Transport Phenomena in Biological Systems

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Lecture - 23
Shell Momentum Balances

Welcome back. In the last lecture, we started the momentum flux. We needed to understand fluids because you would not have had a background in fluids. We looked at first the shear stress and how shear stress can be considered as momentum flux. And then we looked at the most important characterization or very important characterization of fluids in terms of its rheological properties.

Typically a τ shear stress versus shear rate graph, the various types of fluids and the two major types of flows. This is what we have seen in the last class. Today, we will start analyzing the flow. And of course, I do not have to tell you how important flows are right. I think it will come up somewhere, but let me briefly tell you this. There are flows all over the body.

Blood flows, lymph flows, so many things flow in the body. Many flows occur in industries, okay. The industry thrives on fluids moving from one point to the other, for various processing for producing products, and so on so forth. So fundamentally flows are important for us. And that is the reason why we biological engineers are studying this. We are trying to understand so that we can analyze new situations then design new things.

Or even if there are existing situations, we can troubleshoot and then operate things appropriately and so on. Let us move forward. We will start with Shell momentum balances.

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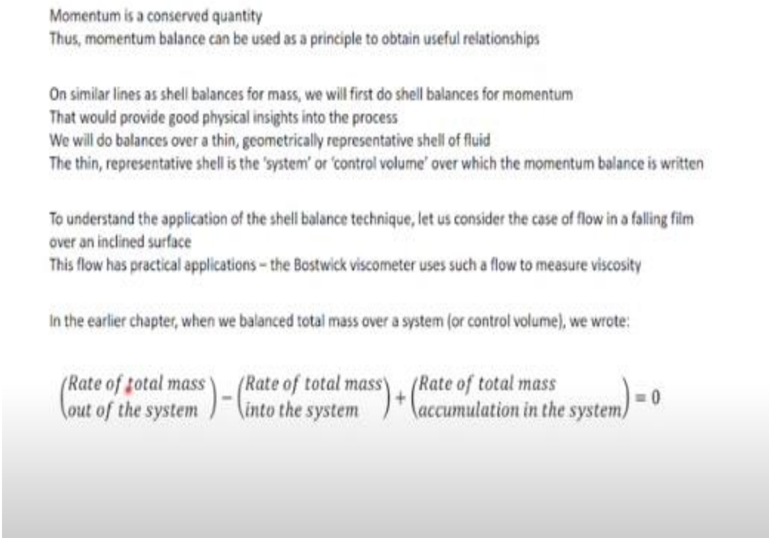

Recall that we have already seen that momentum is a conserved quantity. And therefore momentum balance can be used as a principle to obtain useful relationships. I need to use this I suppose. On similar lines as shell balance, recall shell balance we considered representative shell of the system of interest and then we did balances over that differential shell.

If it is a rectangular Cartesian coordinate system the shell was a cuboid. If it was a cylindrical system it was an annular cylinder and if it is a spherical system it was an annular sphere and so on so forth. So now we are going to do, in the earlier chapter we did mass balances over the shell. And here we are going to do momentum balances. That provides us a lot of insights for analysis, design, and operation.

That is our interest in this. As I already said, we are going to do balances over a thin geometrically representative shell of fluid. And this thin representative shell of fluid will turn out to be the system or the control volume over which the momentum balance is written. And then of course, you can integrate it for the entire thing. So to understand the application of the shell balance principle, I am going to consider a certain situation, reasonably general situation.

This is the case of flow in a falling film over an inclined surface. You have a surface at an angle. There is a thin film of fluid that is flowing over it, and we are going to consider that case. The flow has practical applications itself. This kind of a situation inclined flow, it has practical applications. The Bostwick viscometer is essentially this and that is used to measure viscosities of fluids that uses the same kind of flow and by making certain measurements the viscosity can be calculated.

