

Transport Phenomena in Biological Systems
Prof. G.K. Suriaishkumar
Department of Biotechnology
Indian Institute of Technology-Madras

Lecture - 19
Review of Mass Flux

Welcome. In towards the end of the last lecture, I mentioned that we have completed the second chapter on mass flux. Before we move forward, let us briefly review all that we have done so that it gives you things in perspective, the mass flux aspects in perspective, as well as it helps you revise some of the aspects. It improves the learning and so on so forth. So we will do this after every large chapter.

The first chapter was on mass conservation itself and that is part review, half of it was review. The application to microscopic systems was probably new. That is the reason why we did not spend an entire lecture reviewing that. Very briefly mass conservation, mass can neither be created nor destroyed that converted to a useful form was what we started using rate of input minus rate of output plus rate of generation minus rate of consumption equals the rate of accumulation.

$$\frac{dm}{dt} = r_i - r_o + r_g - r_c$$

We wrote it of a certain way that would be useful. Then we had shown the application to a humidifier, a macroscopic system. And then to a cell, a microscopic system. And that was pretty much what it was at that stage. And also I think we derived the equation of continuity for the total mass or a single component system, total mass. Now let us review the mass flux aspects, .

(Refer Slide Time: 01:55)

As mentioned earlier,

$$\text{Flux of a quantity} = \left(\frac{\text{Quantity moved}}{\text{time}} \right) \left(\frac{1}{\text{Area perpendicular to the direction of movement}} \right)$$

$$\text{Mass flux} = \left(\frac{\text{Mass moved}}{\text{time}} \right) \left(\frac{1}{\text{Area perpendicular to the direction of movement}} \right)$$

In fluid systems,

$$\text{Density} \times \text{velocity} = \frac{\text{kg}}{\text{m}^3} \times \frac{\text{m}}{\text{s}} = \text{kg m}^{-2} \text{ s}^{-1} \text{ is mass flux}$$

So, flux of any quantity we said was the quantity moved per time per unit area perpendicular to the direction of the motion itself. In this case mass flux is mass moved per time per area perpendicular to the direction of motion. And we also said that in fluid systems the density times velocity directly gives us mass flux. Density and velocity are measurable.

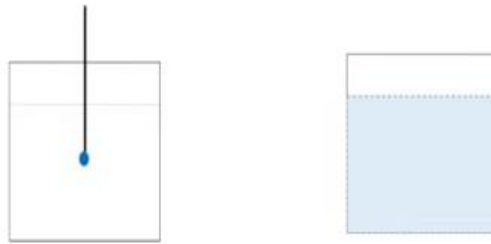
Mass flux formulation which helps us look at various things from the same perspective and that is why we are after a flux kind of a formulation, makes the things general, makes us see relationships between movement of various different conserved quantities and so on so forth. .

(Refer Slide Time: 02:47)

Wide relevance

Flux of substrates and products in bioreactors
Flux of desirable substances in membrane filtration
Glucose flux across the cell
Product flux (e.g. ethanol) out, across the cell
The transport of protein from the site of assembly to the site of function in the cell
The mass flux of oxygen from the blood to the organ where the cells of the organ use it

Let us consider this experiment:



Thermal motion

Net effect: movement of ink molecules from a region of high concentration to the others of lower concentration

Then, we looked at the relevance of mass flux. And then we said how the mass flux comes into being when there is you know the thermal interaction between the various molecules, jiggle things around to effectively result in a motion of the species from a region of high concentration to a region of low concentration, till the concentrations are the same everywhere and in a particular phase.

We also said across phases it is the difference in chemical potential that drives it, but in a phase, the chemical potential can easily be replaced by concentration and that should work.

(Refer Slide Time: 03:36)

What causes the flux?

A driving force

What is the driving force for mass flux?

A difference in concentration over a distance – concentration gradient

Strictly speaking, it is the chemical potential gradient, but for mass flux within the same phase, concentration gradient is sufficient

The concentration difference is 'primarily' or firstly linked to the mass flux

Many driving forces can cause much higher mass flux (e.g. stirring the beaker with ink)
We will see this in the last chapter – multiple driving forces causing the same flux

What causes the flux? That is what we have been talking about, the driving force. A difference in concentration over distance, a concentration gradient. So we also said that,

this is primarily associated with the mass flux and therefore, we call it the primary driving force. Although the primary driving force may not result in the maximum flux. We said that many different driving forces could cause a flux.

For example, many different driving forces could cause mass flux. Concentration gradient can cause bulk motion or convective motion of the fluid itself. If the fluid itself is moving, of course, the species would move along with the fluid much faster. The difference in temperatures or a temperature gradient can cause a mass flux as we will see later and so on so forth.

A difference in electrical potential, electrical potential gradient can cause a mass flux. And many of these driving forces can cause much higher mass fluxes. However, the one that is primarily associated with it is called the primary driving force, which is the concentration gradient here.

(Refer Slide Time: 04:48)

Average velocities

Let us consider a multi-component mixture with many species (components)
Let \vec{v}_i be the velocity of i^{th} species with respect to stationary co-ordinates axes

The mass average velocity for a multi-species mixture with n species can be written as:

$$\vec{v} = \frac{\sum_{i=1}^n \rho_i \vec{v}_i}{\sum_{i=1}^n \rho_i} \quad \text{Eq. 2.1.1. - 1}$$

Note: 'species' are not molecules
Species: A group of molecules of the same species i in a tiny volume element, take the sum of individual velocities of molecules of species i and divide by the number of such molecules in that tiny volume element.

Similarly, a molar average velocity \vec{v}^+ is defined as:

$$\vec{v}^+ = \frac{\sum_{i=1}^n c_i \vec{v}_i}{\sum_{i=1}^n c_i} \quad \text{Eq. 2.1.1. - 2}$$

Then we talked of average velocities, the mass average velocity, the molar average velocity.

(Refer Slide Time: 04:55)

Let us consider the disinfection of a laboratory using formaldehyde vapours. Typically, formalin solutions (~40% w/v of formaldehyde in water) is used to generate formaldehyde vapours that kill micro-organisms in an enclosed space. Care is taken to seal all windows and doors with duct tape to prevent leakage of formaldehyde vapours when the disinfection is carried out. The vapours are generated by the increase in temperature due to the exothermic reaction between the added potassium permanganate (KMnO_4) and formalin.

