Transport Phenomena in Biological Systems

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Lecture – 26
Equation of Motion - Continued

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Let us recall the general momentum balance equation (Eq. 3.3. – 1)

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{out of the system}\end{pmatrix} - \begin{pmatrix}\text{Rate of momentum}\\ \text{into the system}\end{pmatrix} + \begin{pmatrix}\text{Rate of momentum}\\ \text{accumulation in the system}\end{pmatrix} = \begin{pmatrix}\text{Sum of forces acting}\\ \text{on the system}\end{pmatrix}$$

Substitute the various terms for the x-direction, divide by $\Delta x \Delta y \Delta z$

And take the limit as $\Delta x, \Delta y, \Delta z \to 0$ to get

$$\frac{\partial(\rho v_x)}{\partial t} = -\left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x \qquad \text{Eq. 3.4. – 1}$$

Note: Eq. 3.4. – 1 is for the x-direction alone

If we do a similar exercise in the y and z directions, we would get

$$\frac{\partial(\rho v_y)}{\partial t} = -\left(\frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z}\right) - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) - \frac{\partial p}{\partial y} + \rho g_y \qquad \text{Eq. 3.4. – 2}$$

$$\frac{\partial(\rho v_z)}{\partial t} = -\left(\frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z}\right) - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) - \frac{\partial p}{\partial z} + \rho g_z \qquad \text{Eq. 3.4. – 3}$$

Welcome back. In the previous class, we had derived the momentum balance equation,

In compact, vectorial notation

$\frac{\partial(\rho\vec{v})}{\partial t}$	$=$	$-[\vec{\nabla}.\rho\vec{v}\vec{v}]$	$-[\vec{\nabla}.\tilde{\tau}]$	$-\vec{\nabla} p$	$+\rho\vec{g}$
Rate of increase in momentum per unit volume		Rate of gain in momentum by convection per unit volume	Rate of gain in momentum by viscous effects per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

(3.4-4)

This is a short form way of representing or a vectorial way of representing these equations which we are more familiar with in terms of the components in various directions. This entire set is compactly written in that one equation that we saw just before okay. See here you have ($\rho\, v_x v_x$), ($\rho\, v_y v_x$), ($\rho\, v_z v_x$), ($\rho\, v_x v_y$), ($\rho\, v_y v_y$), ($\rho\, v_z v_y$), ($\rho\, v_x v_z$), ($\rho\, v_y v_z$), ($\rho\, v_z v_z$). Therefore, there are 9 terms here. Similarly there are τ_{xx}, τ_{yx}, τ_{zx} and so on 9 terms here okay.

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Vectorially,

$$\frac{\partial(\rho\vec{v})}{\partial t} = -[\vec{\nabla}.\rho\vec{v}\vec{v}] - [\vec{\nabla}.\vec{\tau}] - \vec{\nabla}p + \rho\vec{g} \qquad \text{Eq. 3.4.-4}$$

Rate of increase in momentum per unit volume	Rate of gain in momentum by convection per unit volume	Rate of gain in momentum by viscous effects per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

And that this has been represented as ∇.ρ vv, this is neither a dot product nor a cross product okay. This is a dot product yes, but vv what is this? This has been written this way to represent these 9 terms here, right. So remember this, I will talk more about it. Similarly this τ again represented these nine terms here okay, this is called a tensor. A tensor is represented as this, this as we had come to know is also a tensor. This is the second order tensor; this also is a second order tensor. A vector, a column vector typically that we use or a row vector it is a first order tensor okay, tensor is a more general term and these 2 were quite straightforward, ∇.p and ρg were quite straightforward. So let us talk a little bit more about these new vectorial quantities or tensorial quantities that you have been introduced to as a part of this derivation, let us begin with that.

