Transport Phenomena in Biological Systems
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Lecture – 28
Laminar Flow Through a Pipe

Welcome to the next lecture. We are looking at momentum fluxes. We derived the momentum balance equation, the equation of motion and I showed you how applying the equation of motion in some situations, in fact in many situations can significantly simplify our analysis. The shell balances are robust, however, they will take a lot of effort, they are cumbersome, especially in cylindrical and spherical coordinates.

Whereas the equation of motion we have already put in the effort we have derived those equations, we can directly take those equations and apply them as long as we are clear as to their limitations of application. Today, I am going to talk to you about the analysis of flow through a cylindrical pipe. Flow through a cylindrical pipe is so widely used okay. You just look around yourself, look around within yourself.

There are so many cases of flow through cylindrical pipes. All your vasculature are cylindrical, right. So any flow you know in the body is flow through a cylindrical pipe.

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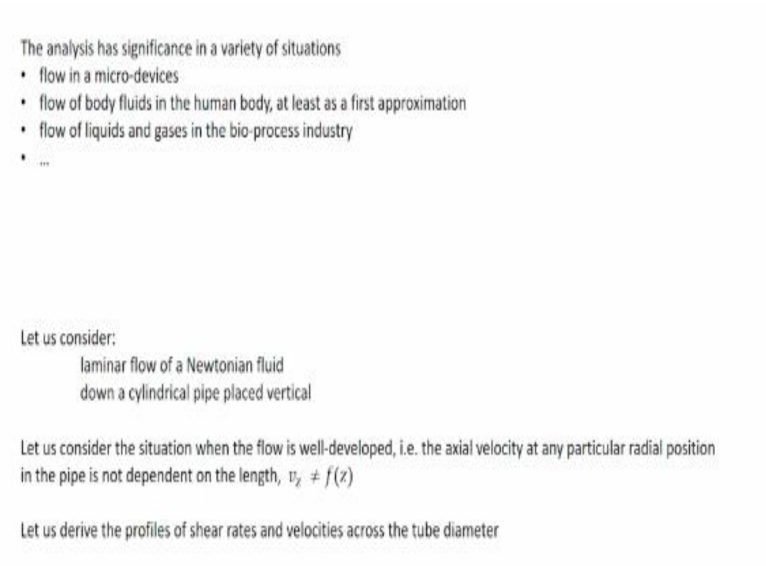

Let me tell you some, so it has significance in a wide variety of situations. The flow in micro devices if you are interested in fabricating micro devices lab-on-a-chip and so on and so forth, various different micro devices they all involve flows which are usually through cylindrical. It is very tiny cylindrical pipes of tiny diameter. Flow of fluids in the human body, at least as a first approximation is that.

Flow of liquids and gases in the bio process industry or all flow through cylindrical pipes, almost all through cylindrical pipes and so on and so forth, so it has by entry levels. So pay a little bit of attention what are we deriving here can be applied in a wide variety of situations. Let us consider laminar flow of a Newtonian fluid okay that is the first assumption. We are looking at laminar flow of a Newtonian fluid.

Down a cylindrical pipe placed vertical okay that is the situation and we are going to consider the situation when the flow is well-developed okay. Suppose you have a pipe of a certain length. At the entrance, there could be some entrance effects and at the exit there could be exit effects okay or the end effects would be there at the two ends. For a majority of the pipe, the flow would be well-developed.

There are estimates by which you can get the length to which the flow needs to travel before it gets to well-developed flow conditions as it is called. You already seen an example of well-developed flow when we said that the velocity of a thin layer of fluid flowing over an inclined plate does not change with the distance okay, which means the flow is well-developed there. There are various situations where the flow is well-developed.

Here, the flow is well-developed for most of the pipe. So the analysis is relevant for the pretty much the entire pipe except for the entrance and exit regions and the entrances and exits could be few with huge lengths in between, so all there it is relevant. This is the definition of well-developed flow. The axial velocity at any particular radial position in the pipe is not dependent on the length.