Let us assume that we are generating formaldehyde vapours in a long cylinder. A = formaldehyde (MW=30) and B = air (MW=29). Let us consider the plane where $x_A = 1/5$. Let us say that at that plane,

$$\vec{v}^* = 7 \text{ units} \quad \vec{v}_A - \vec{v}^* = 8 \text{ units}$$

Find $\vec{v}_A, \vec{v}_B, \vec{v}_B - \vec{v}^*, \vec{v}_A - \vec{v}_B, \vec{v}_A - \vec{v}_B, \vec{v}_B - \vec{v}^*$

Then we worked out a problem to understand these various velocities.
(Refer Slide Time: 04:59)

$$\vec{v}^* = \frac{\sum_{i=1}^n c_i \vec{v}_i}{\sum_{i=1}^n c_i} = \frac{1}{(c_A + c_B)} (c_A \vec{v}_A + c_B \vec{v}_B) = x_A \vec{v}_A + x_B \vec{v}_B$$

x is the mole fraction

From the problem statement we know that at the plane $x_A = \frac{1}{5}$,

$\vec{v}^* = 7$ units (upward direction is taken as positive)

$\vec{v}_A - \vec{v}^* = 8$ units

From the above velocities, we can get

$\vec{v}_A = 8 + \vec{v}^* = 15$ units

Using $\vec{v}^* = x_A \vec{v}_A + x_B \vec{v}_B$ we can get **pause**

$7 = \frac{1}{5} (15) + (1 - \frac{1}{5}) \vec{v}_B$

$\therefore \vec{v}_B = 5$ units

$\therefore \vec{v}_B - \vec{v}^* = -2$ units (opposite direction)

Solution

Essentially a substance moving. Therefore, there is a velocity associated with it. We are trying to get to fluxes. So, flux needs to be written in terms of velocities. That is the reason why we looked at velocities in some detail. And this was the problem that we worked out.

(Refer Slide Time: 05:20)

		<u>Mass flux and Molar flux</u>
The mass flux of a species i, wrt stationary coordinates	$\vec{n}_i = \rho_i \vec{v}_i$	Eq. 2.1.2.-1
The molar flux of a species i, wrt stationary coordinates	$\vec{N}_i = c_i \vec{v}_i$	Eq. 2.1.2.-2
The relative mass flux of a species i, relative to mass average velocity	$\vec{j}_i = \rho_i (\vec{v}_i - \vec{v})$	Eq. 2.1.2.-3
The relative molar flux of a species i, relative to mass average velocity	$\vec{J}_i = c_i (\vec{v}_i - \vec{v})$	Eq. 2.1.2.-4
The relative mass flux of a species i, relative to molar average velocity	$\vec{j}_i^* = \rho_i (\vec{v}_i - \vec{v}^*)$	Eq. 2.1.2.-5
The relative molar flux of a species i, relative to molar average velocity	$\vec{J}_i^* = c_i (\vec{v}_i - \vec{v}^*)$	Eq. 2.1.2.-6
		Commonly used fluxes $\vec{n}_i, \vec{N}_i, \vec{j}_i$ and \vec{J}_i^*

And then the expressions for mass and molar flux in terms of very fundamental quantities, density, velocity and so on so forth. The mass flux and the molar flux capital N_i , they are vectors. **Molar flux is concentration times velocity whereas mass flux is density times velocity.** Then we said that we are typically interested in the motion with respect to other species, not with respect to stationary coordinates.

And therefore, we brought in the relative mass flux $\rho_i v_i$ minus the average velocity, mass average velocity. And then the relative molar flux with respect to the mass average velocity for completeness. Relative mass flux and relative molar flux relative to the molar average velocity.

Of these we said we commonly would be using the mass flux, the molar flux, and the relative mass flux relative to the mass average velocity as well as the last one, the relative molar flux relative to the mass average velocity. These are the ones that are commonly used. There are many other fluxes that are available. You can define and so on so forth. We will not be getting into those.

To know about some of those fluxes that are not covered here, you could look at Bird, Stewart and Lightfoot, one of your main reference books to get an idea of a molal average and so on so forth; molal flux.

(Refer Slide Time: 06:48)

Now, let us look at mass flux. It can be written from Eq. 2.1.2. - 5 and 2.1.1. - 1, as

$$\vec{j}_i = \rho_i v_i - \frac{\rho_i}{\sum_{j=1}^n \rho_j} \sum_{j=1}^n \rho_j \vec{v}_j$$

From the definition of mass fraction, we can write the above as

$$\vec{j}_i = \vec{n}_i - w_i \sum_{j=1}^n \vec{n}_j \quad \text{Eq. 2.1.2. - 9}$$

Or

$$\vec{j}_i = \vec{n}_i - w_i \vec{n}_T \quad \text{Eq. 2.1.2. - 10}$$

\vec{n}_T is the total mass flux

Then, we started finding out the relationships between these various quantities to make sense. We got to some important relationship. This was $\mathbf{j}_i = \mathbf{n}_i - w_i \mathbf{n}_T$ was one of them. **(Refer Slide Time: 07:13)**

Now, let us consider Eq. 2.1.2. - 6

$$\vec{j}_i^* = c_i (\vec{v}_i - \vec{v}^*)$$

Substituting \vec{v}^* using Eq. 2.1.1. - 2, the above equation can be written as

$$\vec{j}_i^* = c_i v_i - \frac{c_i}{\sum_{j=1}^n c_j} \sum_{j=1}^n c_j \vec{v}_j$$

From the definition of mole fraction, we can write

$$\vec{j}_i^* = \vec{N}_i - x_i \sum_{j=1}^n \vec{N}_j \quad \text{Eq. 2.1.2. - 7}$$

$$\text{Or } \vec{j}_i^* = \vec{N}_i - x_i \vec{N}_T \quad \text{Eq. 2.1.2. - 8}$$

\vec{N}_T is the total molar flux

Earlier we $\mathbf{J}_i^* = \mathbf{N}_i - x_i \mathbf{N}_T$, . These two are important relationships where the link between a total flux and the diffusive flux is given. However, we said in throughout this chapter, we will consider only diffusive flux, only the flux that arises out of its primary driving force, a concentration gradient. And therefore, you do not have to worry about the convective bulk motion and so on and so forth.

(Refer Slide Time: 07:49)

Constitutive equation

The **conservation** equation (continuity equation or "equation of change" that we have seen so far is widely applicable.

A relationship exists between the flux of the conserved quantity and the material (constituent) properties of the system of interest
Such a relationship/equation is not as widely applicable as the conservation equation, but is applicable to a class of similar substances
Such an equation is called a '**constitutive** equation'

A combination of
constitutive equations (or equation of state) and
conservation equation (equation of change)
is useful in analysis and design of engineering systems

Then, we looked at something called the constitutive equation. We said a conservation equation is applicable in general pretty much throughout, the for all practical purposes, pretty universal. And whereas a constitutive equation which depends on the constituent nature of the substance. There are, there is a relationship between flux of a conserved quantity and the material properties of the system of interest.