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Let us look at equations 3.4-1, 3.4-2 and 3.4-3 again

We can recognize that $\vec{\tau}$ has 9 terms

τ is a second order tensor with 9 components that can be represented by

$$\vec{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

See Appendix 1 for tensor algebra details

$\vec{v}\vec{v}$ is a new concept

It is neither a dot product nor a cross product

Look at equations 3.4. – 1 to 3 (first terms on the RHS) to understand that $\vec{v}\vec{v}$ has 9 terms

$\vec{v}\vec{v}$ is known as the 'dyadic product' and is a special form of second order tensor

A dyadic product of 2 vectors $\vec{v}$ and $\vec{w}$ is

$$\vec{v}\vec{w} = \begin{pmatrix} v_x w_x & v_x w_y & v_x w_z \\ v_y w_x & v_y w_y & v_y w_z \\ v_z w_x & v_z w_y & v_z w_z \end{pmatrix}$$

See Appendix 1 for dyad algebra details

So as we saw the shear stress has 9 terms, the shear stress tensor τ squiggle hat has 9 terms. So τ is the second order tensor with 9 components that can be represented as τ tensor equals within in a matrix form we typically write the components τ_{xx} τ_{xy} τ_{xz} τ_{yx} τ_{yy} τ_{yz} τ_{zx} τ_{zy} τ_{zz} okay. Please see the appendix 1 of your textbook for a nice development of this.

I have told you many things here itself, if some things are unclear please check the appendix. As I mentioned earlier, vv is a new concept. There is no dot here, there is no cross here, right. It is not a dot product, it is not a cross product, but this vv has resulted in 9 terms okay. So in other words we have represented those 9 terms as v vector v vector written together okay, so this is a new quantity, it is called the dyadic product and it is a special form of the second order tensor.

For example, a dyadic product of 2 vectors v and w that is $v_{xi} + v_{yj} + v_{zk}$ and $w_{xi} + w_{yj} + w_{zk}$. If these are the vectors that are being considered, then vw the dyadic product is nothing but $v_x w_x$ $v_x w_y$ $v_x w_z$ $v_y w_x$ $v_y w_y$ $v_y w_z$ $v_z w_x$ $v_z w_y$ $v_z w_z$ okay. So this is the representation of this, this rows in our manipulations in our derivation and this is called a dyadic product, this is known as dyadic product.

There are a couple of terms in Eq. 3.4-4 that could be new. A review of Eqs. 3.4-1, 3.4-2 and 3.4-3 will reveal that $\tilde{\tau}$ has 9 terms. τ is a second order tensor with 9 components that can be represented by

$$\tilde{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

See Appendix 1 for more on tensor algebra.

Similarly, $\vec{v}\vec{v}$ is a new concept. Note that it is neither a dot product nor a cross product. A review of Eqs. 3.4-1 to 3.4-3 (first terms on the LHS) will reveal that $\vec{v}\vec{v}$ has 9 terms. $\vec{v}\vec{v}$ is known as the 'dyadic product' and is a special form of second order tensor. The dyadic product of two vectors $\vec{v}$ and $\vec{w}$ is

$$\vec{v}\vec{w} = \begin{pmatrix} v_x w_x & v_x w_y & v_x w_z \\ v_y w_x & v_y w_y & v_y w_z \\ v_z w_x & v_z w_y & v_z w_z \end{pmatrix}$$

Again, if you are unclear and for what algebra or what identities are valid for this and so on and so forth, please check the appendix okay. You already know that you need to treat vectors differently from tensors when they are written in a compact form. When you write the components, they are just scalars okay. The components are scalars, so they are just numbers that you can manipulate in the same way.

But when you compact them and write it up a vector form and you compact them and write it up a tensor form, there is a certain algebra that goes along with it which needs to be consistent with what you get when you expand those various components and do manipulations on those components. So that is the reason why a simple algebra that you do with numbers may or may not be valid for the vector algebra.

There is a separate algebra for vectors, and you need to pick that up. You would know, you would have done some vector algebra as a part of your initial math course. The appendix 1 gives a complete set of manipulations as needed by us that is it, it is not comprehensive, as needed by us whatever vector algebra is needed, tensor algebra is needed is given as actually developed in that appendix, please go ahead and see that.

