I break the signal f of x comma y or decompose the signal f of x comma y into its frequency component. So, essentially, we did this in time in spatial domain by using the spatial impulse response right, the direct delta your point impulse response right.

We talked about f of x, y you can pick point. So, essentially, you can see the arbitrary function f of x, y in terms of your impulse response. So, instead of that now, you are breaking, or you are decomposing your f of x comma y into complex exponentials. So, that is what is happening here.

So, you are breaking f of x comma y in terms of complex exponentials again, what is this complex exponential? e power rho when we cover the important signals, we covered complex exponentials and we wrote that in terms of sine and cosine. So, essentially what you have is you can see this as sine and cosine with some frequency which is your spatial frequencies u and v.

So, you have writing or decomposing this f of x comma y and seeing how much of frequency content is there. So, your f of u comma v so, how many frequency content this f of x, y is composed of ok. So, naturally, this is your forward and similarly, if you are given all the frequency composition, then you could get back to your spatial domain f of x comma y ok.

So, very simple concept wise direct extension of your time domain for a frequency domain to the time in 1D one-dimension right, time domain to frequency domain, frequency domain to time domain forward Fourier transform, inverse Fourier transform same thing. Of course if you do that, then you are representing your signal f of x comma y in terms of its constituent frequencies.

And therefore, we can write the you have to have the integrals right, you have to be able to evaluate this and when is it sufficient right? You have to have a sufficient condition when you can compute the integral so that you can calculate your Fourier transforms and that is happening if f of x comma y is continuous or if it is discontinuous, it is only finite discontinuities and in any case, it has to be absolutely integrable only in this case, you can calculate the Fourier transform.

It turns out that most of the transform that we will be applying for this imaging systems or signals that are participating in this undergoing this in the medical imaging systems, it is reasonably good approximation, so, it is it satisfies this. So, most of the time you would not have a problem calculating the Fourier transforms rather it is the sufficient condition will be met ok.

So, it is very powerful as you know I cannot reiterate it enough you if you have done any signal processing with one-dimension, you know the powerfulness of your Fourier transform.

So, same thing extends here also. In fact, this we will not jump the gun ahead of its time, but then, you will notice that understanding of the Fourier transform is very important or you have to be very comfortable because in one of the modalities that we will cover, the signal you will be acquiring can be thought of as directly estimating the frequency response.

So, let us you know frequency domain. So, we will wait until that. So, get a feel for what it is. So, if that is the case right, you if you break the signal f of x, y; f of x comma y into its frequency components so, you could calculate what is this called as your right magnitude,

magnitude response or your magnitude spectra right, this is your real part, this is your imaginary part right so, this is your magnitude spectra.

If you do a square of this, then it becomes power spectra. So, here you are just doing magnitude spectra, and this is your phase. So, you can decompose into magnitude spectra and phase spectra. Notice, you have to need both of them to uniquely identify f of x comma y clear.

So, let us now use this to run through some basic property because we see this transform is very powerful. So, let us see what are the good properties of this Fourier transform, that will make us play with this little more comfortably ok.

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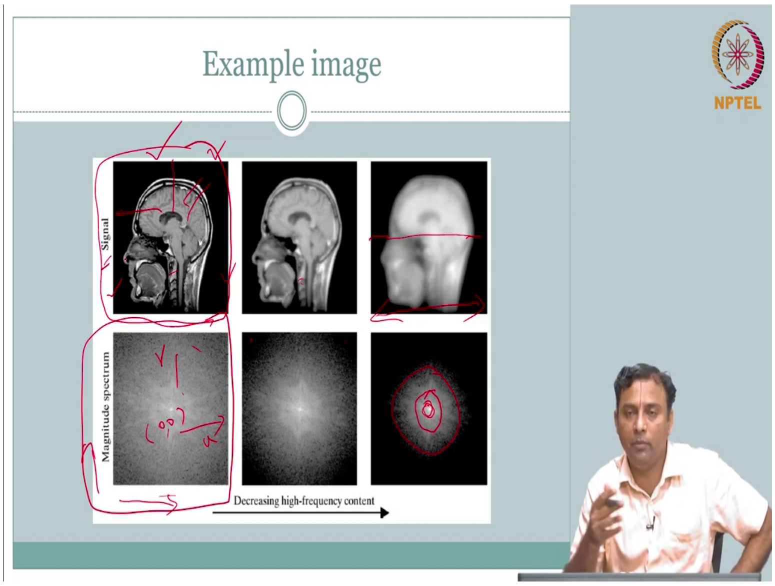

So, before we jump into that just one slide to give you the big picture of what do we mean by frequency components and how to visualize right. So, what do you see in the first row? It says signal just because we have covered signals right. So, what is this signal? It has two-dimensions x and y. Remember, we are not plotting this we are visualizing this two-dimensional signal as an image ok. So, that is a concept that we brush.

