Transport Phenomena in Biological Systems
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Lecture – 25
Equation of Motion

Welcome. Today, we will begin looking at the second approach to solve problems of this kind to analyze situations of this kind and so on so forth. To recall, we are looking at momentum flux, even earlier while doing mass flux, there are two major approaches to solving problems. Here first the problem is reposed, the situation needs to be converted to an appropriate problem, needs to be posed and then we can solve it using these two broad approaches.

One was called the shell balance approach where we wrote balances over a differential shell or a representative shell which depended on the geometry of the system. The other approach is the equation approach, the conservation equation approach, the equation of continuity, mass conservation equation for the earlier flux the mass flux. In this case, it is momentum balance or effectively Newton's second law of motion or say the same thing is called equation of motion in this case.

So let us first derive the equation of motion and then as it happened in the mass flux case you will find that it simplifies the solving of many different situations. Of course, it has some limitations, the same limitations as earlier whenever there is a change in geometry of the shell, if there is a variable let's say a diameter or something like that, these equations or some of these equations may not be applicable. With that, let us begin today's main aspect equation of motion.

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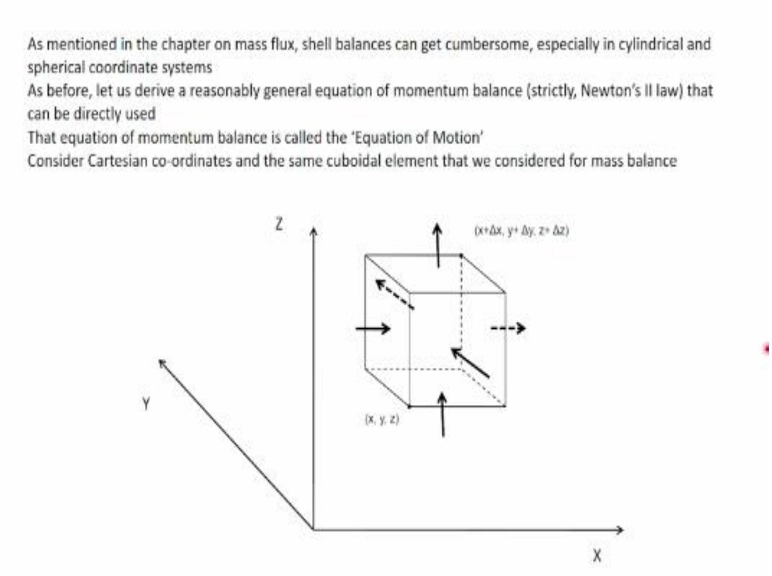

The shell balances can get cumbersome, especially in the cylindrical and spherical coordinate systems. Let us derive a reasonably general equation of momentum balance as I said strictly Newton's second law in the equation of motion. What we are going to do is we are going to derive it as usual on a set of coordinates that we have an intuitive feel for because we have been using those coordinates right from very early on, which are the rectangular Cartesian coordinates.

So this is the same volume element, the control volume or system if you want to call it so. Control volume has fixed volume, system could change volumes that is the only difference between those two terms. Let us take a control volume in the rectangular Cartesian coordinate space. This is x, this is y right-handed Cartesian coordinate system from x. You go from x to y, right-handed screw moves this direction, therefore this is z.

This is the cuboidal control volume. This is let us say x, y, z. The coordinates of this point are (x,y,z). The coordinates of this point are $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ okay, the same kind of the situation as earlier. Earlier, we saw mass moving in all 3 dimensions. We took each dimension separately, here we are going to see momentum moving in all 3 directions. You are going to see those as quantities, conserved quantities of course.

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As discussed during shell momentum balance earlier, momentum flows into and out of the volume element by two means:

- convection (by virtue of fluid flow)
- molecular aspects (by virtue of velocity gradients)

Momentum rate by convection

$(\rho\vec{v})\vec{v}$ is momentum flux (mass flux x velocity; also check through units)

The rate of momentum (momentum per time) is $|\vec{A}|(\rho\vec{v})\vec{v}$ $|\vec{A}|$ = magnitude of the area vector

Units wise: $m^2\left(\frac{kg}{m^3}\frac{m}{s}\right)\frac{m}{s}$

There are three components in the x, y, and z directions to the rate of momentum vector.
Each of those components is, in turn, composed of three other components, as shown next

So the momentum flows into and out of the volume element by 2 major means that we have already seen. One is by convection or by virtue of fluid flow. The second one is by molecular aspects or by virtue of velocity gradients. Velocity gradients are the driving forces for that mechanism for momentum flux. Let us look at them one by one. To recall, we are doing a balance over a cuboidal element in a rectangular Cartesian coordinate space.