The axial velocity at any radial position okay, it is a cylindrical pipe, so we are looking at radial positions of various diameters, various distances from the center, there the velocity at a particular point is the same along the length. The velocity at the center would be different from the velocity near the wall that is perfectly fine, but the velocity near the wall would be the same irrespective of the distance that is traveled in the pipe that is what we call as the well-developed flow okay.

In other words, across a cross-section, there would be variation, but across the length there will not be a variation, good. We are going to derive the profiles of shear rates and velocities across the tube diameter that is our main interest here to get insights and this is useful hugely in analysis, design, operation, and so on.

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The system of interest is cylindrical
Therefore, it is best to use cylindrical coordinates here
Table 3.4. – 2 is relevant

Let us first consider Eq. A2 in Table 3.4. – 2

$(v_r = 0)$ $(v_\theta = 0)$ $(v_\theta = 0)$ $(v_r \neq f(z))$ $(v_r = 0)$

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$

$(v_r = 0)$ $v_r \neq f(\theta)$, $v_r = 0$ $(v_\theta = 0)$ $(v_r \neq f(z))$, $(v_r = 0)$ $(g_r = 0)$

$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \quad \text{Eq. 3.4.2. – 1}$$

$$\frac{\partial p}{\partial r} = 0 \qquad p \neq f(r) \quad \text{Eq. 3.4.2. – 2}$$

The pressure across the cross-section at a particular length in laminar flow through a pipe does not depend on the radial position
This is an important insight

The system of interest is cylindrical and therefore it is best to use a cylindrical coordinate system here and therefore check back, go to your table 3.4 -2 you made a copy of it, take a look at the copy. We are going to look at equation A2 because it is the case of a Newtonian fluid in laminar flow, therefore we can directly use it okay. ρ and the viscosity are constants.

Now, we are going to look at the terms that are relevant. We are looking at steady state case, therefore no time derivatives, the first term goes to 0. There is no velocity in the radial direction, there is velocity only in the axial direction, therefore v_r is 0 that goes to 0, v_θ which is the velocity in the θ direction, there is none, there is no circulation there, therefore v_θ is 0. This term also disappears because v_θ is 0.

Now v_z is there, v_z is certainly there moving down, this is a vertical pipe moving down v_z. The z direction is the vertical axis. However v_r, v_r is anyway 0, we do not have to worry about it. There is no velocity in the radial direction okay. Let me recalibrate here. This is a vertical pipe. This is the axial direction, this is the radial direction. There is v_z, there is no v_r, there is no v_θ okay, good.

So this term goes to 0, $\frac{dp}{dr}$ we do not know much, let us keep that, v_r is 0 therefore this term goes to 0, v_r is 0 so this term goes to 0, v_θ is 0 and so this term goes to 0, v_r is 0 zero this term disappears and gravity acts only in this direction. The r direction is perpendicular to the g_z direction, there is no component of g and therefore g_r is 0. So what do we have here? What is left here? It was nothing else except for this term. So we get this 3.4.2 – 1.

Let us first use Eq. A2 from Table 3.4-2

$$\rho\left(\overset{(SS)\;0}{\cancel{\frac{\partial v_r}{\partial t}}} + \overset{(v_r=0)\;0}{\cancel{v_r\frac{\partial v_r}{\partial r}}} + \overset{(v_\theta=0)\;0}{\cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}} - \overset{(v_\theta=0)\;0}{\cancel{\frac{v_\theta^2}{r}}} + \overset{(v_r \neq f(z))\;(v_r=0)\;0}{\cancel{v_z\frac{\partial v_r}{\partial z}}}\right)$$