Such a relationship is the constitutive equation. And a combination of the constitutive equations or equations of state as they are called, as well as the conservation equations are very helpful in the analysis and design of engineering systems.

(Refer Slide Time: 08:33)

Adolf Fick experimentally found the relationship between molar flux and concentrations in dilute binary solutions of non-reacting solutes (Fick's 1 Law).

Fick's law in 1 dimension, for a species i:

$$\vec{J}_i^* = -D_i \frac{dc_i}{dx} = -c D_i \frac{dx_i}{dx} \quad \text{Eq. 2.2.1. - 1}$$

c is the total concentration

x_i is the mole fraction of i

D_i is the diffusivity of i in the mixture

The species i moves relative to the mixture in the direction of decreasing mole fraction of i

From Eq. 2.2.1. - 1, we can infer that mass flux is proportional to the negative of the concentration gradient

In general, **any flux is proportional to the negative of a certain gradient**
This **gradient** is that of the **primary driving force** for that particular **flux**

We saw the Fick's first law which provides us with a way of getting estimates of the flux in terms of a diffusivity coefficient and so on so forth. This is in one dimension.

(Refer Slide Time: 08:50)

In 3-D, $\vec{j}_i = -c D_i \vec{\nabla} x_i$ Eq. 2.2.1.-2

Now, substitution of Eq. 2.2.1.-2 in Eq. 2.1.2.-8 yields

$$-c D_i \vec{\nabla} x_i = \vec{N}_i - x_i (\vec{N}_T) \quad \text{Eq. 2.2.1.-3}$$

Or $\vec{N}_i = -c D_i \vec{\nabla} x_i + x_i (\vec{N}_T)$ Eq. 2.2.1.-4

(diffusion) (fluid motion/
convective component/
bulk motion)

You could have it in three dimensions. You know the derivative, one derivative gets replaced by a ∇ , which is derivative in all three dimensions. Then yeah this is the total flux as a diffusive flux plus the flux due to fluid motion, convective component, bulk motion. There could be many other things as well, we will see later. Here in this chapter, we are not going to consider this at all.

The aspects have been carefully chosen such that this does not come into the picture for understanding the subject better. In the last chapter, we will bring this up.

(Refer Slide Time: 09:28)

Now, let us look at Fick's law written in terms of the mass fraction instead of mole fraction

$$\vec{j}_i = -\rho D_i \vec{\nabla} w_i \quad \text{Eq. 2.2.1.-5}$$

Eq. 2.2.1.-5 can be derived from the Fick's law in terms of molar quantities (Eq. 2.2.1.-1), by using the definitions of mole, mass fraction, etc.

Substituting Eq. 2.2.1.-5 in Eq. 2.1.2.-10, $\vec{j}_i = \vec{n}_i - w_i \vec{n}_T$ we get

$$-\rho D_i \vec{\nabla} w_i = \vec{n}_i - w_i (\vec{n}_T) \quad \text{Eq. 2.2.1.-6}$$

Reading assignment: Fick's law for concentrated solutions Section 2.2.1.1. in the textbook

Then the mass, the Fick's law written in terms of the mass fraction is where we finished up with the, with that part. And then let me get to wherever I am getting to. Yes, this is

the, we said that you could approach these problems in two different ways. One is a shell balanced approach.

(Refer Slide Time: 10:18)

Generally speaking, there are two approaches to solve the relevant problems

- (i) the shell balance approach and*
- (ii) the application of the relevant conservation equation*

e.g. the equation of continuity in this case of mass conservation

And the other one is application of the relevant conservation equation. A shell balance approach means that you, we apply the material balance or we do a material balance over a thin representative shell. That shell depends on the geometry of the system if it is a rectangular Cartesian coordinate kind of a system. Rather, if it is a rectangular system, then we use a rectangular Cartesian coordinate geometry cuboidal system.

If it is a cylindrical system, we can use cylindrical coordinates. If it is a spherical system, we can use spherical coordinates. Then the equation of continuity is what we get in the case of mass conservation.

(Refer Slide Time: 10:59)

Shell balances

Balances of conserved quantities are made over a representative shell in the system

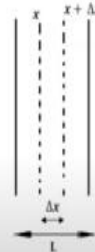
The shell represents the geometry under consideration.

For rectangular Cartesian coordinate systems: the shell could be a cuboid

For cylindrical systems: the shell could be an annular cylinder

For spherical coordinates: the shell could be an annular sphere.

Let us consider a uniform membrane
In that membrane, let us consider a shell of thickness Δx , through
which diffusion occurs normal to the surface area A



I showed you how to do shell balances for a uniform membrane. We chose a shell. And then we wrote our balances over it.

(Refer Slide Time: 11:08)

$$\text{Mass conservation: } \frac{d(m)}{dt} = r_i - r_o + (r_g - r_c)$$

A material balance written over the shell (system) on component i entering at x and leaving at $x + \Delta x$ in terms of molar fluxes:

$$\frac{\partial c_i (MW_i)}{\partial t} A \Delta x = N_i|_x (MW_i) A - N_i|_{x+\Delta x} (MW_i) A + R_i (MW_i) A \Delta x \quad \text{Eq. 2.3.1-1}$$

Let us divide throughout by $(MW_i)A$, a constant in this case **pause**

$$\frac{\partial c_i}{\partial t} = \frac{N_i|_x - N_i|_{x+\Delta x}}{\Delta x} + R_i$$

In the limit $\Delta x \rightarrow 0$, from the definition of the derivative **pause**

$$\frac{\partial c_i}{\partial t} = - \frac{\partial N_i}{\partial x} + R_i \quad \text{Eq. 2.3.1-2}$$

To arrive at some very useful relationships for a membrane, which we used much repeatedly in fact.