So in the earlier chapter we balanced mass as I mentioned. We wrote it as follows in general. The rate of total mass out of the system - the rate of total mass into the system + the rate of total mass accumulated in the system equals zero. Or we said into - out equals accumulation and I have just transposed the equation here for our purposes.

So rate of mass out - rate of mass in + rate of a total accumulation equals zero. This is accumulation in the system.

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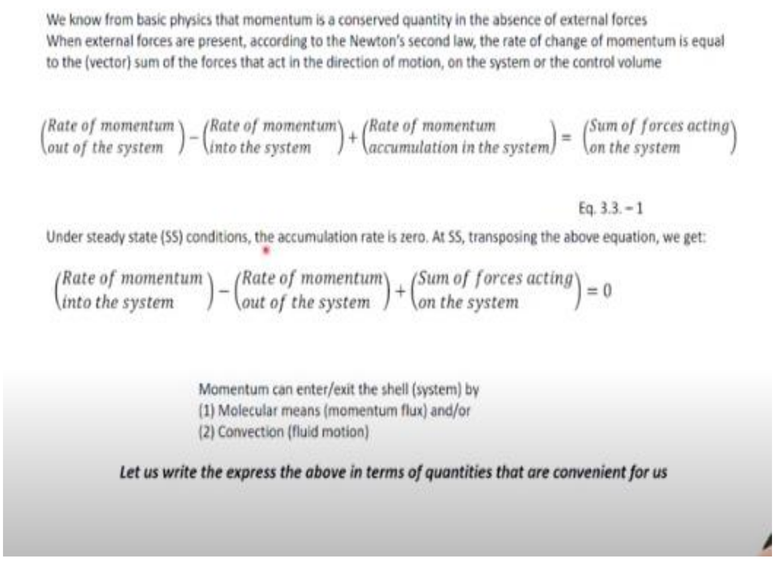

$$\begin{pmatrix}\text{Rate of total mass}\\ \text{out of the system}\end{pmatrix} - \begin{pmatrix}\text{Rate of total mass}\\ \text{into the system}\end{pmatrix} + \begin{pmatrix}\text{Rate of total mass}\\ \text{accumulation in the system}\end{pmatrix} = 0$$

We know from basic physics, that momentum is conserved in the absence of external forces, okay. The rate of change of momentum is directly proportional to the forces acting on it. If there are no forces acting on it, the rate of change of momentum is zero. So when external forces are present, yeah that is what I just mentioned. Then according to Newton's second law, the rate of change of momentum is equal to the vector sum of forces that act in the direction of motion on the system or the control volume, okay. Just note these are very essential things. The vector sum of forces that act in the direction of motion that is all. The other directions, perpendicular directions especially, we are not bothered on the system or the control volume.

So on the same lines as this, rate of total mass out - rate of total mass in + the rate of total mass accumulated in the system. Instead of mass if we use momentum, we get rid of momentum out of the system - rate of momentum into the system, + rate of momentum accumulation in the system equals the sum of forces acting on the system. Instead of zero in it as in the case of mass here it equals the sum of forces acting on the system. Or in the absence of any force acting on the system it is zero. So when we say conservation, we are taking the special case of no forces acting on it. It is strictly speaking Newton's second law that we are applying here. This is equation 3.3-1.

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{out of the system}\end{pmatrix}-\begin{pmatrix}\text{Rate of momentum}\\ \text{into the system}\end{pmatrix}+\begin{pmatrix}\text{Rate of momentum}\\ \text{accumulation in the system}\end{pmatrix}=\begin{pmatrix}\text{Sum of forces acting}\\ \text{on the system}\end{pmatrix} \tag{3.3-1}$$

Under steady state conditions, the accumulation rate is zero, okay. This term can drop out. So at steady state, we are imposing the condition of steady state now. We can write this equation as rate of momentum into the system - rate of momentum out of the system + sum of forces acting on the system essentially I have taken the two terms to the other side. Sum of forces acting on the system equals zero, okay. I hope you are able to see this. This transposes equation and put this term to zero because it is steady state.