$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\overset{(v_r=0)\;0}{\cancel{(r v_r)}}\right) + \overset{v_r \neq f(\theta)\;(v_r=0)\;0}{\cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}} + \overset{(v_\theta=0)\;0}{\cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}} + \overset{(v_r \neq f(z))\;(v_r=0)\;0}{\cancel{\frac{\partial^2 v_r}{\partial z^2}}}\right] + \overset{(g_r=0)\;0}{\cancel{\rho g_r}} \qquad (3.4.2\text{-}1)$$

We get $\frac{\partial p}{\partial r}$= 0 after we cancel all the irrelevant terms okay. What does this mean? Okay this means something very powerful or very insightful. There is no variation in pressure across the radius. This is a vertical pipe right, there is no variation pressure across the radius. The pressure across the cross-section is the same okay. That is something good to know, there is we do not have to worry about pressure varying across the cross-section in cylindrical flow.

The equation reduces to

$$\frac{\partial p}{\partial r} = 0 \text{ or } p \neq f(r) \qquad (3.4.2\text{-}2)$$

This is an important insight, i.e. the pressure across the cross-section of a pipe at a particular length in laminar flow through a pipe does not depend on the radial position.

So p is not a function of r that is what we got. So we will call this equation 3.4.2 – 2. The pressure across the cross-section at a particular length in laminar flow through a pipe does not depend on the radial position okay. Note that we are saying that the pressure across the radius is the same, however, the pressure across this radius could be very different from the pressure across this radius okay. There could be a variation in pressure with the axial distance.

There is no variation in pressure with the radial distance. You need to make these fine distinctions here okay, I mean to have that picture very clearly in mind that is an essential aspect of this course, some more concentration, some better understanding of the physics is absolutely required here. This happens to be a very important insight. We can now just close our eyes and say across the cross-section the pressure does not vary okay, that is very useful.

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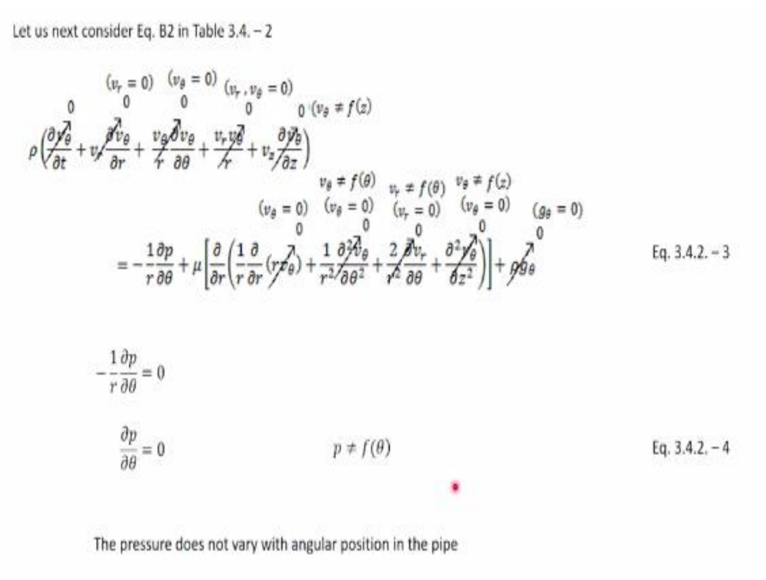

Next let us consider equation B2 in table 3.4 - 2. B 2 is this same equation, yes the cylindrical coordinate system 3.4 - 2 B2, yeah B2 is in terms of the other velocity, this was B1right, this was we considered A2. A2, B2 and C2 okay. Equation A2 we considered first and now we are going to consider B2. This A2 was in terms of v_r and so on so forth, B2 is in terms of v_θ. This is the equation, please verify.