Momentum moves in and out of that space. So first by convection. We have already seen that (ρv) times v is the momentum flux, mass flux into velocity so that becomes momentum flux. I have already shown you how that happens, you can go back and look at it. When we did shell balances, I had spent some time on this term to show you how this becomes momentum flux and you multiply it by the area you get momentum rate.

Let me very briefly recall just the units, just in terms of units. The rate of momentum or momentum per time is area times A(ρv) times v, all these are vectors, A is the magnitude of the area vector. This is given as the magnitude of the vector, this is the symbol for the magnitude of the vector, absolute value. The magnitude of the vector times ρv times v is the rate of momentum flux as I said, yeah this is what I mentioned earlier.

If you check in terms of units, area is meter squared, density is mass per volume (kilogram per meter cube), velocity is meter per second. If you cancel out the various terms here two will get on, two will go away here and then one more will remain here in terms of m. Therefore, this will become m^2, yeah. Let us take this together as momentum, so momentum rate and what it remain here is m^2 at the bottom you will get momentum flux.

For momentum transport by convection, note that $(\rho\vec{v})\vec{v}$ is momentum flux (the units can be written down and checked). Thus, the rate of momentum (momentum per time) is $A(\rho\vec{v})\vec{v}$, where A is the area. The units work out as $\cancel{m}^2\left(\frac{kg}{\cancel{m}^3}\frac{m}{s}\right)\frac{\cancel{m}}{s}$.

Also note that this is a vector, mass is a scalar, so it was much easier to deal with and that is the reason why we looked at it first. It is a scalar, it is easier to deal with it, has an intuitive feel to it, you can easily relate to it and thereby pick up the principles and then move on to momentum. Typically, chemical engineers, aerospace engineers, so on and so forth do momentum balance first and then do the other balances.

I have deliberately shifted this for this reason for ease of understanding, ease of learning the material and so and so at various levels. So there are 3 components here to momentum, the x, y and z directions and each of these components is composed of 3 other components, just remember this I will show you how this happens.

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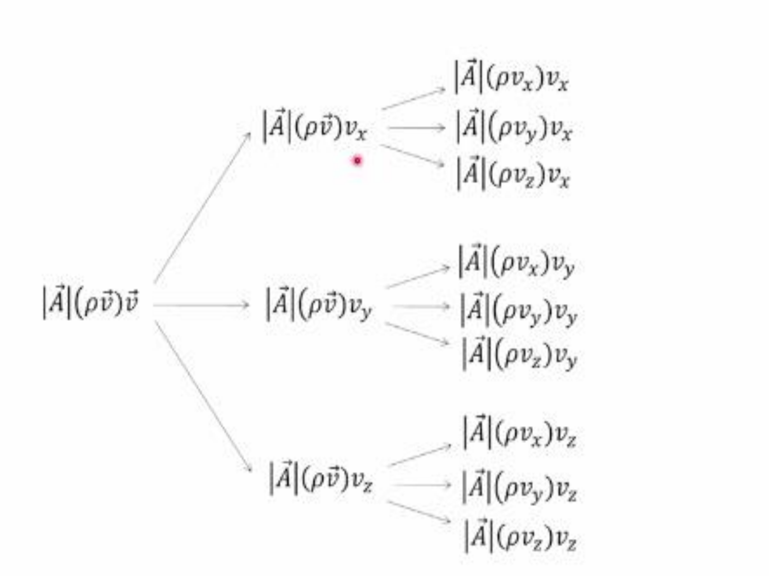
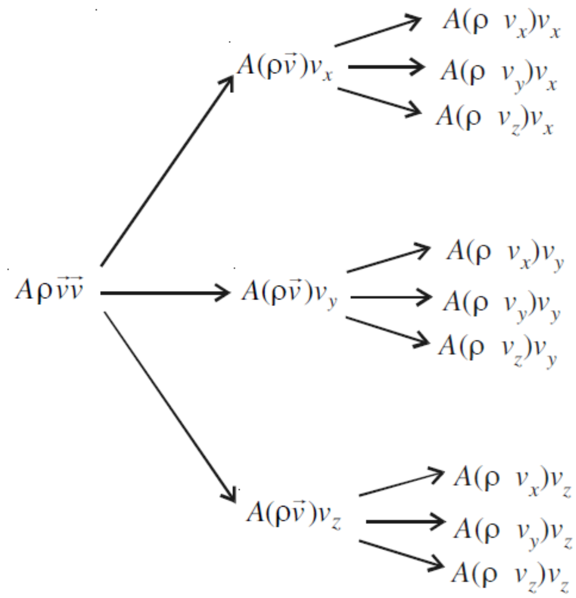