Then, you can see a head right I mean, but then which direction of the head is this? Side views or sagittal, so, right. So, you should be able to appreciate the view you should be appreciate what this two-dimensional signal representation as it is in the form of a image pixel, each of the pixel value has a different shade right black to white depending on the value that it contains. So, that is your image.

So, now what do you see? Here, this is going to be your magnitude spectrum. What does that mean? That means, I have taken the Fourier transform, I have decomposed this 2D signal f of x comma y into capital F of u comma v so, we are obtaining. So, this is your u and v, the two directions of frequency in along x axis and frequency along y axis.

Clearly, you see one thing I mean for most of you if you look at this, you cannot really make out anything I mean this you can really spatial domain, this looks like a head, and we know what it is right. So, there is this is head inside, there is some brain part, and this is nose, this is mouth, I mean we can you know we know what it is whereas, here it is very difficult to interpret.

However, there are some salient points, salient aspects about this that we can still intuitively interpret that is the goal of the slide right here. So, first things first, ok. So, this is the signal, and it contains all of these frequencies meaning all of these frequencies are contributing to getting this image right.

So, what is this frequency? We know, we saw from the visualization remember go back to the review of signals when we talked about how to visualize the frequencies, what do we mean by spatial frequency u and v right, we had that parallel bars that are running black and white, vertical pattern and then that was at an angle so, we calculated u frequency and v frequency what does that? What do we mean by u frequency? That means, how do you have a fluctuation.

What is a fluctuation or oscillation in this case? Black to white or white to black or you know the pattern so, that is your variation, the intensity variation along your x axis likewise your intensity variation along y axis, intensity as an the cyclic variation right black to white to white to black remember how we plotted that. So, essentially, a number of cycles of that that is there is in the y direction is going to be your v frequency.

So, if you look at it, clearly you can see here if I draw there is going to be a white, black so, there is a y component, there is a frequency in the y component, there is a frequency in the y x component right, u direction so, wherever you see vertical lines right or feature here that means, it is going to have variation in the horizontal direction or the x direction. So, that is going to be u frequency.

So, clearly you can see that means, in this space we have several frequencies ok. So, it need not be just u and v that is perpendicular, it can be at any angle. So, here for example, you mean this goes from horizontal to vertical, so, you all see so many angle. So, they it is all perpendicular there so, the frequencies along there is some frequency in the u direction, some frequency in the v direction right. So, some other point, u comma v.

So, remember when we also saw an example of u and v both being non-zero right so, that means, you have both the frequencies. So, that is how this plane is populated. So, how do we appreciate, ok? Even though here it is very difficult to appreciate any pattern, we can still observe few things.

One is there is a bright spot at the center right and probably it gets less denser as you go outwards that is higher frequency. So, this is your frequency axis center is 0 comma 0, 0 comma 0 is the center so, this is your u, this is your v right. So, there is positive frequency, negative frequency, up and down, 0 comma 0 is at the center.

So, clearly what you see is if for example, we remove right frequencies that are above a certain frequency. So, this is black and out right so, that means, the high frequencies are removed. When you remove the high frequency in the frequency content, how does the image look? Wow, it looks blurred right more physically if you want to appreciate the edges are now looking less prominent.

So, for example, you still see the shape of the head, you still see the brain right, you see the structures, but then it is not sharp right look at here and look at here, it is not sharp, the transitions are not as sharp as in this image. So, that means, there is no rapid transition, rapid transitions are minimized.

What do we mean by rapid transitions? That means, the fluctuation from white to black to black to white right that is becoming less prominent that mean, it is slowing down that means, a high frequency if it is transitions are fast that means that is high frequency. So, if you remove the high frequency, the transitions become less visible.