At steady state, the accumulation rate can be set to zero, and the balance becomes

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{into the system}\end{pmatrix}-\begin{pmatrix}\text{Rate of momentum}\\ \text{out of the system}\end{pmatrix}+\begin{pmatrix}\text{Sum of forces acting}\\ \text{on the system}\end{pmatrix}=0$$

Momentum can enter or exit the shell that we are considering by the following mechanisms. By molecular means, which is through momentum flux and or through convective means through fluid motion, okay. When the fluid is moving, there is a certain velocity. There is a certain mass. So there is a certain momentum associated with the fluid.

And therefore, there could be a moment flux that is caused because of the motion of that fluid, right? So that also will play a role in this momentum equation. That is a contributor to the momentum here. So we need to take that into account. So let us rewrite this in terms of the quantities that are convenient to us. This is fine. This is in principle.

This is generally valid. But none of these we can directly measure. We have to measure separate things. So we will write it directly in terms of measurable variables.

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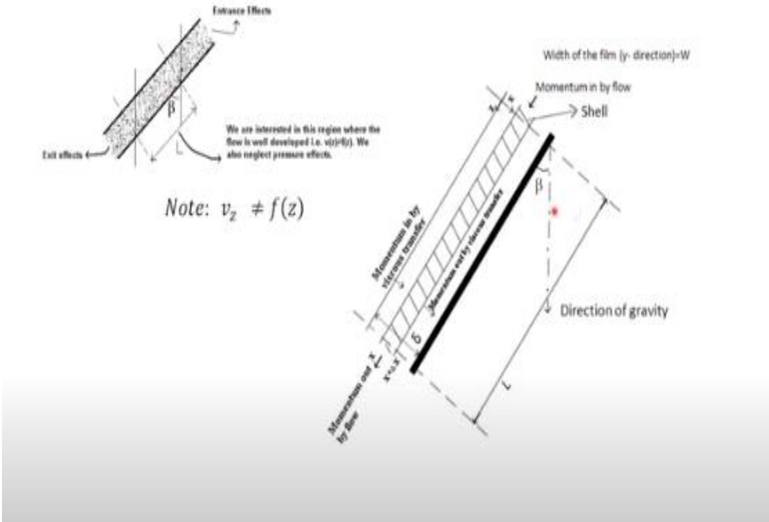

So this is the situation that I have been talking about. Let me give you some more details. We have an inclined plane here, we have a fluid flowing over it, which is given here. This is a very thin layer of fluid. I have just expanded it here. And let us say when it starts flowing in this region, you will have some entrance effects and so on so forth, the non-idealities.

Here when it is moving out, it will have some exit effects. We will analyze the flow when these things do not bother us, okay. That will be for most of the flow essentially, on the entrance and exit. As long as we do not measure things there we are fine. So let us take a section of this flow of length L and this is a section that we are going to analyze.

And at this section the flow is well developed and therefore v_z, this is the z direction here, okay. This is the z direction here. v_z is not a function of z, okay. It has already achieved well developed flow conditions, nothing is going to change with the distance here at a particular point away from the plate. The velocity here will be equal to the velocity here. That is called well developed flow conditions.

That is what we are going to take. I will come to the equation in a little bit. And this is of course, at an angle of β, the plate is at an angle of β. This v_z is not a function of z. It is a well-developed flow conditions which exist. It is not an ideality or anything like that, it actually exists. So a slightly different diagram to understand it understand the momentum flows a little better.

This is the thin plate here. This is the fluid that is flowing over a plate. This is the shell, the hashed area, you know this hashed thing that I have shown here is the shell that I am going to consider a cuboidal shell. So it will be a rectangle in a cross sectional view. There is this dimension into the screen that I am not going to consider here, because there is no variation across that distance.