Fig. 3.4-2 The various components of the momentum rate



Therefore you have 9 components for the momentum rate through convective means. We are going to consider only this to develop in detail and once we write things with this, the other things we can write by extension okay, that will allow us to deal only with much less number of terms, much less number of equations to begin with.

And that will make things much easier and these are just extensions in the y and z directions, instead of v_x you have v_y, instead of v_x you have v_z here. So the same equations would be valid except for these changes here okay. So, we are going to look first in detail at the x-momentum alone.

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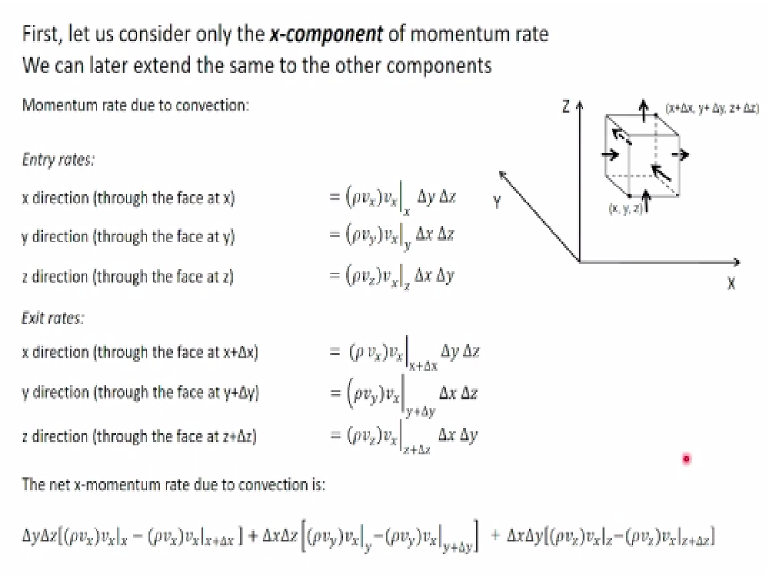

So x component of the momentum rate, we can later extend. This is the same picture for visualization. The momentum rate due to convection, the entry is through the face at x, through the face at y and through the face at z, right.

Entry Rates

x direction (through the face at x) $= (\rho v_x)v_x|_x \;\; \Delta y \Delta z$

y direction (through the face at y) $= (\rho v_y)v_x|_y \;\; \Delta x \Delta z$

z direction (through the face at z) $= (\rho v_z)v_x|_z \;\; \Delta x \Delta y$

Exit Rates

x direction (through the face at $x + \Delta x$) $= (\rho v_x)v_x|_{x+\Delta x} \;\; \Delta y \Delta z$

y direction (through the face at $y + \Delta y$) $= (\rho v_y)v_x|_{y+\Delta y} \;\; \Delta x \Delta z$

z direction (through the face at $z + \Delta z$) $= (\rho v_z)v_x|_{z+\Delta z} \;\; \Delta x \Delta y$

Thus, the net x momentum rate due to convection is

$$\Delta y \Delta z[(\rho v_x)v_x|_x - (\rho v_x)v_x|_{x+\Delta x}] + \Delta x \Delta z[(\rho v_y)v_x|_y - (\rho v_y)v_x|_{y+\Delta y}]$$
$$+ \Delta x \Delta y[(\rho v_z)v_x|_z - (\rho v_z)v_x|_{z+\Delta z}]$$

The net x-momentum due to convection is nothing but the entry minus exit. So entry minus exit for each of these directions taken individually and then summed up is going to give us the net x-momentum. Net momentum consists of all these terms. And similarly you have a term for the y-direction component and the z-direction. The x-momentum in the y-direction the x-momentum in the z-direction that is what it means.