(Refer Slide Time: 11:18)

Here, the flux N_i is only diffusive

$$\bar{N}_i = \bar{J}_i = -D_i \frac{\partial c_i}{\partial x}$$

Thus

$$\frac{\partial c_i}{\partial t} = D_i \frac{\partial^2 c_i}{\partial x^2} + R_i \quad \text{Eq. 2.3.1-3}$$

If there is no net production of i in the volume, $A\Delta x$, by a reaction

$$\frac{\partial c_i}{\partial t} = D_i \frac{\partial^2 c_i}{\partial x^2} \quad \text{Eq. 2.3.1-4}$$

Fick's second law

Under steady-state conditions **pause**


$$0 = D_i \frac{\partial^2 c_i}{\partial x^2} \quad \text{Eq. 2.3.1-5}$$

In 3-D


$$\frac{\partial c_i}{\partial t} = 0 = D_i \nabla^2 c_i \quad \text{Eq. 2.3.1-6}$$

You know this relationship could have been used many times. And then we looked at, we derived the Fick's second law from this relationship. This is the variation of time and the variation of space both given the same relationship here, that is the niceness of it. And under steady state conditions, we get a nice compact relationship using Fick's second law that can be used to analyze it.

(Refer Slide Time: 12:00)



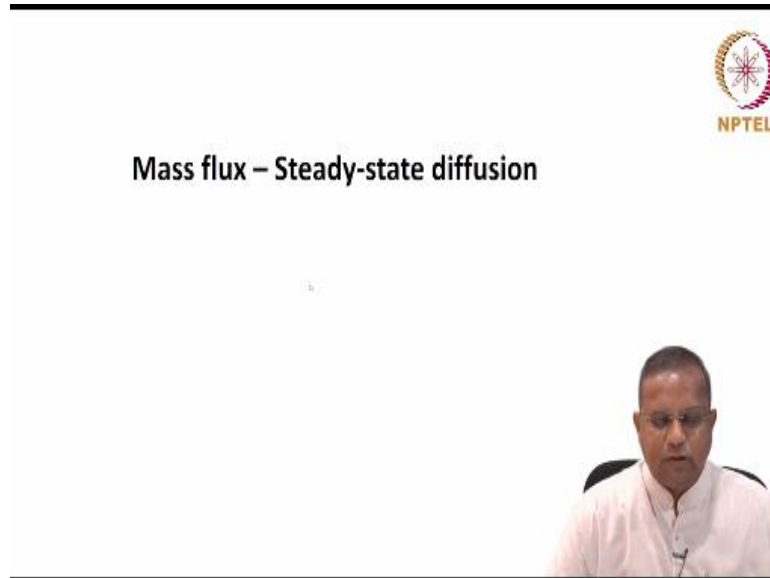
**Mass Flux –
Conservation (continuity) Equation Approach**



Then we looked at the conservation equation approach a continuity equation approach. We derived the conservation equation because we said shell balances could get cumbersome, especially with cylindrical and spherical coordinate systems it could get very cumbersome. Therefore, we derived a reasonably general equation, one very general equation and another at constant c , D_{AB} which is reasonably general as long as there is no change in geometry.

You can, that is one of the limitations. If there is a change in geometry of the system, you cannot use that. Otherwise you can use that. And so we have derived it. We have an equation. All we need to do is take that equation see what terms are relevant and go forward with it, . Let me not show you the derivation here because we are, it is a review. So you can go back and check how we derived it and so on so forth.

(Refer Slide Time: 12:59)



Then, we looked at steady state diffusion. We started applying the conservation equation because we have derived the relationship directly. I asked you to make a copy of these tables, soft copy, hard copy whatever it is and keep it for ready reference because we will be using it many different times as you have already seen. We directly go to those tables pick up the relevant equation and use it, cancel the irrelevant terms and whatever remains is what gives us the basis for analysis.

(Refer Slide Time: 13:26)

The properties of interest at a point in space do not change with time
 The properties of interest are not functions of time – the time derivatives can be set to zero

Highly relevant in many biological situations
 Biological diffusion, say across a membrane can be approximated as steady state diffusion

Steady state diffusion across membranes

Diffusion across membranes: Two mechanisms

1. Dissolve – diffuse mechanism: The solute first dissolves in the membrane and then diffuses through it
2. Diffusion through pores: The solute needs to move through the pores in the membrane

So steady state flux when steady state is when the properties of interest at a time do not change, at a point in space do not change with time and that is relevant for many biological processes. Many biological processes take place at steady state. Here we derived the steady state diffusion across membranes which is highly useful in many different cases.

So, we said that the diffusion occurs due to two mechanisms. One a dissolve diffuse mechanism and then diffusion through pores. We spent a good amount of time on the dissolve diffuse mechanism to arrive at.

(Refer Slide Time: 14:08)

Let us look closer

$$j_i^s = -D_i \text{ eff } \frac{\partial c_m}{\partial x} = \frac{K D_i \text{ eff}}{d} (c_0 - c_L)$$

As the equation indicates, the steady state flux is a constant. The SS flux is independent of position.

If $c_0 > c_L$ the flux is in the positive x direction

If $c_0 < c_L$ the flux is in the negative x direction

$\frac{K D_i \text{ eff}}{d}$ is defined as "permeability", P , of the solute i across the membrane: dissolve (K) – diffuse (D_{i,eff}) mechanism

Note: the permeability is not an intrinsic membrane property since it depends on the thickness of the membrane, d

Also, note that $K = \frac{c_m|_{x=0}}{c_0}$ assumed $= \frac{c_m|_{x=d}}{c_L}$

If $K < 1$, $c_m|_{x=0} < c_0$ and $c_m|_{x=d} < c_L$

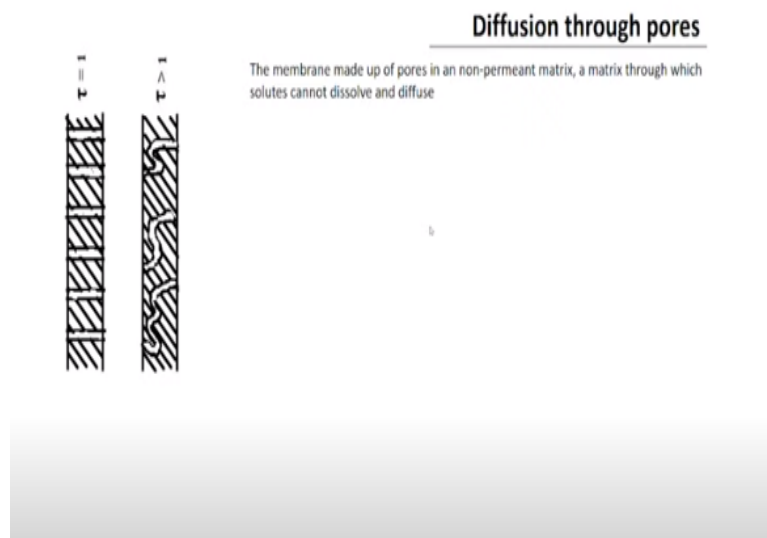
The concentrations on the membrane surfaces are less than those in the fluids

This relationship the molar flux is $-D_i \text{ effective } \frac{\partial c_m}{\partial x}$ or $\frac{k D_i \text{ effective}}{d} (c_0 - c_L)$ which gives the flux across a thin, across a membrane, across a uniform membrane. Then we said that

$\frac{k D_{i \text{ effective}}}{d}$, K is the partition coefficient. $D_{i \text{ effective}}$ is the diffusion coefficient. So the dissolved diffuse mechanism divided by the thickness of the membrane(d) is the permeability.