And therefore two dimensions are good enough for us to get a complete picture of this. So this is the shell here. The shell is of a thickness Δx, x is this direction. Yeah here is given here. x is this direction. z is the direction of flow here. So if this is the (0,0) point that we are going to consider for analysis, the shell is located at a distance of x and has a thickness of Δx.

So there could be momentum transfer through viscous transfer. Viscous is we said that one layer of fluid moving affects the other layer and so on so forth by the viscosity by the shear stress and so on so forth right. That is what we mean by viscous transfer. This was the case of the first situation where we had two parallel plates and one plate moving. Similar thing happens in all these situations.

So there could be a momentum transfer in the x direction by viscous transfer. And there is of course, fluid moving in the z direction. Therefore, the momentum transfer into and out of this system, i.e. the shell would happen by flow. It comes in by flow here and goes out by flow here. So this is the second component. This is by viscous transfer and this is by flow.

And width of the film in the y direction which is this is W. This is the direction of gravity and you have your angle here indicated and the length of this section as L.

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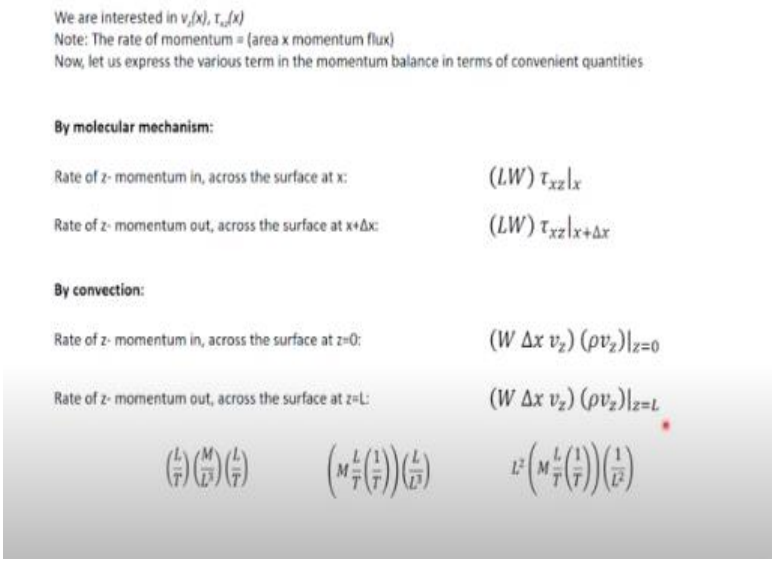

We are interested in knowing what the velocity distribution is with or the velocity variation with the depth of the thin fluid that is flowing, that insight we want. And also we want the shear stress distribution, okay. These two are very important insights that we can get from flow situations which will help us in a variety of ways. And that is the reason why we are focusing on this.

So the velocity distribution, shear stress distribution is a very standard thing that we look for whenever we analyze flow situations. Note τ_{xz}, x is the direction of action in this direction, z is the direction of motion. So this picture you need to keep on making till you become comfortable with it and then it will become second nature to you. Also note the rate of momentum is nothing but area times momentum flux.

The same way mass rate is mass flux times the area. You multiply by the area, area by the momentum flux, we get the rate of momentum. And each term in that momentum balance was rate of momentum. Now we are going to express the various terms in the momentum balance in terms of convenient quantities, measurable quantities, measurable ultimately maybe.

So by molecular mechanism by the viscous effects the rate of z- momentum in across the surface at x okay. Let me show this to you. The rate of x momentum in across the surface at x. This is what we are looking at. See this is our system. We are doing balances over the system. We are looking at how momentum moves in and out. So the momentum by viscous transfer will move in here and move out here.

So the rate of z-momentum in the because it is caused by the motion in the z direction, the effect is in the x direction, but this is because of the motion in the z direction, we call it z-momentum. The rate of z-momentum in across the surface at x is nothing but the momentum flux times the area. Momentum flux τ_{xz} at x that is straightforward now.