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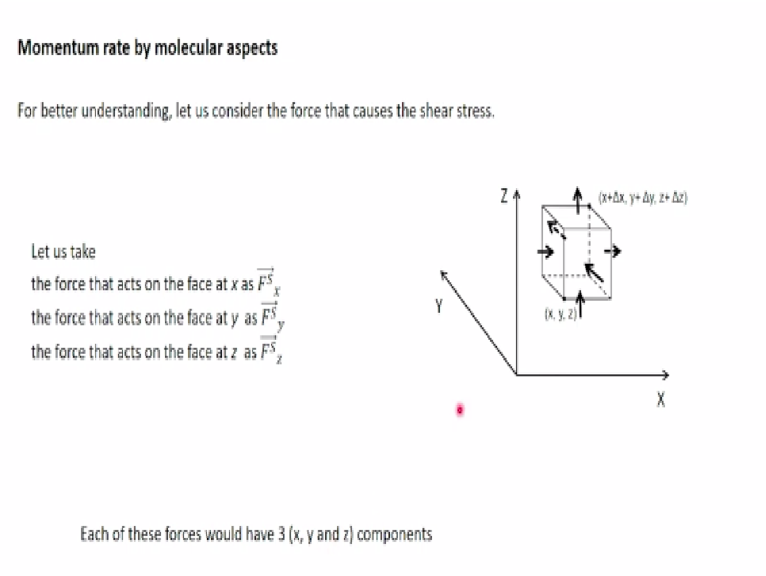

Okay, we are done with one term where I have not shown you, okay we will come to that in a little bit later. Now the momentum rate by molecular aspects due to viscosity between the layers, due to the molecular nature, molecular interactions and so on that is given by the shear stress and to understand this better let us consider the force that is responsible for the shear stress. The force that is responsible for the shear stress is the shear force.

The shear force acts on a surface we all know this and therefore let us take the force that acts on the face at x. This is the face at x, the force that acts on the face at x is F^s at x okay, the surface force at x. Similarly, the force that acts in the face at y is F^s times y which is y is here, this is the face at y. The force that acts on the face is F^s at y and the force that acts on the face at z which is at the bottom here is F^s at z okay.

Each of these forces, these are forces right and each of these forces would have 3 components x, y and z components on their own(i.e. 3*3=9).

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Dividing the force components by the appropriate areas will give the components of the stresses

$$\left.\begin{matrix} F^S{}_{xx} \\ F^S{}_{xy} \\ F^S{}_{xz} \end{matrix}\right\} components\ of\ \overrightarrow{F^S}_x \qquad \left.\begin{matrix} \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{matrix}\right\} components\ of\ \vec{\tau}_x$$

$$\left.\begin{matrix} F^S{}_{yx} \\ F^S{}_{yy} \\ F^S{}_{yz} \end{matrix}\right\} components\ of\ \overrightarrow{F^S}_y \qquad \left.\begin{matrix} \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \end{matrix}\right\} components\ of\ \vec{\tau}_y$$

$$\left.\begin{matrix} F^S{}_{zx} \\ F^S{}_{zy} \\ F^S{}_{zz} \end{matrix}\right\} components\ of\ \overrightarrow{F^S}_z \qquad \left.\begin{matrix} \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \end{matrix}\right\} components\ of\ \vec{\tau}_z$$

τ_{ij} denotes shear stress when $i \neq j$, and it denotes normal stress when $i = j$

Both shear stress and normal stress arise due to molecular aspects

Pressure is not related to shear or normal stresses

And therefore we will end up with $F^s{}_{xx}$ as the x component of the face of the surface force at x, $F^s{}_{xy}$ is the y component of the surface force on the face at x, $F^s{}_{xz}$ is the z component of the surface force on the face at x. So these are the components of F^sx. Similarly the components of F^sy are $F^s{}_{yx}$, $F^s{}_{yy}$, $F^s{}_{yz}$ and the components of the F^sz or the surface force on the face at z is $F^s{}_{zx}$, $F^s{}_{zy}$, $F^s{}_{zz}$ okay.