Permeability is an important parameter for any membrane. And then we looked at some insights .

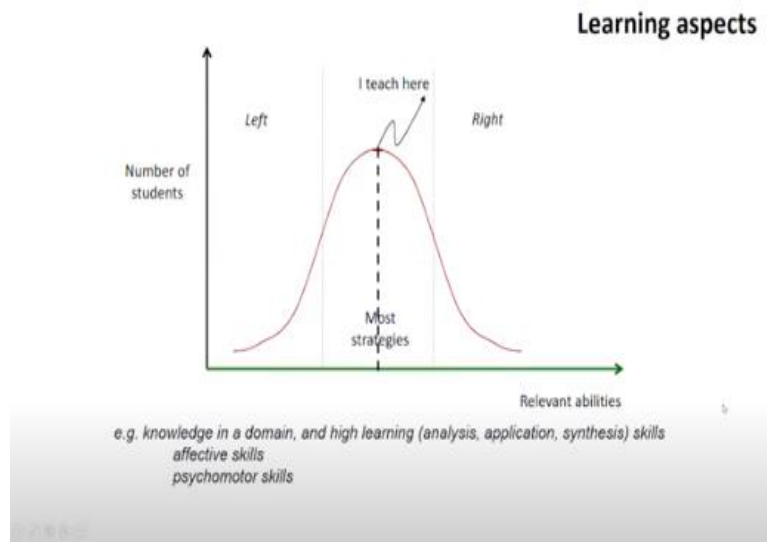
(Refer Slide Time: 14:59)



Then I directly gave you some relationships for the diffusion through pores without the derivation. I had given you a source for the derivation of these equations. We considered two cases there. One is when the pores are large enough compared to the size of the solute. Whereas, the other case was the size of the pores are comparable to the size of the solute. We had two different expressions for that.

Then we looked at a different geometry right. This was to show you how you could apply the conservation equation, equation of continuity to a cylindrical to a system with cylindrical geometry. So steady state diffusion radial diffusion across tubular walls. We had looked at a problem to understand how to use it because just application of the equation.

(Refer Slide Time: 15:59)


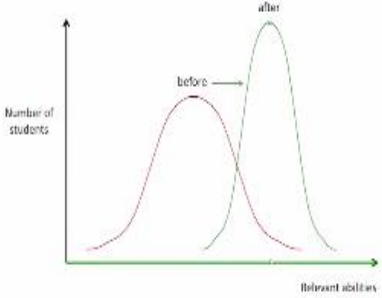




So I gave you the problem first. But before I, before that I talked about learning aspects. I said, usually there is a distribution, a normal or a Gaussian distribution in terms of relevant abilities. I typically teach in the middle, somewhere here. And if you are here, you might feel a little bored, you could probably go through it faster.

If you are here, you might feel a little lost, which means you will have to look at it a few more times. And then you should be able to get it. Because I have worked out every single step, every single mathematical step, which is usually one of the major difficulties for people here or here, sometimes even here because many textbooks do not give you the intervening steps.

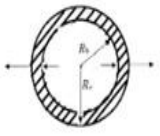
Then you spend a lot of time trying to figure out how we went from one step to another. But that we obviate in this particular presentation as well as the book.

(Refer Slide Time: 16:53)

And then I talked about going from a distribution like this before the course began to a distribution that is much narrower and taller at the end of the course. This is what we are after. That is for the learning aspect.

(Refer Slide Time: 17:07)



Solution

This is a cylindrical system – it will be easier to work with cylindrical co-ordinates
 System: bronchiole wall
 Species: the drug
 A mass balance of the species on our system. Here, we can directly use the equation of continuity in cylindrical co-ordinates equation B2 from Table 2.3.2 – 1

$$\begin{aligned}
 &= 0 \text{ (SS)} \quad = 0 \text{ (} v_r = 0 \text{)} \quad = 0 \text{ (} v_\theta = 0 \text{)} \quad = 0 \text{ (} v_z = 0 \text{)} \quad = 0 \text{ (} c_i \neq f(\theta) \text{)} \\
 &\frac{\partial c_i}{\partial t} + \left(v_r \frac{\partial c_i}{\partial r} + v_\theta \frac{1}{r} \frac{\partial c_i}{\partial \theta} + v_z \frac{\partial c_i}{\partial z} \right) - D_i \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_i}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 c_i}{\partial \theta^2} + \frac{\partial^2 c_i}{\partial z^2} \right) = R_i \quad = 0 \text{ (no rxn)}
 \end{aligned}$$

$$D_i \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_i}{\partial r} \right) \right) = 0$$

r is the only independent variable now – we can replace the partial derivatives with the total derivatives

$$D_i \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{dc_i}{dr} \right) \right) = 0 \quad \text{Eq. 2.4.2 - 1}$$

$c_i = Kc_b \text{ at } r = R_b \quad \text{Eq. 2.4.2 - 2}$

$c_i = Kc_o \text{ at } r = R_o \quad \text{Eq. 2.4.2 - 3}$

K = distribution co-efficient, the ratio of the drug concentrations in the two phases at equilibrium (identify phases in Figure)

And then I showed you how to apply the equation of continuity to a cylindrical system which is again applied to a very relevant situation that of a drug diffusion across a bronchiole wall.

(Refer Slide Time: 17:13)

Solving Eq. 2.4.2. - 1 (derivative in Eq. 2.4.2. - 1 = zero, implies $r \frac{dC_A}{dr} = \text{constant}$, say C_1),

$$C_A = C_1 \ln r + C_2 \quad \text{Eq. 2.4.2 - 4}$$

Substituting the boundary conditions,

$$C_1 = \frac{K(C_b - c_0)}{\ln\left(\frac{R_b}{R_0}\right)}$$

$$C_2 = Kc_0 - K(C_b - c_0) \frac{\ln R_b}{\ln\left(\frac{R_b}{R_0}\right)}$$

By substituting C_1 and C_2 in Eq. 2.4.2 - 4 and by rearranging,


$$C_A = Kc_0 - K(C_b - c_0) \frac{\ln\left(\frac{R_b}{r}\right)}{\ln\left(\frac{R_b}{R_0}\right)} \quad \text{Eq. 2.4.2 - 5}$$

Therefore, the flux at R_b


$$\vec{J}_A = -D_{AB} \left. \frac{\partial C_A}{\partial r} \right|_{r=R_b} = \frac{D_{AB}K(C_b - c_0)}{r \ln\left(\frac{R_b}{R_0}\right)} \Big|_{r=R_b} = \frac{D_{AB}K(C_b - c_0)}{R_b \ln\left(\frac{R_b}{R_0}\right)} \quad \text{Eq. 2.4.2 - 6}$$


We got relationships that give us the concentration profile in the wall of the bronchiole of the drug as well as an expression for the flux at the wall. After this, I showed you the application to a spherical system.

