Transport Phenomena in Biological Systems

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Lecture-13
Steady-state Radial Diffusion

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Welcome back. Today let us start looking at the situation of steady state radial diffusion in spherical pellets, I guess you get the idea. Initially we looked at rectangular Cartesian coordinate system, a flat membrane. And then we looked at a cylindrical membrane for cylindrical coordinates and now spherical pellets to give you an idea as to how to look at a spherical geometry system.

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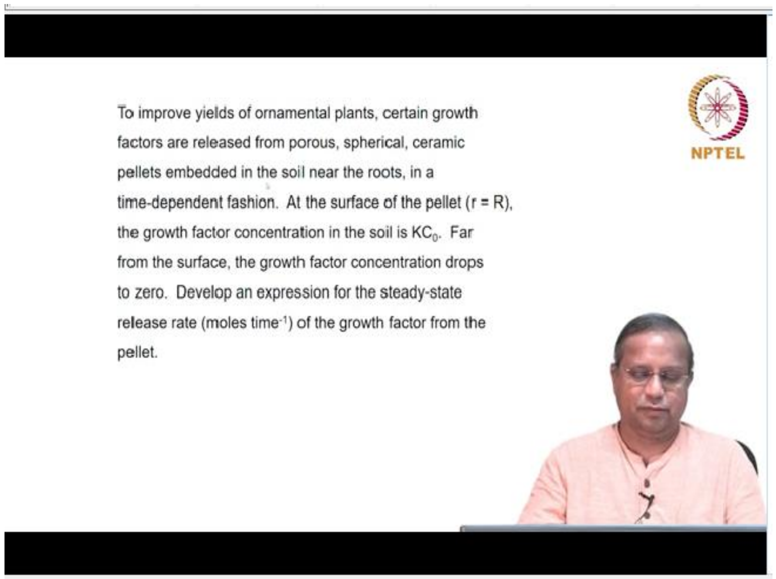

Let me present this problem to you first and then we will figure out ways of working this out problem-based learning, yeah. To improve yields of ornamental plants, certain growth factors are released from porous, spherical, ceramic pellets, embedded in the soil near the roots in a time dependent fashion, at the surface of the pellet, the growth factor concentration in the soil is Kc_0, far from the surface, the growth factor concentration drops to 0. Develop an expression for the steady state release rate (in moles per time) of the growth factor from the pellet. Please read this as many times as you want, pause the video and read this as many times as you want to get a mental picture of what is happening here.

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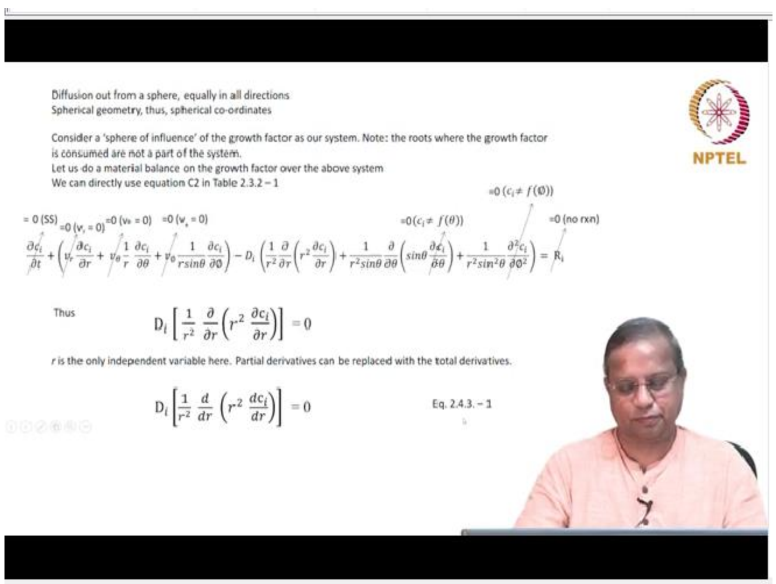

Now with that picture in mind let us start solving this. So, the diffusion of the growth factor is out from the sphere equally in all directions. So, this is coming out of a spherical pellet therefore it is good to use spherical coordinates. So, we will consider a sphere of influence , there is a small pellet which is sending out the growth factor. There is a certain sphere of influence of the growth factor.

And that sphere of influence is what we are going to consider as a system. And in our system, we are mentally going to remove the roots, where the growth factor is consumed and that is not going to be a part of the system, that complicates things and therefore let us mentally remove that as a part of the system, our system does not contain the roots. It is just the sphere of influence without the roots.