Let us next consider Eq. B2 from Table 3.4-2

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{\substack{0\\(SS)}} + \underbrace{v_r \frac{\partial v_\theta}{\partial r}}_{\substack{0\\(v_r=0)}} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{\substack{0\\(v_\theta=0)}} + \underbrace{\frac{v_r v_\theta}{r}}_{\substack{0\\(v_r,\, v_\theta=0)}} + \underbrace{v_z \frac{\partial v_\theta}{\partial z}}_{\substack{0\\(v_\theta \neq f(z))}}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\underbrace{(r v_\theta)}_{\substack{0\\(v_\theta=0)}}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{\substack{0\\v_\theta \neq f(\theta)\\(v_\theta=0)}} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{\substack{0\\v_r \neq f(\theta)\\(v_r=0)}} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{\substack{0\\v_\theta \neq f(z)\\(v_\theta=0)}}\right] + \underbrace{\rho g_\theta}_{\substack{0\\(g_\theta=0)}} \quad (3.4.2\text{-}3)$$

Now we start cancelling the terms, steady state term. As a steady state case, therefore there is no variation with time, this goes to 0, v_r is 0 so this goes to 0, v_θ is 0 so this goes to 0. Same reason, this term disappears, this term disappears, this term disappears, this gone, this gone, this gone and g_t term, there is no g in the θ direction okay. The direction is perpendicular to the vertical direction where it is relevant and therefore this goes to 0.

So the only term that remains is this $-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$. What does this mean that there is no variation in pressure across θ, both put together as there is no variation in pressure across the cross-section, this and the radius θ and the radius put together gives us there is no variation pressure across the cross-section, $\frac{dp}{d\theta} = 0$, p is not a function of θ okay.

So the pressure does not vary with the radial position, the pressure does not vary with the angular position, and therefore the pressure does not vary across a cross section okay.

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Let us consider Eq. C2 in Table 3.4. – 2

$$\rho\left(\overset{0}{\frac{\partial v_z}{\partial t}} + \overset{(v_r=0)\;0}{v_r\frac{\partial v_z}{\partial r}} + \overset{(v_\theta=0)\;0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \overset{(v_z\neq f(z))\;0}{v_z\frac{\partial v_z}{\partial z}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \overset{v_z\neq f(\theta)\;0}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \overset{(v_z\neq f(z))\;0}{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \qquad \text{Eq. 3.4.2. – 5}$$

$$-\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{\partial p}{\partial z} + \rho g_z \qquad \text{Eq. 3.4.2. – 6}$$

Let us define $P = p - \rho g z$

Thus $\frac{\partial p}{\partial z} - \rho g = \frac{\partial (p - \rho g z)}{\partial z} = \frac{\partial P}{\partial z}$

Now let us consider equation C2 okay. We are directly taking these balances and seeing what insights they are giving us. We take equation C2, apply it to here. This is the equation C2 in terms of v_z okay. This is steady state situation, therefore there is no variation with time, the time derivatives are 0, v_r is 0 therefore this term disappears, v_θ is 0 this term disappears. The v_z of course exists, however v_z is not a function of z, well-developed flow okay.

The equation reduces to

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$$

Thus

$$\frac{\partial p}{\partial \theta} = 0$$

or

$$p \neq f(\theta) \qquad (3.4.2\text{-}4)$$

Thus, the pressure does not depend on the angular position in the pipe.

Now, let us consider Eq. C2 from Table 3.4-2

$$\rho\left(\underset{(SS)\ 0}{\frac{\partial v_z}{\partial t}} + \underset{(v_r=0)\ 0}{v_r\frac{\partial v_z}{\partial r}} + \underset{(v_\theta=0)\ 0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \underset{(v_z \neq f(z))\ 0}{v_z\frac{\partial v_z}{\partial z}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underset{v_z \neq f(\theta)\ 0}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \underset{(v_z \neq f(z))\ 0}{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \qquad (3.4.2\text{-}5)$$

Therefore the derivative goes to 0, this term disappears, $\frac{\partial p}{\partial z}$ of course remains. Here v_z of course varies with r okay, at the cross-section there are different velocities, therefore this term remains. The v_z is not a function of θ, this velocity does not vary with θ, therefore this derivative goes to 0. Similarly, v_z is not a function of z, therefore this term goes to 0 and of course g_z is in the same direction as relevant, therefore this term remains. So, what we get is, okay I have just transpose this, I have taken this to the other side .