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Let us recall the general momentum balance equation (Eq. 3.3. – 1)

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{out of the system}\end{pmatrix} - \begin{pmatrix}\text{Rate of momentum}\\ \text{into the system}\end{pmatrix} + \begin{pmatrix}\text{Rate of momentum}\\ \text{accumulation in the system}\end{pmatrix} = \begin{pmatrix}\text{Sum of forces acting}\\ \text{on the system}\end{pmatrix}$$

Substitute the various terms for the x-direction, divide by $\Delta x \Delta y \Delta z$

And take the limit as $\Delta x, \Delta y, \Delta z \to 0$ to get

$$\frac{\partial(\rho v_x)}{\partial t} = -\left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x \qquad \text{Eq. 3.4. – 1}$$

Let us rewrite this equation, what is this equation? If you see here, this was equation 3.4. – 1 okay. So this is the momentum balance for the x momentum or momentum in the x direction, the momentum rate in the x direction. So here we have as I mentioned earlier a product of 3 functions.

The product of which we need to take one at a time to be able to directly use it or it might be much easier to directly use it, so let us simplify that and as you will see it gets simplified significantly.

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let us write Eq. 3.4. – 1 as

$$\frac{\partial(\rho v_x)}{\partial t}+\left(\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}\right)=-\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

The LHS can be expanded as

$$\rho\frac{\partial v_x}{\partial t}+v_x\frac{\partial\rho}{\partial t}+\left(\rho v_x\frac{\partial v_x}{\partial x}+v_x\frac{\partial\rho v_x}{\partial x}+\rho v_y\frac{\partial v_x}{\partial y}+v_x\frac{\partial\rho v_y}{\partial y}+\rho v_z\frac{\partial v_x}{\partial z}+v_x\frac{\partial\rho v_z}{\partial z}\right)$$

$$=\rho\frac{\partial v_x}{\partial t}+v_x\frac{\partial\rho}{\partial t}+v_x\left(\frac{\partial\rho v_x}{\partial x}+\frac{\partial\rho v_y}{\partial y}+\frac{\partial\rho v_z}{\partial z}\right)+\left(\rho v_x\frac{\partial v_x}{\partial x}+\rho v_y\frac{\partial v_x}{\partial y}+\rho v_z\frac{\partial v_x}{\partial z}\right)$$

$$=\rho\frac{\partial v_x}{\partial t}+v_x\frac{\partial\rho}{\partial t}+v_x\left(\rho\frac{\partial v_x}{\partial x}+v_x\frac{\partial\rho}{\partial x}+\rho\frac{\partial v_y}{\partial y}+v_y\frac{\partial\rho}{\partial y}+\rho\frac{\partial v_z}{\partial z}+v_z\frac{\partial\rho}{\partial z}\right)+\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)$$

$$=\left\{v_x\frac{\partial\rho}{\partial t}+\rho v_x\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)+v_x\left(v_x\frac{\partial\rho}{\partial x}+v_y\frac{\partial\rho}{\partial y}+v_z\frac{\partial\rho}{\partial z}\right)\right\}+\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)$$

$$=\{E\}+\rho\frac{Dv_x}{Dt}\qquad\text{where}\qquad E=v_x\left(\frac{\partial\rho}{\partial t}+v_x\frac{\partial\rho}{\partial x}+v_y\frac{\partial\rho}{\partial y}+v_z\frac{\partial\rho}{\partial z}\right)+\rho v_x\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_x}{\partial y}+\frac{\partial v_x}{\partial z}\right)$$

Now, let us consider Eq. 3.4-1 written as

$$\frac{\partial(\rho v_x)}{\partial t}+\left(\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}\right)=-\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

We are going to expand this term essentially as well as this term, this is a product of two functions, this is the product of three functions. All of it is can vary, none of it is a constant, we have not assumed any constancy of density or anything like that to make it generally applicable because the gas the density can vary, right