The length is the length of the plate of our analysis, W is the width. So therefore, that will give you the area. So area times the momentum flux is the momentum rate at in across the surface at x. The rate of z-momentum out across the surface at x + Δx, which is here. This is the surface that we are looking at.

This is x direction, x here, x + Δx here, okay is nothing but the shear stress τ_{xz} at x + Δx times the area, area remains the same. By convection, by bulk motion by the fluid motion, there is another component to momentum in this case. So rate of z-momentum in across the surface at z equals 0, you need to understand this better. So let us get back to that figure. See z equals 0 is what, this here, okay.

By Molecular Mechanism

Rate of z momentum in, across the surface at x: $(LW)\ \tau_{xz}|_x$

Rate of z momentum out, across the surface at $x + \Delta x$: $(LW)\ \tau_{xz}|_{x+\Delta x}$

So flows occurring here the system is limited to this now. Therefore flow is occurring here. So rate of z-momentum in the momentum in the z direction and this is the phase across which it enters. This is the phase across which it leaves. The phase at z and the phase at L; z equals 0 and z equals L. The rate of z-momentum in across the surface at z equals 0 is, let us write it like this.

By Convection

Rate of z momentum in, across the surface at $z = 0$: $(W\ \Delta x\ v_z)\ (\rho v_z)|_{z=0}$
Rate of z momentum out, across the surface at $z = L$: $(W\ \Delta x\ v_z)\ (\rho v_z)|_{z=L}$
Gravity force acting on the fluid in the direction of motion: $(L\ W\ \Delta x)\ (\rho\ g \cos\beta)$

We write it in this way to facilitate us later. So let us keep this same thing. $\rho\ v_z$ is nothing but density times the velocity. So density is mass per unit volume M/L^3. Velocity is (L/T). Let us just consider this v_z alone for now.

For $\rho\ v_z$, $(M)/L^{3}$ * (L/T)
For $v_z\ \rho\ v_z$, (L/T) *$(M)/L^{3}$ * (L/T)
Rearranging this,
$(L/T)*(M/L^3)*(L/T) = (ML/T(1/T))(L/L^3)$
Multiply by area, L^2, we get = $L^2 * [(ML/T)(1/T))(1/L^2)]$ for $W\ \Delta x\ v_z\ \rho\ v_z$
($W*\Delta x$ is the area L^2)

It is very essential to understand this. We will be using this again and again and again in a lot of formulations, a lot of derivations. Now the rate of z-momentum out across the phase at z equals L okay. I think I showed you that. Let me show it to you once more maybe. Here it is moving out at the phase z equals L equals the same format $W\Delta x$ $v_z(\rho\ v_z)$at z equals L.

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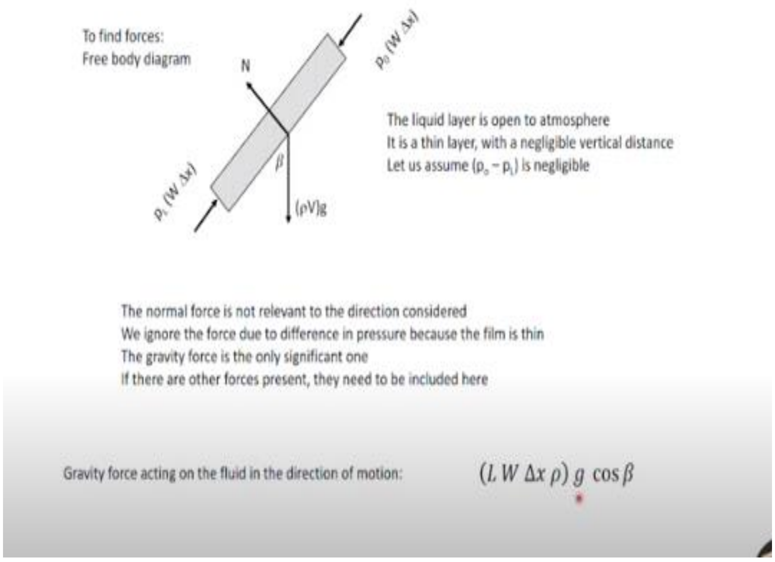

Now let us consider, we need the forces right. We have the input and output terms fine that the momentum in the momentum balance. Right this is rate of momentum into the system, rate of momentum out of the system. We have this third term here, sum of forces. We have accounted for this and this, we need to account for this in the momentum balance. We are just filling in the terms here. Then we are done.

