Transport Phenomena in Biological Systems

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Lecture – 24
Shell Momentum Balances - Continued

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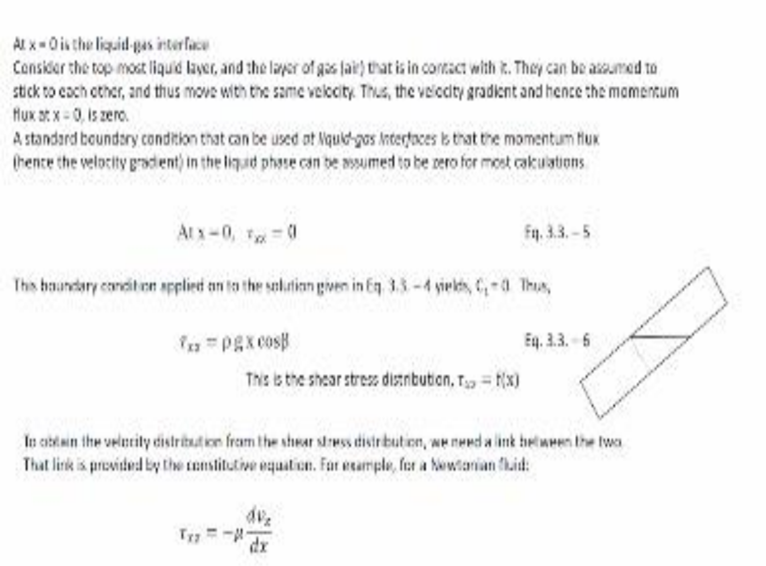

Welcome, back. We are looking at shell balances in momentum flux. In the previous class, we looked at the case of a thin layer of fluid moving over a flat plate and we tried to write shell momentum balances for that situation. We arrived at the expression. We considered the contributions of momentum flux through molecular means or viscous means and also through convective means and then we looked at the forces.

We put in the various terms into the momentum balance equation then we arrived at

$$\tau_{xz} = \rho g x \cos\beta$$

which is nothing but a linear variation along the x direction as the x changes, it goes from 0 value as indicated by this to a maximum value somewhere here okay. So, this is what we have. Therefore, we have this shear stress distribution in this particular situation. We also need the velocity distribution.

How do you go from the shear stress distribution to the velocity distribution? Yes, we need a relationship between the velocity and shear stress. What is that relationship? You have already seen this, can you think about it? Let us say for a Newtonian fluid, can you think about it? Yes, it is a Newton's law viscosity. Newton's law viscosity says that shear stress equals viscosity times the shear rate, right.

So we need a relationship between shear stress and shear rate to get the velocity distribution or you could also do it that way, there are various ways of doing it, this is one way of doing it. We get the shear stress profile and then use the relationship, a fluid property relationship, fluid constitutive equation between shear stress and shear rate and get the velocity profile. Let us do that in this class.

For Newtonian fluid, we know that,

$$\boldsymbol{\tau_{xz} = \mu\left(\frac{dv_z}{dx}\right)}$$

Where the subscripts are relevant to the directions here z and x.

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Substituting the constitutive equation into Eq. 3.3. – 6, we get

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos\beta}{\mu}\right)x \qquad \text{Eq. 3.3. – 7}$$

The solution of the above D.E. is

$$v_z = -\left(\frac{\rho g \cos\beta}{2\mu}\right)x^2 + C_2 \qquad \text{Eq. 3.3. – 8}$$

C_2 can be found by another standard boundary condition: at the solid- fluid interface, the fluid velocity equals the velocity with which the surface itself is moving

The fluid is assumed to cling to any solid surface with which it is in contact ('no-slip' boundary condition)

$$\text{At } x = \delta, \quad v_z = 0 \qquad \text{Eq. 3.3. – 9}$$

By substituting the boundary condition into the solution, Eq. 3.3. – 8, we get

So, if we substitute this constitutive equation into 3.3 - 6 which is the expression for the shear rate. Let us call this equation 3.3 – 7. So now if we integrate this, we directly get v_z as simple as that and if we solve this differential equation, then we get 3.3-8 as shown below