The LHS can be expanded as

$$\rho\frac{\partial v_x}{\partial t}+v_x\frac{\partial\rho}{\partial t}+\left(\rho v_x\frac{\partial v_x}{\partial x}+v_x\frac{\partial\rho v_x}{\partial x}+\rho v_y\frac{\partial v_x}{\partial y}+v_x\frac{\partial\rho v_y}{\partial y}+\rho v_z\frac{\partial v_x}{\partial z}+v_x\frac{\partial\rho v_z}{\partial z}\right)$$

$$=\rho\frac{\partial v_x}{\partial t}+v_x\frac{\partial\rho}{\partial t}+v_x\left(\frac{\partial\rho v_x}{\partial x}+\frac{\partial\rho v_y}{\partial y}+\frac{\partial\rho v_z}{\partial z}\right)+\left(\rho v_x\frac{\partial v_x}{\partial x}+\rho v_y\frac{\partial v_y}{\partial y}+\rho v_z\frac{\partial v_x}{\partial z}\right)$$

$$=\rho\frac{\partial v_x}{\partial t}+v_x\frac{\partial\rho}{\partial t}+v_x\left(\rho\frac{\partial v_x}{\partial x}+v_x\frac{\partial\rho}{\partial x}+\rho\frac{\partial v_y}{\partial y}+v_y\frac{\partial\rho}{\partial y}+\rho\frac{\partial v_z}{\partial z}+v_z\frac{\partial\rho}{\partial z}\right)$$

$$+\rho\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)$$

Now, let us combine this in a slightly different fashion to effect some simplifications, a lot of simplification happens. I am just keeping your $v_x \frac{\partial \rho}{\partial t}$ here and combining $\rho \frac{\partial v_x}{\partial t}$ along with this term okay. You see why I am doing this because $\frac{\partial v_x}{\partial t}$ this is nothing but the substantial derivative okay, density is outside here. So, let me combine this with this to write this as a substantial derivative.

$$= \left\{ v_x \frac{\partial \rho}{\partial t} + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + v_x \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) \right\}$$

$$+ \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)$$

$$= \{E\} + \rho \frac{D v_x}{Dt}$$

where

$$E = v_x \left(\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

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Using the equation of continuity $\quad \frac{D\rho}{Dt} = -\rho\,(\vec{\nabla}.\vec{v})$

the first term on the RHS of the previous equation can be written as the negative of the second term on the RHS. Thus,

$$E = v_x \left[-\rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \right] + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Thus, Eq. 3.4. – 1 can be written as

$$\rho \frac{D v_x}{Dt} = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

The other two components (y and z) of momentum rate are expressed as above and added together, to get

$$\rho \frac{D\vec{v}}{Dt} = -[\vec{\nabla} \cdot \vec{\tau}] - \vec{\nabla} p + \rho \vec{g} \qquad \text{Eq. 3.4.} - 5$$

$\frac{mass}{volume} \times accelaration$ | Viscous forces on the element per unit volume | Pressure force on the element per unit volume | Gravitational force on the element per unit volume

Now, when we use the equation of continuity, the complete form. So the first term on, I think that is the we are looking at the left hand side, the left hand side of the equation when we use equal to whatever left hand side right hand side that is only side that we are looking at can be written as a negative of the second term on the right hand side.

$$E = v_x\left[-\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\right]+\rho v_x\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)=0$$

Thus, Eq. 3.4-1 can be written as

$$\rho\frac{Dv_x}{Dt} = -\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

And you have the right hand side of the original balance equation on the x direction momentum balance and you could express the other components just by extension. If you do not believe me, you can go back and re-derive the whole thing for the y-direction alone, for the z direction alone you will find that you will end up as extensions of this you will have to write the appropriate indices and you will be fine.

The other two components (y and z) of momentum rate can be similarly expressed and added together, to get a 3-D representation

$\rho\frac{D\vec{v}}{Dt}$	$=$	$-[\vec{\nabla}.\tilde{\tau}]$	$-\vec{\nabla}p$	$+\rho\vec{g}$
$\frac{\text{Mass}}{\text{Volume}}\times\text{Acceleration}$		Viscous forces on the element per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

(3.4-5)

Now if you use the equation of continuity, we can write the substantial derivative of

$\frac{D\rho}{Dt} = -\rho\ \nabla.\mathbf{v}$ that is what the equation of continuity tells us, general equation of continuity with no assumptions here. So the first term on the right hand side can be written as the negative of the second term on the right hand side okay.