To account for the forces, let us draw a free body diagram. How we are doing, okay we are fine. So free body diagram here. This is the shell, right? A fluid that we are considering therefore this is x, this is x + Δx, z equals 0, z equals L. There is we have cut out the shell from the fluid. The free body diagram shows the forces that are acting. There will be a pressure force that is acting in this direction. And therefore, W*Δx is the area, area times the P_0 pressure **= P_0 (WΔx)** This gives you the force in this direction. There is another force the pressure force that is acting in this direction. W*Δx is the area times P_L the pressure here is P_L = **P_L (WΔx)**, that is this force. And then of course, there is a normal force here N.

There is of course a gravitational force here which is mass, (ρ V). ρ is the density times the volume of this, times the acceleration due to gravity, g which is given by (ρ V g). And of course, this is at an angle of β to the vertical okay. So these are the various forces that are acting. However, note that this z is the direction of motion.

And we are only worried about the forces in the direction of motion for us to do a momentum balance or to consider how the movement in that direction is affected by the forces. So we are worried only about the components in the z direction. Also note that the liquid layer is open to the atmosphere right? It is a thin layer with negligible vertical distance. Okay, I already said it is a very thin layer.

That is the way Bostwick viscometer works by the way. It is a thin layer with negligible vertical distance. Now note when the liquid is open to the atmosphere, the pressure at the top surface is going to be that the atmospheric pressure. And the layer is very thin. Therefore there is the hydrostatic pressure because of that thickness of fluid can be neglected, okay. That is the reason that is one of the simplifications that we have because it is a very thin layer.

And therefore, the difference between this pressure and this pressure can be considered negligible. Or in other words, P_0 can be taken to be the same as P_L in this particular case, because the layer of fluid is very thin. And because of that, this force will cancel with this force. However, and of course, the normal force is not in the or is in the perpendicular direction to the direction of motion.

Therefore, this will not contribute. However, there is a gravitational force here and there will be a component in the direction of motion that will contribute. So normal force is not relevant because of the direction being normal or perpendicular to the direction of motion in this case. We are ignoring the force due to pressure difference because I feel the film is thin. I have already told you this.

And therefore, the gravity force is the only significant one. If other forces are present, let us say some electrical force is present, some mechanical some magnetic force is present, some other forces are present, this is the place to include them. So this analysis considers only these forces, normal force, gravitational force, the pressure forces which we have neglected because of the thin filament and so on so forth.

In addition to this, if there are other forces then you need to include that for your situation. Gravity force acting on the fluid in the direction of motion is going to be the component of this in this direction. This $\cos\beta$ is the going to be the component, you all know that. **Thus gravity force acting on the fluid in the direction of motion = $(LW\Delta x\,\rho)\cos\beta$**