To extreme cases, if you really cut down right, then you notice still you are able to see that this is a head, this is a nose, more or less you are able to see the structure, but it looks more or less homogeneously white, you really do not see much transitions, you have some transitions which are captured, but predominantly what this says is all of the energy is if it is contained only at the center that is your dc value or 0 frequency.

If there is 0 frequency, what is the meaning? There is no oscillation, meaning there is no change, there is no fluctuation in the intensity that means, if this is only one value right, if there is high contribution from 0 value, 0 comma 0 there is no frequency, this should look flat, this should look same value throughout which is pretty much what it is.

Of course, you have some frequency is still left and therefore, you see settled. So, if I go ahead and you know filter it further and use only this frequency, this will become like a disk right white disk that means, you just have one black to white transition another black white to black transition within this whole space whatever this length scale is.

So, you have one fluctuation per length right you see or no's fluctuation per length will become 0 value right that will be your only one point. So, if you have only one point, one frequency at 0 comma 0 which is your dc frequency and that contains all the energy that means, your spatial is going to be flat equal to 1 everywhere right that is what you expect.

So, the main point that I would like for you to appreciate here is so, looking at this image, physical dimensions, the length dimensions in the spatial domain, you can now, perhaps eyeball and appreciate what do we mean by spatial frequencies ok. In time domain maybe we are trained, you could look at a signal form, the wave form and start to count how many you know cycles are there and you can guess the frequency.

So, here as well you should be able to visualize what it is even though the 2D spectrum is going to be not really having much pattern right, it is going to be you know have a dc and then high frequency content, the pattern will be visible here ok good. So, we now know what is frequency you know the Fourier transform and how do we get from spatial domain to frequency domain and now, let us move on to look at some of the properties ok.

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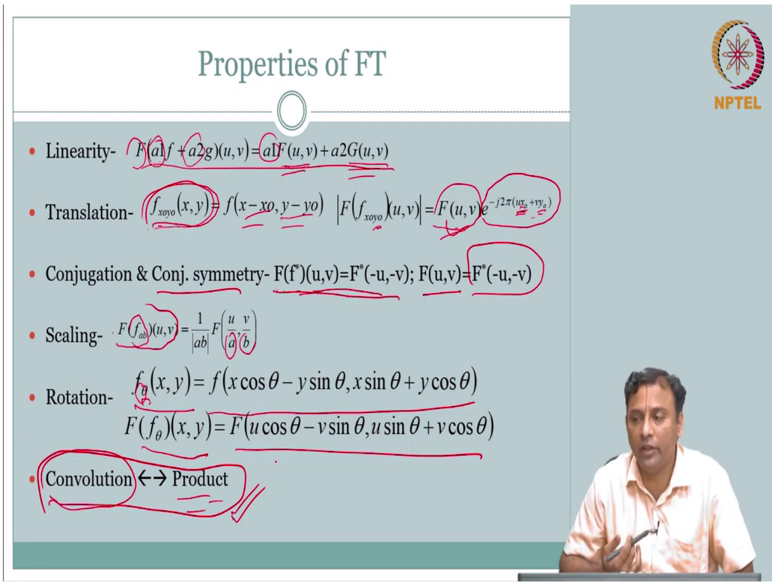

So, the way we would like to do is since this is a review, I would just list this out that will give you few seconds to perhaps recall, connect with what the property name would be. All of these that we are going to cover should not be the first time that you are looking at this or hearing about this. If that is the case, then I sincerely recommend you go back, and review right read one-dimension first and then, look at this ok. So, what is this? In fact, some of this that we will cover we just reviewed for systems anyway.

So, what is this? This says this is the Fourier transform operator. So, this is your F of a1f that is I have a function small f, I have a function small g f of x comma y, g of x comma y, a1 of f of x comma y plus a 2 of g of x comma y if I send them through Fourier transform right, if I take the Fourier transform of sum of these two signals right weighted by a1 and a2 respectively that is equal to a1 F of u comma v that is your Fourier transform of f of x comma v is F of u comma v.