So this is what our thing in the flower brackets was and this can be written as, this is the only term that will remain here okay because this can be written as the negative of the second term, the first term on the right hand side can be written as the negative of the second term on the right hand side and therefore E becomes 0 okay. The E, first term is nothing but the negative of the second term using the equation of continuity. I would like you to work this out and convince yourself that this is indeed 0 and therefore your equation of motion in the x direction becomes, I would like you to see tables 3.4 – 1 to 3, have I shown you that already? No I have not shown you that, let me show you that.

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Using the equation of continuity $\frac{D\rho}{Dt} = -\rho\,(\vec{\nabla}.\vec{v})$

the first term on the RHS of the previous equation can be written as the negative of the second term on the RHS. Thus,

$$E = v_x\left[-\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\right]+\rho v_x\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right) = 0$$

Thus, Eq. 3.4. – 1 can be written as

$$\rho\frac{Dv_x}{Dt} = -\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

The other two components (y and z) of momentum rate are expressed as above and added together, to get

$$\rho\frac{D\vec{v}}{Dt} = -[\vec{\nabla}\,.\,\tilde{\tau}] - \vec{\nabla}p + \rho\vec{g} \quad \text{Eq. 3.4. – 5}$$

$\rho\frac{D\vec{v}}{Dt}$	$-[\vec{\nabla}\,.\,\tilde{\tau}]$	$\vec{\nabla}p$	$\rho\vec{g}$
$\frac{mass}{volume}\times accelaration$	Viscous forces on the element per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

See Tables 3.4. – 1 to 3

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TABLE 3.4. – 1 The equations of motion in rectangular Cartesian coordinates

X direction:

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x}-\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho g_x \quad \textbf{(A1)}$$

For a Newtonian fluid with constant ρ and μ:

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x \quad \textbf{(A2)}$$

Y direction:

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y}-\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho g_y \quad \textbf{(B1)}$$

For a Newtonian fluid with constant ρ and μ:

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y \quad \textbf{(B2)}$$

So see here the equation of motion written and Cartesian coordinates, let us focus on A1, B1 and C1 for the time being okay. So this is the expanded form of the vectorial rotation that we wrote. We have written it like this so that we can use it directly. We have v_x v_y v_z we can use this one directly okay. So note that A1, B1 and C1 gave us those 3 terms that we derived okay, so that will come back to this in a little bit.

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If the interest is in finding velocity distributions, we need to substitute the stresses in terms of velocity gradients and fluid properties.

We need to realize that the simple relationship between shear stress and a single shear rate in the 2-D form of the Newton's law of viscosity,

$$\tau_{yx} = \mu\left(-\frac{dv_x}{dy}\right)$$

was for an initial understanding

In 3-D, multiple velocity gradients would determine a shear stress

The equations given in Table 3.4. – 4 to 6 are the components of the stress tensor for a Newtonian fluid in laminar flow in the three coordinate systems are needed for a complete representation of the dependences of shear stress on various shear rates.

Now, let us move forward. Let us do a few more things here, let us try to understand a few more things here. Now using the equation of continuity which is I think this is what I showed you earlier, yeah this is what we did earlier okay. We have this equation in terms of, of course there is a velocity here, there is a shear stress, there is a pressure and there is gravity. Shear stresses are not easy to measure okay.

So if you have a handle on shear stresses, then you could use this equation, very general equation valid all the time okay. If you have only the velocities, then we need probably a different form to be directly useful okay. So if our interest is in finding the velocity distribution, we need to substitute the stresses in terms of the velocity gradients and fluid properties. You already know how to do that.

We have the Newton's law of viscosity that gives us the relationship between shear stress and shear rate. We looked at it in one dimension, we can extend it to 3 dimensions. So some relationship between shear stress and shear rate is needed to convert the previous equation which is in terms of the shear stress to an equation in terms of velocities okay. Of course that would that would limit the applicability of the equation.