(Refer Slide Time: 17:51)



Steady-state radial diffusion in spherical pellets





That was this aspect. Spherical pellets. Radial, steady state radial diffusion in spherical aspects. We had used the spherical coordinate, the equation of continuity in spherical coordinates to solve this problem.

(Refer Slide Time: 18:04)

To improve yields of ornamental plants, certain growth factors are released from porous, spherical, ceramic pellets embedded in the soil near the roots, in a time-dependent fashion. At the surface of the pellet ($r = R$), the growth factor concentration in the soil is KC_0 . Far from the surface, the growth factor concentration drops to zero. Develop an expression for the steady-state release rate (moles time^{-1}) of the growth factor from the pellet.

Diffusion out from a sphere, equally in all directions
Spherical geometry, thus, spherical co-ordinates

Consider a 'sphere of influence' of the growth factor as our system. Note: the roots where the growth factor is consumed are not a part of the system.

Let us do a material balance on the growth factor over the above system

We can directly use equation C2 in Table 2.3.2 - 1

$$\frac{\partial c_i}{\partial t} + \left(v_r \frac{\partial c_i}{\partial r} + v_\theta \frac{1}{r} \frac{\partial c_i}{\partial \theta} + v_\phi \frac{1}{r \sin \theta} \frac{\partial c_i}{\partial \phi} \right) - D_i \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c_i}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial c_i}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 c_i}{\partial \phi^2} \right) = R_i$$

$\begin{matrix} =0 (c_i \neq f(\theta)) \\ =0 (c_i \neq f(\theta)) \\ =0 (no \ rxn) \end{matrix}$

Thus
$$D_i \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c_i}{\partial r} \right) \right] = 0$$

r is the only independent variable here. Partial derivatives can be replaced with the total derivatives.

$$D_i \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dc_i}{dr} \right) \right] = 0 \quad \text{Eq. 2.4.3. - 1}$$

And thereby we picked up how the concentration of the growth factor varies in the sphere of influence of the pellet that is releasing the growth factor.

(Refer Slide Time: 18:18)

Solution to the above differential equation

$$c_i = A - \frac{B}{r} \quad \text{Eq. 2.4.3. - 2}$$

A and B can be found with the boundary conditions:

$$\text{At } r = R \quad c_i = c_o \quad \text{Eq. 2.4.3. - 3}$$


$$\text{At } r = \infty \quad c_i = 0 \quad \text{Eq. 2.4.3. - 4}$$

Substituting the above BCs in Eq. 2.4.3. - 2, $c_o = A - \frac{B}{R}$ and $A = 0$ Thus




And that is what we did. I do not think I need to get into details. You can look at the details. Then, after that, we looked at steady state. Yeah. No I think I still have a reaction term.

(Refer Slide Time: 18:44)



*Steady-state radial diffusion in spherical pellets
with reaction*



We took a spherical pellet and then we looked at inside the pellet. The only difference between the previous times is that the previous times there was no reaction term. And here there is a reaction term. We had looked at the case of an immobilized enzyme and the reaction occurring there and the relevant aspects.

(Refer Slide Time: 19:02)

Let us consider an enzyme reaction for which the enzyme is immobilized on a porous spherical pellet. The pores could have a high surface area say $250 \text{ m}^2\text{g}^{-1}$. The pellet itself is placed in a fluid environment. Since the enzyme is immobilized inside the pores of the pellet, the transfer of substrate to the site of the immobilized enzyme through the pores, and the transport of product out of the pores are expected to play a major role in determining the process kinetics. Thus, the transport inside the pores needs to be considered rather than the transport to and from the surface of the pellet.

Derive an expression for the 'effectiveness factor', which gives a measure of how much the reaction is hindered due to immobilization.



So this is the immobilized enzyme pellet and the substrate needs to go through the various pores to reach the point of the enzyme and the product needs to move out.

(Refer Slide Time: 19:14)

Spherical geometry – so, let us use spherical co-ordinates

System: spherical pellet

Let us do a material balance on the substrate (concentration = s) over the above system

We can directly use equation C2 in Table 2.3.2 – 1

$$\frac{\partial c_1}{\partial t} + \left(v_r \frac{\partial c_1}{\partial r} + v_\theta \frac{1}{r} \frac{\partial c_1}{\partial \theta} + v_\phi \frac{1}{r \sin \theta} \frac{\partial c_1}{\partial \phi} \right) - D_1 \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c_1}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial c_1}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 c_1}{\partial \phi^2} \right) = R_1$$

$= 0 (c_1 \neq f(\theta))$

For an enzyme catalyzed reaction, the Michaelis Menten equation is a good first approximation for the reaction rate. Also, r is the only variable. Thus, the partial derivatives can be replaced with total derivatives

$$D_{eff} \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{ds}{dr} \right) \right] = \frac{v_{max} s}{K_m + s}$$

D_{eff} = effective diffusivity

$$- D_{eff} \left[\frac{d^2 s}{dr^2} + \frac{2}{r} \frac{ds}{dr} \right] = \frac{v_{max} s}{K_m + s}$$

Eq. 2.4.3.1 – 1

We looked at those movements, the concentration profiles of the substrate and so on so forth along with the assumption that a Michaelis Menten equation gives us the reaction rates.

(Refer Slide Time: 19:33)

For writing suitable boundary conditions, let us consider the following:

We have radial symmetry

Thus, the substrate concentration at the centre must be the same value irrespective of the radial direction followed to approach it

There cannot be a discontinuity in the substrate concentration at the centre, irrespective of the radial direction of approach

The only way that can happen is if the derivative of the substrate concentration at the centre is zero

$$\text{At } r = 0, \frac{ds}{dr} = 0 \quad \text{Eq. 2.4.3.1. - 2}$$

$$r = R, s = s_0 \quad \text{Eq. 2.4.3.1. - 3}$$

The above equation can be solved to get the substrate concentration profile at steady-state, and insights drawn

Let us use this opportunity to present a more generally applicable solution methodology, in terms of **non-dimensional variables**.