Fourier transform of g of x comma y is G of u comma v. So, thus this ring a bell when I give input as weighted sum of two functions, the output of that is, output of the individual ones through that system with the corresponding sum of respective outputs weighted by the corresponding coefficients, what is that? Linearity ok.

So, this again is going to be very handy. So, the idea is when we did impulse function, time domain, we use this linearity property very much right. So, you will see that means Fourier transform also obeys this linearity. So, analysis of a system using the equal and frequency representation of the signal will be very very tractable as well.

So, next so if I have similarly, if I have f of x comma y, the Fourier transform is capital F of u comma y. So, if I send right if x f of x comma y if I shift the signal so, f of x minus x naught, y minus y naught we just shifted signal. If I shift in spatial domain by x naught and y naught in their respective axis, then what do I see?

What do I get? If I take the Fourier transform like this is the magnitude so, your magnitude of F of f naught x naught y naught just to represent this u comma v is nothing but F of u comma v fine more importantly, there is e power minus j 2 pi x naught y naught.

That means, if I shift to the signal, the magnitude of the Fourier transform does not change ok, the magnitude of the Fourier transform of the shifted signal does not change it is still F of u comma v, but what changes? It reflects in the phase shift or right your frequency here right u of x so, exponential so, there you have a shift. So, that is your phase shift.

Remember, we talked about magnitude plot and phase, magnitude spectra and phase spectra. So, magnitude does not change. So, if I shift the signal in spatial domain, the magnitude of the Fourier transform remains same only the phase shift. So, this folds as a phase shift very similar to in one-dimension if you do time delay right at least that is the usually, we if you do time delay, you will have phase shift similar thing right. So, nothing much to that.

So, these will be very helpful in playing with this thing. So, this is your translation property. So, if you translate in space, it will have a phase shift, magnitude will not change in the Fourier domain ok. So, then properties of conjugation and conjugate symmetry.

So, if you have a complex sorry if you have if capital F of u comma v right is the frequency Fourier transform of a complex signal F of x comma y for example, right, then the Fourier transform of F asterisks, so, this is your complex conjugate this asterisks of u comma v is F complex conjugate of minus u comma minus v. So, this is your conjugation property.

Likewise, if you a F of x comma y is a real valued right and F of u comma v is the Fourier transform of that, then you have F of u comma v will be equal to F asterisks complex conjugate of minus u comma minus v. So, this is your conjugate symmetry property. So, all this even function, odd function right, conjugate real part, imaginary part, what is the phase symmetry, magnitude is all this very similar to your 1D equivalent nothing new here. So, it might help for you to brush that up ok.

So; so, far it is just direct extension of what we know in 1D. This is another thing right if you have, what is this representation? It basically says the small f is just spatial domain signal right. So, in the spatial domain f of x comma y, if a and b are the respective values of scaling so, f a x f then b y so, that is what this a, b denote.

So, if you have f of x comma y and the Fourier transform is capital F of u comma y, if I now have a input signal which is scaled f a x comma b y right, what will be the Fourier transform of that? Fourier transform of that will be f of a comma b u of e equal to see the inverse right. So, this is your scaling property ok.

So, you can similar to what we covered you could see what happens if a equal to b equal to minus 1 right. So, it will become f of minus x comma minus y equal to capital F of right Fourier transform will be capital F of minus u comma minus v that is if you reverse your x and y your Fourier transform also gets reversed ok. So, this is your scaling property very similar nothing new here ok that is fine. This is something that is interesting.

Again, now that we are going to two-dimensions right, in one-dimension this is you can probably have only symmetry around the axis right whereas now, what do you see here this is my f of x comma y, regular f of x comma y I have added a theta here. So, what does this mean? If I have f of x comma y that is my signal, f theta of x comma y represents a signal which says it is a rotated version of f of x comma y right. So, f of x comma y if it is there, your Fourier transform is capital F of u comma v.

Now, if I represent f theta of x comma y which represents a signal f of x comma y that is rotated about the origin by theta right f of x cos theta minus y sin theta comma x sin theta plus y cos theta so, it is making some angle so, it rotating by theta degree, then what happens to the Fourier or this frequency spectrum representation of the signal? If you rotate in spatial domain, what do you think is going to happen in the frequency domain?