And now let us do a material balance on the growth factor over the above system. We are not worried about the circular pellet itself, we are not looking inside this particular pellet, something is coming out equally in all directions from the spherical pellet and the region of influence is spherical and the spherical region of influence is what we are going to consider here. And of course, we have mentally removed the roots, where things are conceived.

This is spherical geometry we are doing a material balance therefore; we can go to table 2.3.2 - 1, hopefully you have made a copy of that. Please do if you have not done so and then you could look at equation, C2 for spherical coordinates in that. And that equation would be this.

$$\frac{\partial c_i}{\partial t} + v_r \frac{\partial c_i}{\partial r} + v_\theta \frac{1}{r} \frac{\partial c_i}{\partial \theta} + v_\varphi \frac{1}{r\, sin\theta} \frac{\partial c_i}{\partial \varphi} - Di\left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c_i}{\partial r}\right) + \frac{1}{r^2 sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial c_i}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta} \frac{\partial^2 c_i}{\partial \varphi^2}\right) = R_i.$$

I hope you know what r, θ, φ are. r is the radial coordinate and θ is the vertical angle and φ is the other angle, these 3 you need, to completely specify a point in a spherical coordinate system. So, r, θ and φ.

If you are unclear about this spherical system, please go to the appendix of the book, it is given there clearly and the way to get from the rectangular Cartesian coordinate system to the spherical coordinate system is also given there in the appendix A. You can check it out. Let us see what terms are relevant here. It is, we are looking at steady state conditions for our analysis and therefore the first term, which is a derivative with respect to time goes to 0.

As explained in the previous class there is no v_r, there is no v_θ, there is no v_φ because there is no bulk motion or convective motion of the fluid here. Therefore, v_r is 0, v_θ is 0, v_φ is 0, then c_i is not a function of θ, there has to be a symmetry in all directions. Therefore, the derivative with respect to θ of c_i is 0. Similarly, there is no variation with the φ direction. Therefore, this term is also 0.

And there is no reaction that is occurring here it is just diffusing out, we have carefully removed the roots and therefore we do not worry about any reaction that is occurring here. That is one of the reasons why we mentally removed the roots from a system, we do not account for this term here. Therefore, what remains is $Di\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c_i}{\partial r}\right)=0$. And r is the only independent variable here and therefore we can replace the partials by the total derivative, what we get $Di\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_i}{dr}\right)=0$.This is the equation of interest here. Equation 2.4.3 – 1.

$$Di\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_i}{dr}\right)=0 \qquad 2.4.3-1$$

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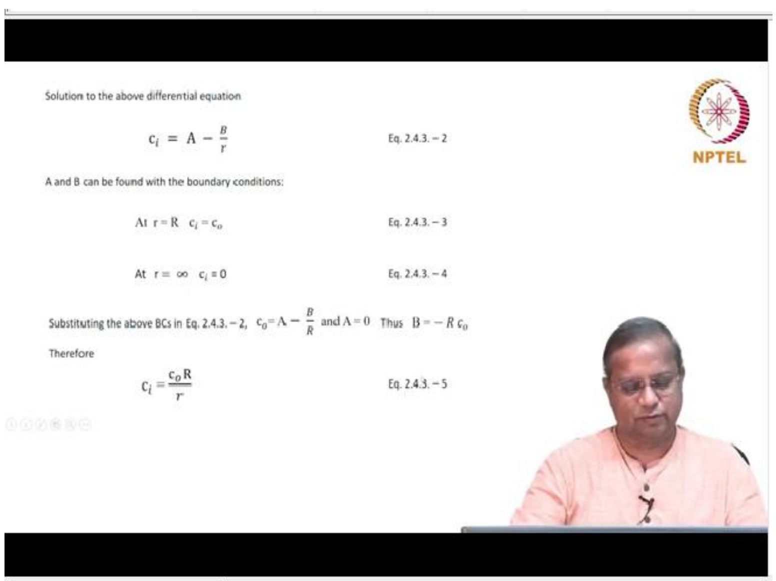

Now we will have to solve this differential equation. The solution to the differential equation is $\mathbf{c_i = A - \frac{B}{r}}$ $(\mathbf{2.4.3-2})$. Recall this from your earlier classes or you could also see this here you would find that this derivative equals 0 and therefore, this has to be a constant and so on and so forth. So, $c_i = A - \frac{B}{r}$ is a solution here, we need to of course find A and B which are constants.