By substituting the constitutive equation in Eq. 3.3-6, we get

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos\beta}{\mu}\right)x \qquad (3.3\text{-}7)$$

The solution of the above DE is

$$v_z = -\left(\frac{\rho g \cos\beta}{2\mu}\right)x^2 + C_2 \qquad (3.3\text{-}8)$$

How do you find the constant of integration? We need a boundary condition. Now what is the boundary condition? Okay, now pay some close attention here because this is also another standard boundary condition that can be used under these situations. It is called the no-slip boundary condition at the solid-fluid surface. The earlier standard condition was the gas-liquid surface, the 2-fluid surface.

The gas-liquid surface v_z since the layer that is closest to the liquid is layer of gas that is close to the liquid is also moving at the same velocity as that of the liquid. The velocity gradient there is the 0 and therefore the shear stress there is 0. Here at this solid-fluid surface, the velocity equals the velocity at which the surface itself is moving because the last layer of the fluid or the layer closest to that of the surface attaches itself to the surface, clings to the surface, does not slip over the surface and therefore its velocity is the same as that of the surface, okay that is what is called the no-slip boundary condition.

The fluid is assumed to cling to any solid surface with which it is in contact and therefore it moves at the same velocity as that of the solid surface. Here, the boundary condition can be written at $x = \delta$, okay the x is moving.

The x coordinate is in this direction, $x = 0$ is the surface of the liquid layer, $x = \delta$ is the wall or the solid surface and there v_z, the velocity of the liquid equals 0. Note the liquid is a system, therefore the velocity of the liquid is 0.

$x = \delta$, $v_z = 0$ Equation 3.3 – 9.

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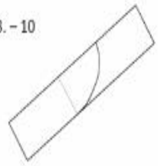
$$C_2 = \left(\frac{\rho\, g \cos\beta}{2\mu}\right)\delta^2$$

Therefore,

$$v_z = \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \qquad \text{Eq. 3.3.} - 10$$



Where does the maximum in velocity occur?

At x = 0

$$v_{z,max} = \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu} \qquad \text{Eq. 3.3.} - 11$$

The average velocity over a cross-section of a film can be found through

$$v_{z,avg} = \frac{\int_0^W \int_0^\delta v_z dx dy}{\int_0^W \int_0^\delta dx dy} = \frac{1}{\delta}\int_0^\delta v_z\, dx \qquad \text{Eq. 3.3.} - 12$$

(W can be cancelled in the numerator and the denominator)

By substituting the boundary condition into the solution, Eq. 3.3-8, we get

$$C_2 = \left(\frac{\rho\, g \cos\beta}{2\mu}\right)\delta^2$$

Therefore

$$v_z = \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \qquad (3.3\text{-}10)$$

We will call this equation 3.3 – 10. Okay I should have shown you. I would like you to stop the video here, pause the video here, go to any graphing software, put in some numbers or put in some functionality here, put in some number for this and see how v_z varies with z.

See how the velocity profile is in that thin layer of fluid, it will be interesting when you do that and then come and check this, pause go ahead. If you did that, you would have found that the velocity profile is parabolic here at x^2 variation. Therefore as x varies, you have v_z varying in a parabolic fashion. Here it is 0 and it goes to the maximum somewhere here okay. This is the velocity representation at 0 here, it is the maximum here.

It is some sort of a parabolic velocity profile x^2 term here, $1 - x^2$, so you have just the reverse, but the pattern is the same. Where does the maximum velocity occur you can take a look at this and say that it occurs at the top surface. If you look at the expression here it is $(1 - x)/\delta^2$.

This takes a value of 1 if it is 0 and therefore the maximum velocity happens when x is 0 which is this surface here. So that is the way we analyze things.