Well, very interesting, very nicely if you rotate it here in this direction in your spatial domain, your it is equivalent to rotating your frequency domain. So, if your F of this function right F of rotated version. So, you have a Fourier spectra when it is capital F of u comma v, when the signal is f of x comma y.

If you rotate the signal right at angle theta, f of x comma y is rotated by theta, then the original spectra capital F of u comma y is also rotated by the same amount of theta. So, essentially, if you do the space, it makes some sense right if you intuitively look at the examples that we covered, there was horizontal pattern, black and white stripes were horizontal pattern.

If I now so, when we did that what did we say? We said the frequency is changing only in u direction, it is constant in v direction so, v was equal to 0 and u had some frequency I think 1 or for the 1, 2 and 4 is what we covered right. So, imagine instead of vertical line, you rotate that what will happen? It becomes horizontal line.

So, now, your frequency domain whatever was there in the u will become to v, v will become to u right, you are rotating the frequency, so, you will have u equal to 0 right the frequency along y direction if you go turn it right whatever was vertical, if you turn in to horizontal, now, you will have variation only in the y direction and it will be same in the x direction, so, your u will become 0, v will have those frequency.

So, it will look like you are shifting, or you are turning, rotating the frequency domain ok. So, this is called as your rotation property. So, these are all very interesting and important because you will encounter because we talk about two-dimension, if it is going to be rotation right, we talked about without jumping into detail to just appreciate why you might encounter this, we talked about views, different views remember, projection, different views so, you can already start to imagine you will encounter rotation quite frequently ok.

Is there anything else we want to do here? Last but not the least, very powerful right. What does this say? When you do convolution so, it is very similar to your time domain, when you do time domain convolution in the time domain same effect is obtained when you do multiplication of their respective frequency domain right in the frequency domain so, similarly here.

So, whatever if you do convolution; if you do convolution operator in spatial domain, you can get the same effect by doing multiplication or product, taking the product. So, if you have two signals and you convolve, you get an output. So, instead of doing convolution in spatial domain, what I can do is take the Fourier transform of the first signal, take the Fourier transform of the second signal and then, multiply them, I will get same effect, I operation on the signal.

So, the net output signal will be the equivalent. So, this is again a very powerful whether you want to implement your algorithms in time domain or here in this case spatial domain or in the frequency domain. So, this back and forth you will have to do several times ok. So, this is a very interesting property.

Likewise, if you have multiplication right, if you have two signals which are multiplied in spatial domain, then you could take their respective Fourier transforms and do a convolution in the frequency domain, you will get the same output signal by these operations ok. So, that is something that is very powerful you will encounter that at least at the time of implementation.

When you are; when you are implementing certain algorithms, when you are doing this operators whether do you want to use frequency domain operators or time domain operators, spatial domain operators which is efficient which is accurate all these things, you will encounter ok.

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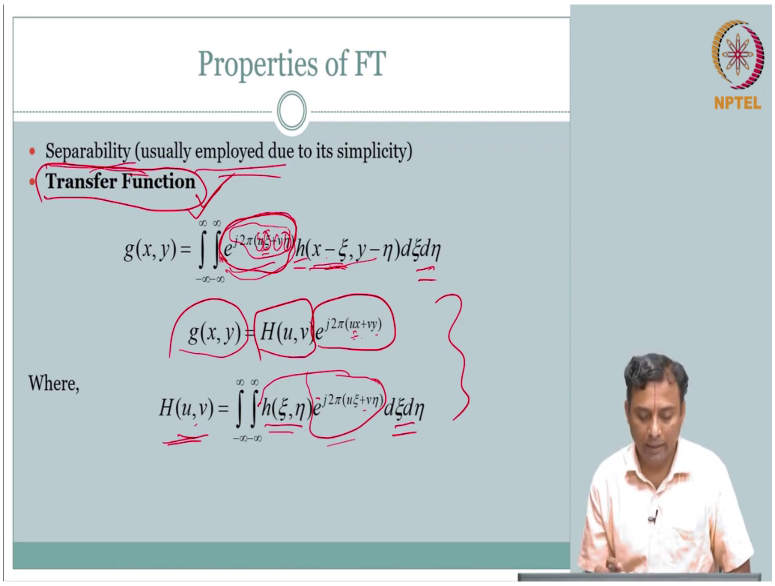

So, just to state another useful one is like I said for systems also right. If you have F capital F of u comma v as the 2D Fourier transform of signal small f of x comma y right that is why you have to do the Fourier transform operator. But instead of doing the two-dimensional Fourier transform, you can also do two one-dimensional Fourier transform meaning you can take the Fourier transform along the x direction and then, cascade it with do Fourier transform again on the y direction.