Once you use Newton's law of viscosity, the equation that you are going to derive will be applicable only for Newtonian fluids, right, that is that goes without saying, whereas the first equation is applicable for any kind of fluid okay, just remember that, but neutral fluids are so widely used that it is very useful to have an equation for Newtonian fluids alone. So let us do that.

This is the Newton's law of viscosity $\tau_{yx} = \mu(-\frac{dv_x}{dy})$ and this again was for an initial understanding, this was only in 2 dimensions okay. As we can see there are contributions of one to the other and so on and so forth. Therefore the velocity gradients in probably two different velocity gradients could contribute to a particular shear stress and so on and so forth this could happen and it is non-trivial.

There was a paper that was published a while ago that has actually worked this out for Newtonian fluids and they have come up with certain relationships between the shear stresses and shear rates. So this is only for an initial understanding, in 3 dimensions the multiple velocity gradients would determine a shear stress as I just mentioned and equations given in table 3.4 – 4 to 6 are the components of the stress tensor for a Newtonian fluid in laminar flow in the three coordinate systems.

Substituting the expressions from Table 3.4-4 in the momentum balances for the three directions, we get

$$\rho\frac{Dv_x}{Dt} = \frac{\partial}{\partial x}\left[2\mu\frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] - \frac{\partial p}{\partial x} + \rho g_x \qquad (3.4\text{-}6)$$

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TABLE 3.4. – 4 Components of the stress tensor for Newtonian fluids in rectangular Cartesian coordinates

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \qquad (A)$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \qquad (B)$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \qquad (C)$$

$$\tau_{xx} = -2\mu\frac{\partial v_x}{\partial x} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad (D)$$

$$\tau_{yy} = -2\mu\frac{\partial v_y}{\partial y} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad (E)$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad (F)$$

Let us look at one coordinate system to understand that, I will show you that. Okay here, here we go. See here let us just look at rectangular Cartesian coordinate system to begin with. You can see that the shear stress τ_{yx} okay, this happens to be equal τ_{xy}, but this is actually a sum of 2 different velocity gradients in 3 dimensions. Similarly, it happens for the other shear stress the third shear stress and the normal stresses are expressed in this complex form.

We are not going to get into why it happens this way, this is non-trivial therefore it is not a part of an initial course. You can go to that paper and check how this is derived and so on and so forth, it is non-trivial. Therefore, let us take it on face value for our purposes that τ_{xx} is given in terms of these velocity gradients, τ_{yy} is given in terms of these velocity gradients and τ_{zz} is given in terms of these velocity gradients. So, this is the relationship between shear stress and shear rates or the velocity gradients in a 3-dimensional case.

Table 3.4-4 Components of the stress tensor for Newtonian fluids in rectangular Cartesian coordinates

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \quad \text{(A)}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \quad \text{(B)}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \quad \text{(C)}$$

$$\tau_{xx} = -2\mu\frac{\partial v_x}{\partial x} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \quad \text{(D)}$$

$$\tau_{yy} = -2\mu\frac{\partial v_y}{\partial y} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \quad \text{(E)}$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \quad \text{(F)}$$

(Refer Slide Time: 26:35)

$$\rho \frac{Dv_x}{Dt} = \frac{\partial}{\partial x}\left[2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] - \frac{\partial p}{\partial x} + \rho g_x \qquad \text{Eq. 3.4.} - 6$$

$$\rho \frac{Dv_y}{Dt} = \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] - \frac{\partial p}{\partial y} + \rho g_y \qquad \text{Eq. 3.4.} - 7$$

$$\rho \frac{Dv_z}{Dt} = \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right] - \frac{\partial p}{\partial z} + \rho g_z \qquad \text{Eq. 3.4.} - 8$$

The above equations of motion Eq. 3.4. – 6 to 8,
equation of state, $p = f(\rho)$, and
variation of $\mu = f(\rho)$

completely determine the pressure, density and velocity components in a Newtonian fluid in laminar flow.