It can be seen that the maximum velocity occurs at $x = 0$. Therefore

$$v_{z,\max} = \frac{\rho g \delta^2 \cos\beta}{2\mu} \qquad (3.3\text{-}11)$$

We would also like to know the average velocity over the cross section okay. So the average velocity over the cross section of the film because the velocity is varying at different points in the film, right.

At the bottom most it is 0, at the top most is a maximum. There is a certain variation across the film. Therefore, if you want to find out the average, you take area weighted average and that is nothing but v_z averages. You take a double integral of this y varying from 0 to w, x varying from 0 to δ, y is in this direction varying from 0 to w, of course x is in this direction varying from 0 to δ,

Now, the average velocity over a cross-section of a film can be computed using

$$v_{z,\text{avg}} = \frac{\int_0^w \int_0^\delta v_z dx dy}{\int_0^w \int_0^\delta dx dy} = \frac{1}{\delta}\int_0^\delta v_z dx \qquad (3.3\text{-}12)$$

So as you can see the w is the same at both ends, therefore I mean the numerator and denominator that will be cancel out and the remaining would turn out to be Equation 3.3 - 12.

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By substituting Eq. 3.3. – 10 in Eq. 3.3. – 12, we get

$$v_{z,avg} = \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu} \int_0^1 \left[1 - \left(\frac{x}{\delta}\right)^2\right] d\left(\frac{x}{\delta}\right)$$

$$= \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu} \left[\left(\frac{x}{\delta}\right) - \frac{1}{3}\left(\frac{x}{\delta}\right)^3\right]_0^1$$

$$v_{z,avg} = \frac{\rho\, g\, \delta^2 \cos\beta}{3\mu} \qquad \text{Eq. 3.3.} - 13$$

The volume flow rate, Q is given by

$$Q = \int_0^W \int_0^\delta v_z\, dx dy = W\delta\, v_{z,avg} = W\delta\, \frac{\rho\, g\, \delta^2 \cos\beta}{3\mu} \qquad \text{Eq. 3.3.} - 14$$

(since W can be cancelled in the numerator and the denominator). By substituting Eq. 3.3-10 in Eq. 3.3-12, we get

$$v_{z,\text{avg}} = \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu} \int_0^1 \left[1 - \left(\frac{x}{\delta}\right)^2\right] d\left(\frac{x}{\delta}\right)$$

$$= \frac{\rho\, g\, \delta^2 \cos\beta}{2\mu} \left[\left(\frac{x}{\delta}\right) - \frac{1}{3}\left(\frac{x}{\delta}\right)^3\right]_0^1$$

$$= \frac{\rho\, g\, \delta^2 \cos\beta}{3\mu} \qquad (3.3\text{-}13)$$

This is the average velocity and we are usually interested in the average velocity, also we are interested in the volumetric flow rate. The volumetric flow rate is nothing but the average velocity times the area, right.

So area times the velocity is flow rate or formally speaking you could go from 0 to w, 0 to δ, So at every point you take the velocity multiplied by the area and take the total of the whole thing. So if you do that you will get,

The volume flow rate Q is given by

$$Q = \int_0^W \int_0^\delta v_z\, dx dy = W \delta v_{z,\text{avg}} = W\delta \frac{\rho\, g\, \delta^2 \cos\beta}{3\mu} \qquad (3.3\text{-}14)$$

Therefore, we found initially the shear stress profile, in this case the velocity profile.

And then the maximum velocity expression, the average velocity expression and the volumetric flow rate expression. So this you can do for many different situations and these are the kind of insights that we are looking for when we analyze fluid systems for our purposes. I think we will stop here, the previous class was something new where a lot of understanding needed to happen and you needed to get a picture.

We completed that here in terms of the various parameters that we can calculate. Let us take a break here. When we come back, we can start out fresh and take things forward. This is shell balance, of course next we are going to do the conservation equation application indirectly, before that we are going to derive the conservation equation of a form that will be useful to us and then we will apply it.

I am going to show you that if we apply it to the same case in one step you get the answer rather than going through all this. See you then. See you in the next class, bye.