So, you can accomplish the two-dimensions, you can separate the two-dimension as two one-dimension that is what we mean by separability and this again will be used just for simplicity right instead accomplishing two-dimensions, you can do it as two one-dimensional operator. So, you can separate the two-dimensional Fourier transform into two one-dimensional Fourier transforms ok.

This is what I was waiting to get after, transfer function, all the other properties are fine, but this is a very important one. Why is this important one? The name is same transfer function you would have heard, why what did we what do you mean by transfer function from your previous prerequisites?

That means, it is transferring, this transfer function transverse something from input to the output this is in the what we talked about system also, the system h transverse the input or you know operates on the input to get the output.

Here, it is just Fourier transform so, you can say transfer function, it transfers something from input to output what is that it is transferring? It is transferring the frequency content right. So, essentially, transfer function is a important concept, we saw the system impulse right a function for a response in the spatial domain.

So, if you are going to do spatial domain analysis of the system right you do all that, you can use the impulse response whereas, if I want to do the analysis in the equivalent domain of the frequency, then I need something to analyze it, I need to be able to characterize the system in that domain right.

So, the h we were able to characterize in the spatial domain. If I know my if I present impulse respond, the systems output is considered as the system impulse response or system points spread function specifically for the two-dimensions right. So, that can be used to analyze the system in spatial domain.

So, if I want to do that in the frequency domain, then we need to characterize the system in the frequency domain. So, how do we do that? Think about how we did in the spatial domain same equivalent we should do. There what did we? We presented this.

We presented the system right with a impulse right, a impulse we presented it and whatever came out we said that is how it responds to that impulse and therefore, any arbitrary signal can be viewed as collection of shifted impulse with that function and therefore, we used all the linear, spatial in variation LSI right, shift in variance that is what we did.

So, here likewise, how do I decompose the signal? There we were able to do f of x comma y was we use the impulse shift and we got different locations we be how did we do here? When we talked about frequency response, we talked about there should be a basis, there should be a frequency that we should apply right.

So, if I now give a system into the system instead of an impulse right, I need a impulse in the frequency domain, what will be an impulse in the frequency domain? A sinusoid a particular frequency or a cosinusoid a particular frequency right any oscillations with one particular frequency will have frequency representation will have only one point; it will be existent only at that frequency right.

So, if I present the system right; if I present the system with a typical sinusoid or cosinusoid, so, we will write it as a complex exponential right.

So, if I hit the system with a complex exponential right, we were calling this as f of x comma y f of right remember and h of and then, your double integral we got the output that is how we wrote when we talked about spatial impulse that time we call the signal as f of x comma y to be generic that input signal is now complex exponential why do we pick complex exponential?

Because we know this is sinusoid or cosinusoid with some frequency specific frequency u and v. So, this is our just substitution right we put this. So, from now we need to do some simplification to understand what this can be viewed as ok.

Quickly looking at this, nothing is obvious just we can recognize that, we did not do anything fancy so far instead of what we use to do f of x comma y, we have replaced that f of x comma y, the input signal with complex exponential that is all we have done, and this is your system impulse response right.

So, this is your system, so, input system this is your output so, nothing fancy. So, now what we will do is its we will do some maneuvering and I will show the result here, but you what you can start to think is ok, we can do some change of variables right I can look at this and say I look x minus epsilon, I have a epsilon here right so, maybe I can do a change of variable so, I can do u of this epsilon I can write it as here just do variable substitution.

So, the idea would be I can break this exponential right here right. If I break these two, epsilon eta x minus epsilon y minus eta so, I will do change of variables so that this whatever you have subtraction here will go there why is that important for us? If exponential if you have four variables plus, minus, you can basically it is easy to split right exponential dot exponential you will have can split it multiple exponentials that is a logic.