If we do this if we substitute these velocity gradients and write our equations of motion, then we would get for the x momentum alone this relationship when you convert the shear stresses in terms of the velocity gradients okay.

$$\rho \frac{Dv_y}{Dt} = \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] - \frac{\partial p}{\partial y} + \rho g_y \qquad (3.4\text{-}7)$$

$$\rho \frac{Dv_z}{Dt} = \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu(\vec{\nabla}.\vec{v})\right] - \frac{\partial p}{\partial z} + \rho g_z \qquad (3.4\text{-}8)$$

The equations of motion (Eqs. 3.4-6 to 3.4-8), equation of state, $p = f(\rho)$, and variation of $\mu = f(\rho)$ completely determine the pressure, density and velocity components in the flowing Newtonian fluid.

So, these would be the equations 3.4. – 6, 3.4. – 7, 3.4. – 8 for a Newtonian fluid because when we looked at the relationship between shear stress and shear rates that limited to a Newtonian fluid in this case, you could have different relationships for different fluids, so these equations would be valid only for a Newtonian fluid. So, these above equations of motion, these are the equations of motion.

And the equations of state are the relationship between pv and t or p as a function of a ρ and the variation of viscosity with density completely determine the pressure, density, and velocity components of a Newtonian fluid in laminar flow okay. Just remember this, this is for some sort of a big picture, if you understand that it was fine, otherwise just keep it in your mind, at some point in time this will become clear to you okay, much after you get into the field and so on so, just remembers this.

(Refer Slide Time: 28:27)

When ρ and μ are constant, since $\vec{\nabla}.\vec{v} = 0$ according to the continuity equation, the equation of motion can be written as

$$\rho \frac{D\vec{v}}{Dt} = \mu \nabla^2 \vec{v} - \vec{\nabla} p + \rho \vec{g} \qquad \text{Eq. 3.4.} - 9$$

This is the famous Navier - Stokes equation

If viscous effects are also not important, $\vec{\nabla}.\vec{\tau} = 0$. Then, Eq. 3.4. – 5 becomes

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \rho \vec{g} \qquad \text{Eq. 3.4.} - 10$$

This is called the Euler equation

When ρ and μ are constant, since $\vec{\nabla}.\vec{v} = 0$ according to the continuity equation, the equation of motion can be written as

$$\rho \frac{D\vec{v}}{Dt} = \mu \vec{\nabla}^2 \vec{v} - \vec{\nabla} p + \rho \vec{g} \qquad (3.4\text{-}9)$$

Equation 3.4-9 is called the Navier-Stokes equation.

This is a nice equation, this is for a Newtonian fluid of course, not as general as the previous equation but that is okay it is very useful. This is equation 3.4. – 9. This is the famous Navier-Stokes equation. For a Newtonian fluid in laminar flow and so on and so forth that is, I mean that is actually a limited equation, a very famous equation, aerospace people use it significantly and so on and so forth, that equation is only a special form of the more general momentum balance equation.

Now if the viscous effects are not important, then the ∇.τ term can be put to **zero** 0 because the shear stress is not going to come in at all okay. This is an idealized situation not a real situation, idealized situation which becomes useful in many analyses and in such a case your equation of motion becomes

If viscous effects are not important, $\vec{\nabla} \cdot \tilde{\tau} = 0$. Then, Eq. 3.4-5 becomes

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \rho \vec{g} \qquad (3.4\text{-}10)$$

Equation 3.4-10 is called the Euler equation.

So, these are 2 special equations that come out of the equation of motion for a Newtonian fluid. Okay, I think before I take a break, let me show you the equations and also the complete set of equations in this table okay. **(Video Starts: 30:46)** This will be available to you for download. So please make a copy of the entire set of tables here and keep them as a part of your notes. You have 3 equations of motion tables, 3.4. –1, I think 3.4. – 2 and 3.4. – 3.