Now let us define a capital P as pressure.

While considering the terms in the above equation, $v_z \neq f(\theta)$ because the flow, in this case, occurs in cylindrical layers. In other words, the axial velocities at all points at a particular radius, and length do not vary with θ.

$$-\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{\partial p}{\partial z} + \rho g_z \qquad (3.4.2\text{-}6)$$

Let us define

$$P = p - \rho g_z z$$

Since $g_z = g$, we can write

$$\frac{\partial p}{\partial z} - \rho g = \frac{\partial (P - \rho g z)}{\partial z} = \frac{\partial P}{\partial z}$$

So, this can be replaced with this okay, I have combined terms. The use of this would become apparent later maybe, but take it on face value for now. And we know from these equations when we substituted earlier that p is not a function of r and p is not a function of θ.

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Since $g_z = g$, We can write Eq. 3.4.2 – 6 as

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\partial P}{\partial z} \qquad \text{Eq. 3.4.2.} - 7$$

We know from equations 3.4.2. – 2 and 3.4.2. – 4 that $p \neq f(r)$ and $p \neq f(\theta)$.

Thus $P = p + \rho g z \neq f(r)$ and $\neq f(\theta)$

Since $P = f(z)$ alone, the partial derivative on the RHS can be replaced by an ordinary derivative

Similarly v_z and r are only $f(r)$ and they are not $f(\theta)$ or $f(z)$
Thus the partial derivative on the LHS can also be replaced by ordinary derivative

With the above, the equation 3.4.2. – 7 can be written as

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dP}{dz} \qquad \text{Eq. 3.4.2.} - 8$$

Let us define

$$P = p - \rho g_z z$$

Since $g_z = g$, we can write

$$\frac{\partial p}{\partial z} - \rho g = \frac{\partial (P - \rho g z)}{\partial z} = \frac{\partial P}{\partial z}$$

Therefore

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\partial P}{\partial z} \qquad (3.4.2\text{-}7)$$

We know from Eqs. 3.4.2-2 and 3.4.2-4 that $p \neq f(r)$ and $p \neq f(\theta)$.
Thus, $P = p + \rho g z \neq f(r)$ and $\neq f(\theta)$.

Therefore, the capital P which we defined as p – $\rho\, g_z$ is also not a function of r and not a function of θ because this is the combination that we are looking at. Since capital P is a function of z alone, we could replace the partial derivative here $\frac{\partial P}{\partial z}$ by $\frac{DP}{Dz}$

Similarly v_z and r are functions of r alone, r is r and v_z varies only with r, it does not vary with z well-developed flow or with θ. Therefore the partial derivative on the LHS also can be replaced by the ordinary derivative. So this r becomes the only variable here, so you could replace the partial with the total. So in the partial derivative equations are much more difficult to solve as you seen earlier.

But this physical situation allows us to directly write these partial derivatives as total derivatives because of the variables that are involved have come down to one each on both sides. So, we could write 3.4.2 – 7

Since $p = f(z)$ alone, the partial derivative on the RHS can be replaced by an ordinary derivative.

Similarly, v_z and r are only $f(r)$; they are not $f(\theta)$ or $f(z)$. Thus the partial derivative on the LHS can also be replaced by ordinary derivative, and the equation becomes

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dP}{dz} \qquad (3.4.2\text{-}8)$$

Let me see what I have okay. Let us take a break here, it is good to take a break here, we have been with some intense stuff for about 20 minutes.

I think it is best to take a break here. Let us continue with the derivation of the relevant relationships for laminar flow of a Newtonian fluid through a cylindrical pipe placed vertically. See you then.