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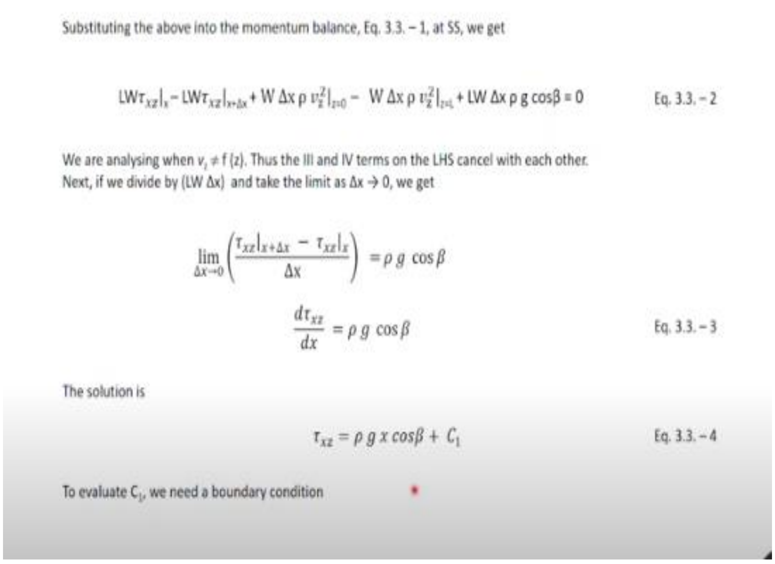

Now we substitute back into the momentum balance input output forces. So the input due to the molecular term across x.

Substituting the above in the momentum balance, Eq. 3.3-1, at steady state, we get

$$LW\,\tau_{xz}\big|_x - LW\,\tau_{xz}\big|_{x+\Delta x} + W\,\Delta x\,\rho\,v_z^2\big|_{z=0} - W\,\Delta x\,\rho\,v_z^2\big|_{z=L} + LW\,\Delta x\,\rho\,g\cos\beta = 0 \qquad (3.3\text{-}2)$$

And as we saw the force, sum of the forces came down to only the component of the gravitational force. LW Δx ρ g cosβ. So all this put together is zero. That is nothing but momentum balance, okay. So we took the second law of motion, momentum balance and then we have just filled in the terms that is all we have done. Only thing is that we have filled in the terms in terms of variables that we are comfortable with. Let us call this equation 3.3-2.

We are analyzing the situation when the velocity in the z direction is not a function of z. And therefore, these two terms are the same. So the third and fourth terms will cancel. Now if we divide throughout by LW Δx and take the limit as Δx tends to 0. Why do you not do that? Why do you not pause the video here, do that and tell me what you get. Same scheme as some of the earlier things that we have done.

So it must be familiar by now. Why do you not do that practice? Let me know. Pause. Go ahead, please. If you did that, you would find that we are dividing by LW Δx and then taking the limit. Therefore, what we have here is τ_{zx} at x + Δx - this because there will be a negative term here. I have taken this to the other side.

Since we have chosen conditions such that $v_z \neq f(z)$, the third and fourth terms on the LHS cancel each other. Then, if we divide the equation by $LW\Delta x$ and take the limit as $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \left(\frac{\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_{x}}{\Delta x} \right) = \rho\, g \cos\beta$$

i.e.

$$\frac{d\tau_{xz}}{dx} = \rho\, g \cos\beta \qquad (3.3\text{-}3)$$

The solution of the above first order differential equation (DE) is

$$\tau_{xz} = \rho\ g\ x\ \cos\beta + C_1 \qquad (3.3\text{-}4)$$

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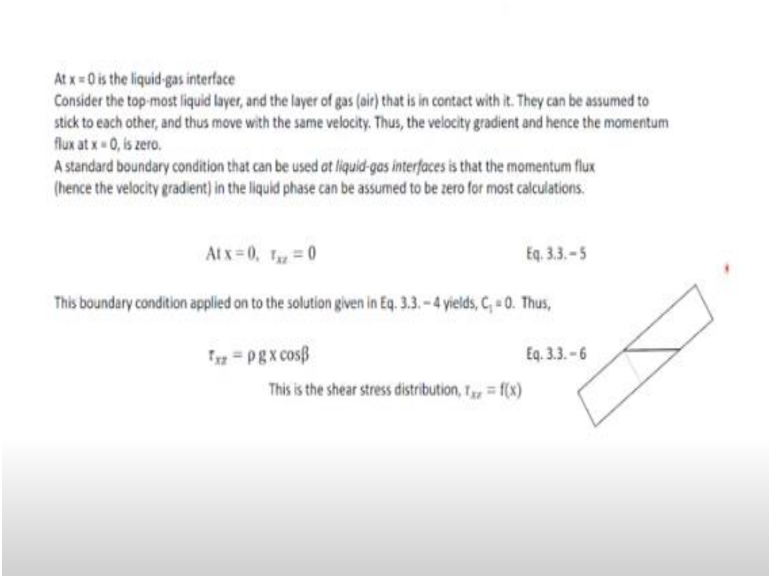