And to find out those constants we use the boundary conditions at r = R, when r equals the radius of the pellet that is releasing at the center. Then $c_i = c_0$. That is the concentration at which things come out. Let us call this 2.4.3 - 3 and if r = ∞, concentration at the sphere of influence ultimately goes to 0, the growth factor concentration goes to 0.

$r = R$, $c_i = c_0$ 2.4.3 – 3

$r = \infty$, $c_i = 0$ 2.4.3-4

2.4.3-4 implies that the concentration of the growth factor at very large distance from the center, there is no growth factor present and therefore the concentration is 0. We substitute the above boundary conditions into this, we would get $c_0 = \mathbf{A - \frac{B}{R}}$ (from 2.4.3-3). And therefore A = 0 when you solve 2.4.3-2 by substituting **$r = \infty$, $c_i = 0$** from 2.4.3-4 and then by substituting A=0 in 2.4.3-3, you get B = - Rc_0 and therefore we get $\mathbf{c_i = c_0 \frac{R}{r}}$ **2.4.3 - 5.**

Please plot this, c_0 is a constant, R is a constant, the small r is a variable here, plot c_i verses r and see how it looks, that is quite a good exercise to do to learn to visualize how the variations are, will call this 2.4.3 - 5.

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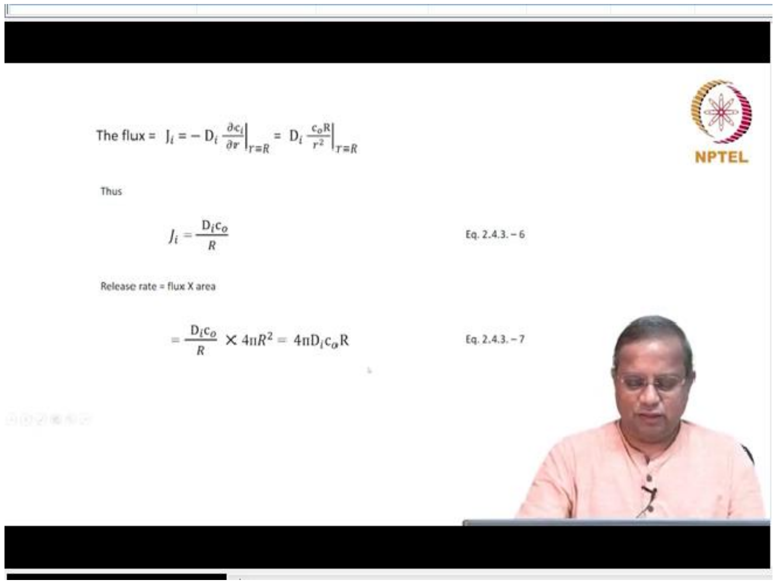

We have the concentration profile, the movement is only through diffusion and therefore the flux is only a diffusive flux. Therefore, the flux J_i is given as $-D_i\left(\frac{\partial c_i}{\partial r}\right)\Big|_{r = R}$. We have an expression for c_i we have just derived an expression for c_i, $c_i = c_0\frac{R}{r}$. We differentiate $c_0\frac{R}{r}$ with respect to r, we get $-c_0\frac{R}{R^2}$ at r = R, upon cancelling R, we get here $-\frac{c0}{R}$ and the flux equals

$J_i = D_i \frac{c_0}{R}$ 2.4.3-6

So, this is the flux that we wanted to find as well as the concentration profile in the sphere of influence. We will call this equation 2.4.3 - 6. We were asked to find the release rate, we needed the flux to find out the release rate, because we know that the rate is nothing but the flux times the area, i.e. amount per time per area is flux, we multiply that by the area to mass rate.

Then you get amount per time, which is the rate and in this case the release rate. So, flux multiplied by the spherical area in the sphere of influence gives the release rate. We get $D_i(c_0/R) * 4\pi R^2 = D_i c_0 4\pi R$ We want the release rate and therefore we are multiplying by the area relevant at that point(a spherical area). And if you simplify this, you are going to get $4\pi D_i\ c_0 R$. So, **Rate = $4\pi D_i c_0 R$**. So, that is, let us call this equation 2.4.3 - 7, it is always nice to number equations.
Of course, as mentioned earlier, these numbers refer to the equation numbers the textbook. So, they may not be sequential in this particular presentation, but they correspond to the equation

numbers in the book, which makes it easier in the long run, when you refer to the book. I think that solves this problem we found the release rate that we needed to find as to do this we needed the shell constant, the concentration profile.

And the flux, both of which we found. So, this was an application of the material balance expression equation of continuity to spherical coordinate system, I think we should stop here in this class and continue with other aspects when we begin the next class. See you then.