Now, let us divide these forces by the area of those faces and if we divide the shear force by the area you get the shear rate. So the corresponding shear rate our τ_{xx}, τ_{xy}, τ_{xz} components of τ_x, τ_{yx}, τ_{yy}, τ_{yz} the components of τ_y and τ_{zx}, τ_{zy}, τ_{zz} the components of τ_z okay. So, we have 9 components for the surface forces or the surface stress for the shear stresses that act on this volume element from the x-direction alone or in the x-direction alone okay.

Also note that this superscript has a nice way of revealing itself, T_{ij} denotes shear stress when i is not equal to j and it denotes normal stress when i = j. So τ_{xx} is the shear force, we will understand this a little better as we go along, but now understand this as a component, one of the components that arise. This is the x component of the stress due to the force in the x-direction or due to the x component of the force in the x-direction that is what this is.

Do not worry about these details, as you get deeper and deeper it will become a part of it okay, but pick this up very clearly, once you pick this up clearly then everything else is just methodically working things. To repeat we are looking only at the momentum rate only in the x-direction that itself has three-three components and those components are the ones that we are writing down. Both shear stress as well as normal stress arise due to molecular aspects.

There is sometimes a confusion between pressure and normal stress okay, do not fall for that, they are 2 different aspects altogether. This normal stress arises due to molecular aspects, pressure is a separate aspect, just take it on face value now and then whenever the confusion arises remember this, you can add them because they are in the same direction. The components of the same direction, they become additive, but they are 2 different aspects that is what it says here pressure is not related to shear or normal stresses.

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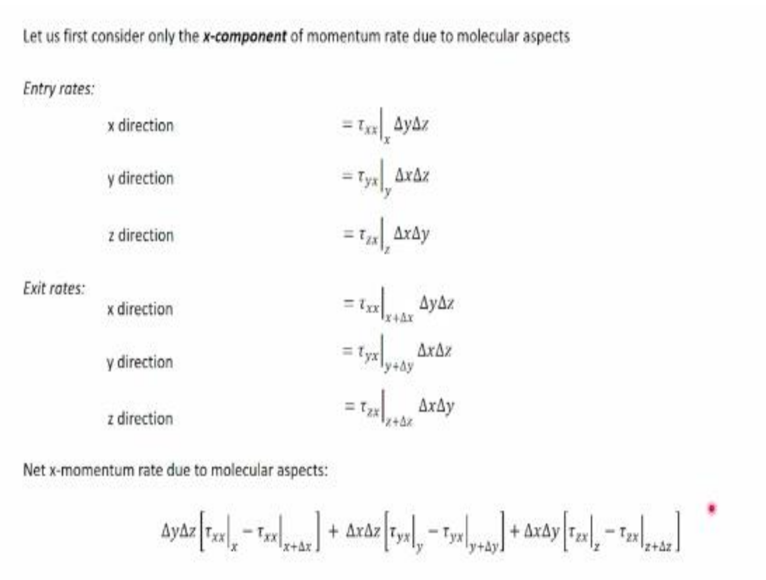

Let us consider only the x component of momentum due to the molecular aspects. What we have written here is all the components as we wrote for the convective contribution also and we are going to consider only the x component τ_{xx}, τ_{xy}, τ_{xz} and then by extension we are going to look at these two, they become very simple extensions later. So this way we need to worry only about 3 equations rather than 9 equations at a time which can be confusing.

The entry rates, again we are looking at entry minus exit because we are going to substitute this in the momentum balance equation. So for the x-direction is $\tau_{xx}|_x \Delta y \Delta z$, y-direction is $\tau_{yx}|_y \Delta x \Delta z$, and the entry rate in the z-direction is nothing but the shear stress times the area the momentum flux times the area that is momentum rate, in this case $\tau_{zx}|_z \Delta x \Delta y$.

The exit rates are, do you want to write them or let me show you one, give you some time to write, in the x-direction that is $\tau_{xx}|_{x + \Delta x} \Delta y \Delta z$. Pause the video here please and then write the other 2 terms for the y-direction and for the z-direction, go ahead please. You would have obtained for the y-direction $\tau_{yx}|_{y + \Delta y} \Delta x \Delta z$ and for the z-direction $\tau_{zx}|_{z + \Delta z} \Delta x \Delta y$ okay.