And then we looked at something called an effectiveness factor, which is what we derived as a part, which is what we have found as a part of this exercise, this problem that is.

(Refer Slide Time: 19:42)

Let us divide throughout by $\frac{s_0 D_{eff}}{R^2}$

$$\frac{d^2 x}{dy^2} + \frac{2}{y} \frac{dx}{dy} = - \frac{v_{max}' x R^2}{D_{eff} (K_m' + s_0 x)}$$

Let us group the RHS term as

$$\frac{d^2 x}{dy^2} + \frac{2}{y} \frac{dx}{dy} = - \frac{v_{max}'}{D_{eff} K_m'} R^2 \left[\frac{x}{1 + \frac{s_0}{K_m'} x} \right] \quad \text{Eq. 2.4.3.1. - 7}$$

Let us define a couple of non-dimensional parameters – we will see the utility of non-dimensional parameters throughout the course

$$M_T = \frac{R^3 \left(\frac{v_{max}'}{K_m'} \right) s_0}{R D_{eff} s_0} = \frac{\text{'a' reaction rate}}{\text{'a' diffusion rate}} = \text{Thiele modulus} \quad \text{Eq. 2.4.3.1. - 8}$$

$$\beta = \frac{s_0}{K_m'} \quad \text{Eq. 2.4.3.1. - 9}$$

The reaction rate is a first order reaction when $s_0 \ll K_m'$

Thus $\beta = \frac{s_0}{K_m'}$ accounts for deviation from first order kinetics

For larger values of β the reaction is zero order and for smaller values of β the reaction is first order.

And also we looked at the use of non-dimensional variables to generalize the solution.

(Refer Slide Time: 19:50)

Let us divide through by $\frac{s_0 D_{eff}}{r^2}$

$$\frac{d^2x}{dy^2} + \frac{2}{y} \frac{dx}{dy} = - \frac{v_{max}^i R^2}{D_{eff}(K_m^i + s_0 x)}$$

Let us group the RHS term as

$$\frac{d^2x}{dy^2} + \frac{2}{y} \frac{dx}{dy} = - \frac{v_{max}^i}{D_{eff} K_m^i} R^2 \left[\frac{x}{1 + \frac{s_0}{K_m^i} x} \right] \quad \text{Eq. 2.4.3.1.-7}$$

Let us define a couple of non-dimensional parameters – we will see the utility of non-dimensional parameters throughout the course

$$M_T = \frac{R^3 \left(\frac{v_{max}^i}{K_m^i} \right) s_0}{R D_{eff} s_0} = \frac{\text{'a' reaction rate}}{\text{'a' diffusion rate}} = \text{Thiele modulus} \quad \text{Eq. 2.4.3.1.-8}$$

$$\beta = \frac{s_0}{K_m^i} \quad \text{Eq. 2.4.3.1.-9}$$

The reaction rate is a first order reaction when $s_0 \ll K_m^i$

Thus $\beta = \frac{s_0}{K_m^i}$ accounts for deviation from first order kinetics

For larger values of β the reaction is zero order and for smaller values of β the reaction is first order.

And coming to the, a Thiele modulus is also very generally used, widely used. It is nothing but the ratio of a reaction rate to a diffusion rate, which lets us compare these two rates in a system, the Thiele modulus.

(Refer Slide Time: 20:07)

In terms of the non-dimensional variables and parameters, Eq. 2.4.3.1.-1 can be written as

$$\frac{d^2x}{dy^2} + \frac{2}{y} \frac{dx}{dy} = - \frac{R^3 \left(\frac{v_{max}^i}{K_m^i} \right) s_0}{R D_{eff} s_0} \left[\frac{x}{1 + \frac{s_0}{K_m^i} x} \right] = -M_T \left[\frac{x}{1 + \beta x} \right] \quad \text{Eq. 2.4.3.1.-10}$$

$$\text{BCs} \quad \text{at } y = 1, x = 1 \quad \text{Eq. 2.4.3.1.-11}$$

$$\text{at } y = 0, \frac{dx}{dy} = 0 \quad \text{Eq. 2.4.3.1.-12}$$

Solution to Eq. 2.4.3.1.-10 would give x vs. y and thus, s vs. r

The usual interest (as well as the problem need) is in knowing how much the reaction is hindered due to immobilization

An **effectiveness factor** that gives us a measure of the hindrance can be defined as:

$$\xi_{\theta} = \frac{\text{Actual reaction rate}}{\text{Reaction rate in the absence of mass transfer resistance}}$$

And then where are we with the effectiveness factor? Yeah, it is an actual reaction rate divided by the reaction rate in the absence of mass transfer resistance. We said that when we immobilize an enzyme we could lose the speed aspects, the kinetic aspects because of the immobilization. However, there are very many advantages to immobilization. That is why we immobilize it.

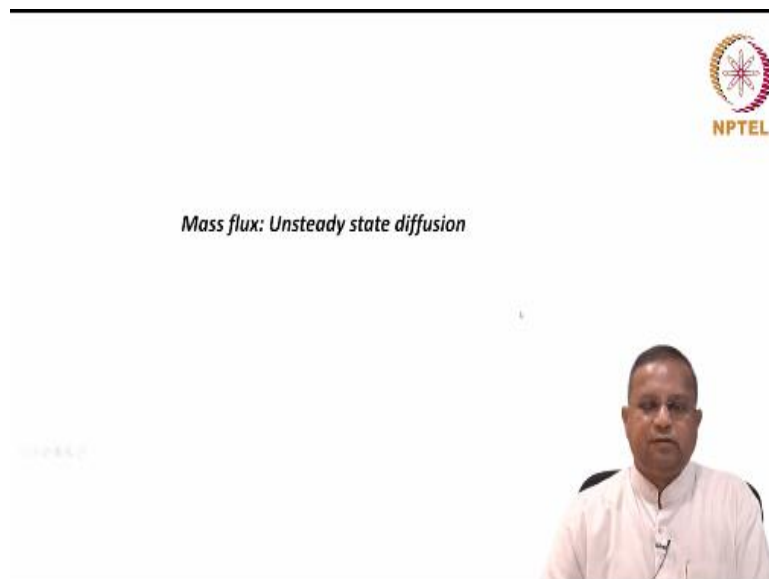
And therefore, when we immobilize it, we would like to know how much of loss is occurring. And the effectiveness factor is one of the parameters that gives us that. It is

the ratio of the actual reaction rate which is actually happening to a hypothetical reaction rate which is the reaction rate in the absence of mass transfer resistance.