Table 3.4-1 The equations of motion in rectangular Cartesian coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x \qquad \text{(A1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x \qquad \text{(A2)}$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y \qquad \text{(B1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y \qquad \text{(B2)}$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \qquad \text{(C1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z \qquad \text{(C2)}$$

Table 3.4-2 The equations of motion in cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial r}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r \qquad \text{(A1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r \qquad \text{(A2)}$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=$$
$$-\frac{1}{r}\frac{\partial p}{\partial \theta}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right)+\rho g_\theta \qquad \text{(B1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=$$
$$-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta \qquad \text{(B2)}$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial z}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z \qquad \text{(C1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$
$$-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z \qquad \text{(C2)}$$

Table 3.4-3 The equations of motion in spherical coordinates*

r direction

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=-\frac{\partial p}{\partial r}$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})+\frac{1}{r\sin\theta}\frac{\partial(\tau_{r\theta}\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi}-\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)+\rho g_r \qquad \text{(A1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)=-\frac{\partial p}{\partial r}$$

$$+\mu\left(\nabla^2 v_r-\frac{2}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2}{r^2}v_\theta\cot\theta-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)+\rho g_r \qquad \text{(A2)}$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial(\tau_{\theta\theta}\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi}+\frac{\tau_{r\theta}}{r}-\frac{\cot\theta}{r}\tau_{\phi\phi}\right)+\rho g_\theta \qquad \text{(B1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$+\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right)+\rho g_\theta \qquad \text{(B2)}$$

ϕ direction

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi})+\frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi}+\frac{\tau_{r\phi}}{r}+\frac{2\cot\theta}{r}\tau_{\theta\phi}\right)+\rho g_\phi \qquad \text{(C1)}$$

For a Newtonian fluid with constant ρ and μ

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

$$+\mu\left(\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right)+\rho g_\phi \qquad \text{(C2)}$$

*Note that $\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$

The difference between these tables is the coordinate system, the first one is for rectangular Cartesian coordinates, the second one is for cylindrical coordinates, the third one is for spherical coordinates. The equation of motion and these three coordinates in terms of the individual components okay. For example if you have cylindrical coordinates, you will have r theta z okay, r theta and z, and if you have spherical coordinates you will have r theta and phi okay.

So, these equations are written in terms of those. The conversion has taken place already using the principles that are given in the appendix and also as we saw earlier the 3.4 – 4 is the components of the stress tensor for Newtonian fluids in Cartesian coordinates first and for cylindrical coordinates next and for spherical coordinates. This you would not need too much.

However, you will need the first 3 tables 1, 2, and 3 of 3.4 **(Video Ends: 32:14)** and therefore please make a copy of all the 6 tables and keep it as a part of your easy access folder okay, keep it in your easy access folder, hard copy, soft copy whatever it is. So, let us finish up. Today, we completed the discussion on the equation of motion derivation, the understanding of the various terms and so on and so forth how they came about.

And we did that because the equation of motion is much easier to apply in many different situations compared to shell balances for any situation, momentum rate in this case. In the next class, I am going to show you the application of the equation of motion to various situations, we will start in a few classes from now on over the next few classes I am going to show you the various application okay. See you then.

Table 3.4-5 Components of the stress tensor for Newtonian fluids in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \quad \text{(A)}$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \quad \text{(B)}$$

$$\tau_{zr} = \tau_{rz} = -\mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \quad \text{(C)}$$

$$\tau_{rr} = -2\mu\frac{\partial v_r}{\partial r} + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(D)}$$

$$\tau_{\theta\theta} = -2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(E)}$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right) \quad \text{(F)}$$

Components for the stress tensor for Newtonian fluids in spherical coordinates(3.4-6):

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \quad \text{(A)}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \quad \text{(B)}$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right) \quad \text{(C)}$$

$$\tau_{rr} = -2\mu\frac{\partial v_r}{\partial r} + \frac{2}{3}\mu\left(\frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) \quad \text{(D)}$$

$$\tau_{\theta\theta} = -2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \frac{2}{3}\mu\left(\frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) \quad \text{(E)}$$

$$\tau_{\phi\phi} = -2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) + \frac{2}{3}\mu\left(\frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) \quad \text{(F)}$$