So, we will just do some change of variables, I would let you to work here, do the change of variable and you will quickly realize its just one step wonder. If you do the change of variables and rewrite right, you can get to this form of g of x comma y as H of u comma v times e power e of j 2 pi into u x plus v y. What is this H of u comma v? H of u comma v is nothing the Fourier transform of h of this guy ok. So, this is not that.

So, you can actually look at it how we you know one step in between that I missed, you should be able to eyeball, and convince yourself that yes so, if I have this, I can change this to epsilon and eta, I am just going to use the same variables just a dummy variables right. So, I will change this to epsilon and eta so, that is how you get this, and this will happen x minus epsilon and x y minus eta right.

So, then I can obviously, group x and y right together and eta and epsilon together ok. So, there should be a negative here right if you do that x minus eta so, there will be a negative here. So, if you do that, you can quickly recognize this term times this term in this double integral form, you can recognize it as H of u comma v, of course, the variables are only eta and epsilon. So, you will have here 2 pi u x v y right that will just come out because that is not the running variable here ok.

So, just substitute; substitute variable and you will quickly be able to group the terms or at least look at the grouping and identify that the double integral of h of epsilon eta right e power minus j 2 pi this guy is nothing but the Fourier transform of h. What is h? System function. So, essentially h capital H of u comma v is the frequency response of your impulse function or your you know system function which we call as the transfer function. Why is that? Because it is transferring the frequency content from input to output ok.

So, I would let you do this one step in between and convince that you are able to get the frequency response or the you can spot the H of u comma v ok.

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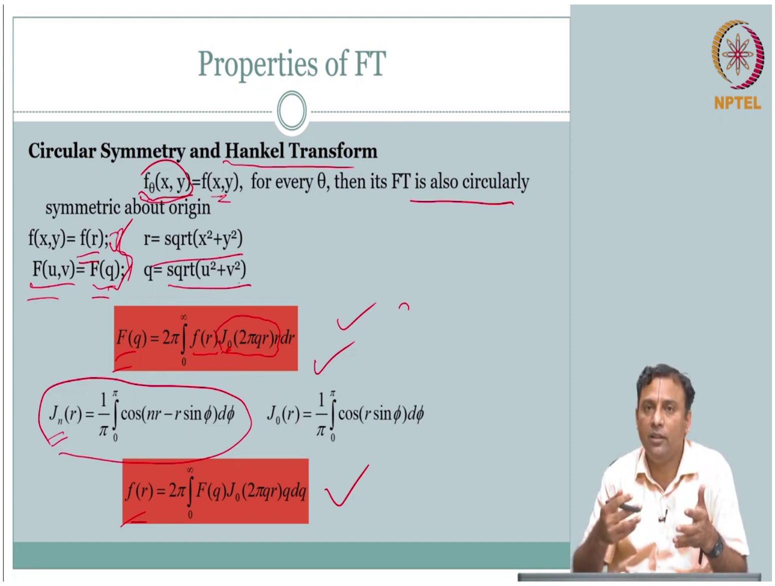

So, of course, there are several important properties, but like I said this is only for quick review that we are interested in. So, one of the other important property is we talked about circular symmetry right. We talked about this f theta of x comma y is nothing but if you have a f of x comma y and you have f theta of x comma y is the same f of x comma y that is rotated right.

So, if this is the case, f theta of x comma y is equal to f of x comma y that means, it is circularly symmetric, no matter which angle you look at the function looks same. So, whatever you have f of x comma y, it does not matter which angle you are looking at right, then it is circularly symmetric whether you look at it at 90 degrees, 0 degrees, it does not matter right, the f of x y is same.

So, it is f theta of x comma y if it is same so, any same as your f of x comma y at for any theta so, that is your spatial domain signal, if that is the spatial domain signal which we call as circularly symmetric, then our interest is in the Fourier transform. How does the Fourier transform of such a signal look right?

How will it look? That will also be circularly symmetric about the origin. I mean roughly just so, that you get comfortable you are able to position the concept in two-dimension when we do 1D right, if you have a Gaussian shaped time domain signal, what will be your of frequency spectra the shape of the frequency spectra that will also be Gaussian right that is something that you know.

So, similarly here, if my spatial domain is circularly symmetry because you have one more dimension circularly symmetric.