So these are the entry and exit rates. So the net x-momentum due to molecular aspects alone is entry minus exit, therefore Δx Δy Δz times (τ_{xx} at x) – (τ_{xx} at x + Δx), similarly you have a term for the y component and term for the z component.

$$\begin{aligned}
&\textit{Entry Rates}\\
&x \text{ direction} = \tau_{xx}|_x \Delta y \Delta z\\
&y \text{ direction} = \tau_{yx}|_y \Delta x \Delta z\\
&z \text{ direction} = \tau_{zx}|_z \Delta x \Delta y
\end{aligned}$$

$$\begin{aligned}
&\textit{Exit Rates}\\
&x \text{ direction} = \tau_{xx}|_{x+\Delta x} \Delta y \Delta z\\
&y \text{ direction} = \tau_{yx}|_{y+\Delta y} \Delta x \Delta z\\
&z \text{ direction} = \tau_{zx}|_{z+\Delta z} \Delta x \Delta y
\end{aligned}$$

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Forces:

We will consider two important forces that usually act:

- fluid pressure
- gravity

If there are other forces acting on the volume element, we need to consider them as additive terms in each direction.

Resultant force in the x-direction:

$$\Delta y\, \Delta z \left(p\big|_x - p\big|_{x+\Delta x}\right) + \rho g_x \Delta x \Delta y \Delta z \qquad\qquad p = f(\rho, T)$$

Accumulation:

Accumulation of *x*-momentum within the volume element:

$$\Delta x \Delta y \Delta z \left(\frac{\partial \rho v_x}{\partial t}\right)$$

And if you recall the momentum balance, it was momentum out minus momentum in plus the sum of the forces equals 0 in the case of steady state, right. So we need the forces' term. So the forces, what are the forces that we are going to consider? We will usually consider 2 important forces that act, any other force is not going to be a part of this derivation, in other words this derivation considers only 2 forces.

If there are additional forces, you need to go back to the derivation states, add those forces at this stage and then re-derive the whole thing. They will just be additive terms ultimately, quite easy to figure out but for you to be sure you need to add those forces here. The forces that we will consider are fluid pressure and gravity, which will occur in many different situations.

If you have an electrical force, if you have a magnetic force, if you have a surface tension force and so on and so forth, you need to add it at this stage. So since we are considering only the x-momentum, let us look at the resultant force in the x-direction. It is the pressure times the area would be the pressure force, (pressure at x) – (the pressure at the face x + Δx) times the area, the area is the same Δy Δz.

Let us consider two important forces that usually act on the volume element, namely fluid pressure force and gravity. If there are other forces acting on the volume element, we need to consider them as additive terms in each direction.

The resultant force in the x direction is

$$\Delta y \Delta z (p|_x - p|_{x+\Delta x}) + \rho g_x \Delta x \Delta y \Delta z$$

where $p = f(\rho, T)$.

This gives us the pressure force and mass times the acceleration due to gravity gives us the gravitational force. So density is mass per unit volume multiplied by the volume of the element ΔxΔyΔz. Therefore, this and this put together gives you mass(ρ*ΔxΔyΔz), mass times g_x, g_x is the component of the acceleration due to gravity in the x-direction and also note that pressure is a function of density and temperature.

The accumulation of the x momentum within the volume element is

$$\Delta x \Delta y \Delta z \left(\frac{\partial \rho v_x}{\partial t} \right)$$

Okay let us still keep the condition, let us not put it to 0 as yet because we are trying to derive a general enough equation, therefore the accumulation on the right hand side of that equation in the x-direction within the volume element within the system is nothing but the density times velocity and the derivative taken with respect to time. So density times the velocity is momentum. So if density is mass per unit volume, therefore you need to multiply it by the volume to get mass and then mass times v_x will give you the momentum in the x-direction that is getting accumulated in the volume element $\frac{\partial}{\partial t}$ represents the time variation, so that is the accumulation rate.

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Let us recall the general momentum balance equation (Eq. 3.3. – 1)

$$\begin{pmatrix}Rate\ of\ momentum\\ out\ of\ the\ system\end{pmatrix} - \begin{pmatrix}Rate\ of\ momentum\\ into\ the\ system\end{pmatrix} + \begin{pmatrix}Rate\ of\ momentum\\ accumulation\ in\ the\ system\end{pmatrix} = \begin{pmatrix}Sum\ of\ forces\ acting\\ on\ the\ system\end{pmatrix}$$

Substitute the various terms for the x-direction, divide by $\Delta x \Delta y \Delta z$