To find the constant of integration, before that let us call this equation 3.3-4. To find the constant of integration, we need boundary conditions. So what is the boundary condition here? Please pay some close attention because this boundary condition we are going to use again and again and again.

At x equals 0 which is the surface of the liquid, the gas liquid interface okay. The liquid layer is moving and there is air above it gas, liquid gas interface. The layer of air that is in close contact with the fluid will also move at the same velocity as that of the fluid under no-slip conditions okay. Or let us say yeah this is air so it does not really matter.

So the there is a fluid moving, there is a topmost layer. And the layer of air, the layer of gas above it will also be pulled along at the same velocity, okay. And this is the condition at the gas liquid interface. When this happens, there is no velocity gradient there. There is no change in velocity with respect to distance. And therefore, there is no momentum flux at x equals 0.

The momentum flux at x equals 0 is 0. Momentum flux as we all know is shear rate and shear rate is proportional to the velocity gradient. If the velocity gradient is 0 the shear rate is 0 and therefore, the momentum flux is 0. So this is the boundary condition, a very important boundary condition.

And the standard boundary condition that can be used at the liquid-gas interfaces throughout is that the momentum flux hence the velocity gradient in the liquid phase can be assumed to be zero for most calculations.

Notice that $x = 0$ is the liquid-gas interface. A standard boundary condition that can be used at *liquid-gas interfaces* is that the momentum flux (hence the velocity gradient) in the liquid phase can be assumed to be zero for most calculations. i.e.

$$\text{at } x = 0, \ \tau_{xz} = 0 \quad (3.3\text{-}5)$$

This boundary condition applied on to the solution given in Eq. 3.3-4 yields, $C_1 = 0$. Thus

$$\tau_{xz} = \rho \ g \ x \cos\beta \quad (3.3\text{-}6)$$

Thus, we have the shear stress distribution, i.e. $\tau_{xz} = f(x)$.

So we have an expression for the shear stress in the fluid we are drawing insights into the fluid flow behavior. We have an expression we have this is a function of x. So if you visualize, this is the equation 3.3-6, the shear stress distribution is known from this. If you visualize, you see it is varying from zero value to a maximum value.

It is a linear variation here, x is a linear function, (ρ g cosβ)x. So it is a linear variation from the value of zero to some maximum value at the solid surface over which it is flowing, okay. So we have the shear stress distribution. We are going to we said we look for shear stress distribution and velocity distribution. When we meet in the next class, we have been at this for some time.

So when we meet in the next class, we will look at the velocity distribution. We are drawing insights, insights into what is actually happening. And those insights are going to be very useful in design of new things and understanding old things, even operation and so on so forth. That is the reason why we are doing this.

The initial what shall I say initial perception of some students is what is the point in knowing about velocities? Let they be whatever they are. What is the point about knowing the shear stress distributions? As you will see here, these give you how the fluid behaves. The fluid behaves determines so many different things. Suppose you want to carry out certain things in a miniature level.

The flows will be micro flows. Do you get the same level of mixing that is required when you convert a flow from a normal situation to a micro flow? That is a question that you need to answer if you are going to design such micro flows. And to design that you need to understand how the flow is happening, how the velocity behaves or the shear stress behaves and so on so forth.

That is the kind of insight that you are getting here, okay. And that is the reason why we need to understand this better. Let us meet in the next class and take things forward. See you then.