So, imagine you have a bell right; you have a bell that means, your frequency response right that is also going to look like a bell that is what this physic I mean the that is how you intuitively get a feel for what this circularly symmetric means that much is straightforward not a big deal, but then, we will have to look at little more detail as to how do we compute this two-dimensional Fourier transform.

So, if I have f of x comma y as circularly symmetric that is f theta of x comma y that is the input signal special case where the input signal is circularly symmetric, it turns out the Fourier transform is also circularly symmetric. So, we can write instead of two-dimensions if such a case instead of two-dimensions right.

r comma theta if theta is same irrespective of theta the signal is same, then we can reduce the two-dimensions into one-dimension. So, you can essentially try to write f of x comma y as f of r only a function of radius right, r is equal to square root of x square because this theta is circularly symmetric, it does not matter ok. So, if this is.

So, we said if this is the case, a Fourier transform is also symmetric, circularly symmetric. Therefore, we should have a equivalent there. So, your F of u comma v is also going to be symmetric and therefore, there also we could write it in terms of only one parameter which is your radius parameter in the frequency domain.

So, you can get your small q a square root of u square plus v, u and v are your spatial frequencies right u comma v your spectra axis are u and v horizontal fluctuations in the x direction and frequency in the y direction. So, much as good straightforward right ok, I see the benefit of having a circularly symmetric case because if the signal is circularly symmetric, the Fourier transform is also circularly symmetric. If it is circularly symmetric, the two-dimensional way, two variables I can reduce it to one variable.

And therefore, I can get one-dimensional f of r is one-dimensional, F of q is one-dimensional, but pretty much what is not said here what is not taken for granted what you should not get confuse we have not really related f of r and capital F of q. We know the two-dimensions are related right.

The two-dimensions f of x comma y and F capital F of u comma v are related by Fourier transform, but what about when you reduce the variable because it is circularly symmetric, you get f of r one-dimension, capital F of q one-dimension.

Is there a relationship between these two, right? Of course, it is there mathematically there is a, but it is not what you think it is not same F of q is not Fourier transform of f of r instead F of q is you have 2 pi 0 to infinity of f of r. Remember in Fourier transform, we had complex exponentials right the signal was multiplied with e power minus j, but here what do you see?

The variable this is dr, but what do you see? J naught 2 pi with an argument of 2 pi q r. So, this is nothing but a 0th order Bessel function of first kind what is. So, this is a unique function. So, we can arrive at that by considering your Bessel function of the nth order of first kind is having this relationship right.

So, if this is the Bessel function, then what do we want? We want 0th order. So, if this is for n, then we can substitute 0 right we could get J naught of r is equal to 0 to pi of cos of r sin phi d phi ok. So, this is called as your Hankel transform that relates your in a circularly symmetric case right of if the signal is circularly symmetric, the Hankel transform relates the spatial domain and the frequency domain, the one-dimensional right are equivalent is related through what is called as Hankel transform.

Why is this important? I mean if you want to calculate the two-dimensional frequency response, you could calculate it through in in this case of circular symmetric, then you could actually calculate the Hankel transform and then arrive at your 2D Fourier transform ok. So, this is forward Hankel transform, this is going to be your inverse Hankel transform ok.

So, I think this probably is most for most of you these two, this Hankel transform and the relationship with circular symmetry might be new, first time, but it is very intuitive, you do not of course, it these might seem little scary, but if you really look at it this is not circularly symmetric right is this weird condition? No, actually if you look at it a patient this is the system right the imaging system, the patient goes in right no matter how you orient yourself, the output should also come out correctly.

My head if I go in particular direction, I get the head. If you go in another direction that should also be right so, circularly so, in most of the time, if that is rotation in variance is expected which is what will happen, then your system response right your system function your PSF or point spread function or your impulse point impulse response to that, that will be circularly symmetric as per. So, this is something that you will encounter ok.

I think this is a good point to stop our review of signals in systems. From the next lecture, we will jump into some basic concepts behind how do we talk about quantify, understand and what do you mean by image quality because at the end of this, we are going to use all these measures, matrix to communicate what a image quality is and how do we characterize the image by its image quality and how do we relate the parameters from physics, from the instrumentation, from the image signal processing, how does it affect the end image quality ok.

So, for now, let us stop here. I will see you next time in the review of image quality parameters.

Thank you.