And take the limit as $\Delta x, \Delta y, \Delta z \to 0$ to get

$$\frac{\partial(\rho v_x)}{\partial t} = -\left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x \qquad \text{Eq. 3.4. – 1}$$

Note: Eq. 3.4. – 1 is for the x-direction alone

If we do a similar exercise in the y and z directions, we would get

$$\frac{\partial(\rho v_y)}{\partial t} = -\left(\frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z}\right) - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) - \frac{\partial p}{\partial y} + \rho g_y \qquad \text{Eq. 3.4. – 2}$$

$$\frac{\partial(\rho v_z)}{\partial t} = -\left(\frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z}\right) - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) - \frac{\partial p}{\partial z} + \rho g_z \qquad \text{Eq. 3.4. – 3}$$

Now the general momentum balance equation is what, 3.3 – 1. Rate of momentum out of the system – rate of momentum into the system + rate of momentum accumulation in the system = the sum forces acting on the system. This we took together first for the convective component and then the molecular component, this we saw net sum of forces and we have a term for this.

We just go and plug those various terms into this and that is all there is we are doing a momentum balance. We just split those various contributions and they were available in an easily understandable form in terms of the variables that can be measured, they were written in terms of the variables that can be measured and now we are putting them back into the overall fundamental momentum balance equation, the second law of motion, Newton's second law of motion.

If you do that and you know that usual trick you divide by Δx Δy Δz, then take the limit as Δx tends to 0, Δy tends to 0, Δz tends to 0 okay. Why do not I ask you to pause the video here, this is the equation that we are interested in, we have already gotten this term separately, this term separately, and this term separately, why do not you go and substitute these terms divide throughout by Δx, Δy, Δz.

Then take the limit as Δx tends to 0, Δy tends to 0, Δz tends to 0 and see what you get okay, then of course we will compare and see whether you got whatever you are supposed to get okay. Pause, go ahead please. This is what you would have gotten for the x-momentum when you substituted it. So this is the momentum balance for the x-momentum alone. However, this is a 3-dimensional situation as we said, it has 3 different components, we can write the other components by this extension. You can go and check yourself, do the whole thing again for the other component and see whether you get this, but you can take it from me that it is just going to be an extension.

If we substitute the above terms for the x direction in the general momentum balance equation Eq. 3.3-1, divide by $\Delta x \Delta y \Delta z$, and take the limit as Δx, Δy, and $\Delta z \rightarrow 0$, we get

$$\frac{\partial(\rho v_x)}{\partial t} = -\left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x \quad (3.4\text{-}1)$$

A similar exercise in the y and z directions would give

$$\frac{\partial(\rho v_y)}{\partial t} = -\left(\frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z}\right) - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) - \frac{\partial p}{\partial y} + \rho g_y \quad (3.4\text{-}2)$$

$$\frac{\partial(\rho v_z)}{\partial t} = -\left(\frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z}\right) - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) - \frac{\partial p}{\partial z} + \rho g_z \quad (3.4\text{-}3)$$

So, all these 3 equations put together is the overall momentum balance equation of motion or you can call them equations of motion in these various directions, right. This is the equation of motion.

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In compact, vectorial notation

$\frac{\partial(\rho\vec{v})}{\partial t}$	=	$-[\vec{\nabla}.\rho\vec{v}\vec{v}]$	$-[\vec{\nabla}.\tilde{\tau}]$	$-\vec{\nabla}\, p$	$+\rho\,\vec{g}$
Rate of increase in momentum per unit volume		Rate of gain in momentum by convection per unit volume	Rate of gain in momentum by viscous effects per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

(3.4-4)

I think we need to take a break here, we have been at quite an intense derivation for quite a long period of time. Let us take a break here I think or before that let me say what these are? This is nothing but rate of increase in momentum per unit volume in the control volume in the system.

This is rate of gain in momentum by convection per unit volume. This is rate of gain and momentum by viscous effects per unit volume. This is pressure force on the element per unit volume and this is gravitational force on the element per unit volume okay. So, these are the various terms that come together in momentum balance in this relationship. Yeah, let us take a break here.

When we come back, let us continue with some more aspects of this equation of motion. There are lot of insights to derive from this equation of motion before we start applying this. There are a lot of things that have left hanging there, there were 9 terms, what kind of a quantity is that okay, those things we will talk about in some detail in the next class. See you then.