For example, if there is no mass transfer resistance, then the concentration of the substrate would be the same as that in the solution throughout. So that gives us the limiting case. So that gives us a parameter by which we can assess the effectiveness of our design maybe if you are designing the immobilized enzyme pellet. And you can use it in many different ways.

And then we looked at the unsteady state case, unsteady state diffusion spherical pellets yes.

(Refer Slide Time: 21:42)



Unsteady state is this. Yes. Unsteady state diffusion. The only difference here is you have an additional time variation term. And that term complicated the mathematics significantly. We went through all the details.

(Refer Slide Time: 21:57)

The concentration of the diffusing molecule, at a particular location, changes with time

Surface modifying agents (SMA)s are used to promote/dissuade cellular growth on a surface of interest – it could be the hull of a ship or a container handling cell solutions. The surface is sometimes exposed to the SMA containing solution for a certain period to effect the modification.

Let us take the case of a thin surface sorbing SMA from a solution with SMA concentration, c_s . The thin surface is placed on the bottom of the vessel containing the SMA solution. There is no movement in the solution after the placement of the surface.

Let us consider the case where the amount of SMA sorbed is a very small fraction of the total SMA amount present in solution. In such cases the SMA concentration far from the surface does not appreciably change (why?).

Find the SMA concentration in solution as a function of time.

We were looking at the concentration profiles in solution of a surface modifying agent when you place a surface to be modified at the bottom and then you allow the surface modifying agent to move through the liquid to react at the surface and modify it. We had considered an appropriate system to allow us for the analysis to get the various concentration profiles of the surface modifying agent variation with time different curve for different times.

(Refer Slide Time: 22:28)

The solution to the above differential equation with the initial and boundary conditions will provide the SMA concentration profiles in the solution above the surface at various times

To make the solution more generally applicable, let us express the equation in terms of dimensionless variables

Let us define

$$\theta = \frac{C_s - C_0}{C_s - C_0} \quad \text{Eq. 2.5. - 5}$$

$$\eta = \frac{z}{\sqrt{4 D_1 t}} \quad \text{Eq. 2.5. - 6}$$

The variable, η , has been constructed to allow the possibility of conversion of the partial differential equation (PDE) to an ordinary differential equation (ODE) by combining both the independent terms z and t .

We have constructed $\theta = f(\eta)$ and $\eta = f(z, t)$

While using the chain rule, $\frac{\partial \theta}{\partial z}$ can be replaced by $\frac{d\theta}{d\eta}$ without any loss.

And we had also used non-dimensional variables to solve it. The solution, the analytical solution that we usually prefer is very involved. We also used non-dimensional variables. I showed you that. Let us go quickly to yeah, that was our solution which is involved.

(Refer Slide Time: 22:54)

Or $\theta = 1 - \text{erf}(\eta)$
 $\theta = \text{erfc}(\eta)$

$\text{erfc}(\eta)$ is the complementary error function, which is defined as $1 - \text{erf}(\eta)$

Replacing the non-dimensional variables with their dimensional equivalents, we get

$$\frac{C_1 - C_0}{C_s - C_0} = \text{erfc}\left(\frac{z}{\sqrt{4 D_1 t}}\right) \quad \text{Eq. 2.5. - 10}$$

Yeah, this is if you look at this as the concentration axis and this as the distance axis in the liquid. So here it is close to 0 and then at time 1 is this. At time 2, which is greater than time 1, it is this. Time 3, which is greater than both, other two, it is this and so on so forth. So, at various different times this profile evolves, . So that is what we found.

(Refer Slide Time: 23:30)



Pseudo-steady state approximation (PSSA)



Then, we finally looked at the pseudo steady state approximation. We said when you have two processes,

(Refer Slide Time: 23:35)

Pseudo-steady state approximation (PSSA) is a view/technique that can be used to simplify the analysis, and the mathematical complexity *when comparing two processes of widely varying rates*

To understand the pseudo steady state approximation, let us consider the process of car manufacture. Let us focus on three of the processes as shown below.

Process	Making the bolts that are used in the engine	Making the engine	Making the whole car
Characteristic rates	Say, 1 bolt per 5 seconds	Say, 1 engine per 1 hour	Say, 1 car per 24 hours

If we focus on engine making, whether the rate of bolt making is 5 s^{-1} or 8 s^{-1} or 2 s^{-1} , ... does not affect the rate of engine making.

If our interest is engine-making, the process of bolt-making is fast enough to be considered at pseudo-steady state, i.e. the changes in the rate of bolt-making (unsteady aspects) will not much affect the rate of engine-making.

Also the rate of whole-car-making is so slow, that it is not even relevant to the rate of engine-making.

Thus, for the interest at hand, i.e. engine-making, the process of whole-car-making can be taken as 'frozen'.

I think I should spend a little bit of time on this new concept, maybe. A pseudo steady state approximation PSSA is a view or a technique that can be used to simplify the analysis and mathematical complexity when comparing two processes of widely varying rates, . This is important, this is the only place we could use this. And then I gave you an example of bolt making and engine making.

The variation in the rate of bolt making does not impinge the rate of engine making. If our interest is in the rate of engine making, we could assume that the bolt making process is at steady state or is at pseudo steady state. It does not really matter whether it is actually at steady state or not. It is at pseudo steady state and that significantly simplifies the analysis.

We can, we do not have to consider the time variation. And that as you can see, as we have already seen, simplifies the analysis significantly.

(Refer Slide Time: 24:32)

Now, let us consider some cellular processes:

Process	Enzyme action	Cell growth/division	Natural mutation
Characteristic rates	One in every 10^3 s	One in every 10^2 s	One in every 10^8 s

If we are interested in cell growth/division, the enzyme action can be taken to be at pseudo steady state, and natural mutation can be considered 'frozen'.

Now, let us consider a thin membrane through which diffusion of a species occurs



Let us take the membrane as the system

Let us say the interest is in the changes in the species concentration in the solutions that are separated by the membrane

If the diffusion through the membrane is fast enough compared to the changes in the concentration of the species in the solutions separated by the membrane, then the diffusion through the membrane can be assumed to take place under steady-state conditions.

We looked at the application of that to a particular problem of determining the permeability of a coating layer to a certain model, protein albumin, . We saw the strategy of getting to the permeability is quite involved. We looked at it over two classes, . So that is what we looked at in terms of mass flux. When we begin the next class, we will start looking at momentum flux. See you then.