Transport Phenomena in Biological Systems

Prof. G. K. Suraishkumar
Department of Biotechnology Bhupat and Jyoti Mehta School of Biosciences Building
Indian Institute of Technology - Madras

Lecture – 29
Laminar Flow Through a Pipe - Continued

(Refer Slide Time: 00:16)

Since $g_z = g$, We can write Eq. 3.4.2 – 6 as

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\partial P}{\partial z} \qquad \text{Eq. 3.4.2.} - 7$$

We know from equations 3.4.2. – 2 and 3.4.2. – 4 that $p \neq f(r)$ and $p \neq f(\vartheta)$.

Thus $P = p + \rho g z \neq f(r)$ and $\neq f(\vartheta)$

Since $P = f(z)$ alone, the partial derivative on the RHS can be replaced by an ordinary derivative

Similarly v_z and r are only $f(r)$ and they are not $f(\vartheta)$ or $f(z)$
Thus the partial derivative on the LHS can also be replaced by ordinary derivative

With the above, the equation 3.4.2. – 7 can be written as

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dP}{dz} \qquad \text{Eq. 3.4.2.} - 8$$

Welcome back. We are in the middle of the derivation of the relevant expressions for fluid flow of a Newtonian fluid through a laminar flow of a Newtonian fluid through a pipe cylindrical pipe that is placed vertically. We drew some good insights. We saw that the pressure is not a function of radius, the pressure is not a function of θ, and therefore the pressure is not a function of the cross section.

The pressure is the same across the cross section however, the pressure could be different at different cross sections okay. It needs to be different at different cross sections and this is where we stopped last time

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dP}{dz} \qquad (3.4.2\text{-}8)$$

(Refer Slide Time: 01:20)

Now, note that the LHS is a function of r and RHS is a function of z, i.e.

$$\frac{\mu}{r}\frac{d\,f(r)}{dr} = \frac{d\,f(z)}{dz} \qquad \text{Eq. 3.4.2.} - 9$$

From mathematics, we know that this is possible only if each derivative equals a constant, say C_1

First, let us consider the RHS of Eq. 3.4.2. – 8

$$\frac{dP}{dz} = C_1 \qquad \text{Eq. 3.4.2.} - 10$$

Therefore, $$P = C_1 z + C_2 \qquad \text{Eq. 3.4.2.} - 11$$

The relevant boundary conditions are

at z = 0 $\quad P = P_0$

at z = L $\quad P = P_L$

The left hand side is a function of r alone, the right hand side is a function of z alone and this is an ordinary differential equation, how do you solve this? Recall from your math classes as to how to go about solving this situation. You have one side as a function of one variable alone, the other side as a function of the other variable alone, what would be the solution? They both are derivatives, what would be the solution?

The solution is, if you recall it is fine, otherwise I will tell you what it is. You have it of this form μ by r times derivative of a function of r with respect to r = a derivative of a function of z with respect to z. This is possible only if each derivative equals the same constant okay that is the only solution that is possible, this is what math tells us, would have told you when you went through a math course.

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{dP}{dz} \qquad (3.4.2\text{-}8)$$

Besides, the LHS is a function of r and the RHS is a function of z, i.e.

$$\frac{\mu}{r}\frac{df(r)}{dr}=\frac{df(z)}{dz} \qquad (3.4.2\text{-}9)$$

This is possible only if each derivative equals a constant, say C_1.

Let us take the RHS of Eq. 3.4.2-8 first

$$\frac{dP}{dz}=C_1 \qquad (3.4.2\text{-}10)$$

Then

$$P = C_1 z + C_2 \qquad (3.4.2\text{-}11)$$

The relevant boundary conditions are

$$\text{At } z = 0, \quad P = P_0$$
$$\text{At } z = L, \quad P = P_L$$

Using the BCs we get

$$C_2 = P_0$$

$$C_1 = \frac{P_L - P_0}{L}$$

Therefore, $$P = \left(\frac{P_L - P_0}{L}\right)z + P_0 \qquad \text{Eq. 3.4.2.} - 12$$

Next, let consider the LHS and equate it to the same C_1

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1 = \frac{\Delta P}{L} \qquad \text{Note: } \Delta P = P_L - P_0$$

Thus $$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{\Delta P}{L}\times\frac{r}{\mu}$$

Integrating, we get $$r\frac{dv_z}{dr} = \frac{\Delta P}{L}\frac{r^2}{2\mu} + C_{2a}$$

Thus

$$C_2 = P_0$$

$$C_1 = \frac{P_L - P_0}{L}$$

Therefore

$$P = \left(\frac{P_L - P_0}{L}\right)z + P_0 \qquad (3.4.2\text{-}12)$$

(Refer Slide Time: 05:38)

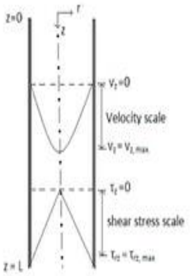
At r = 0, C_{2a} must be equal to 0 (Since vz is finite, $\frac{dv_z}{dr} = 0$, Therefore $C_{2a} = 0$). Thus,

$$\frac{dv_z}{dr} = \frac{\Delta P\, r}{2\mu L} \quad \text{Eq. 3.4.2. – 13}$$

Integrating, we get

$$v_z = \frac{\Delta P\, r^2}{4\mu L} + C_3 \quad \text{Eq. 3.4.2. – 14}$$

Now, using the BC that at r = R, $v_z = 0$ ('no-slip boundary condition'), we get



$$C_3 = -\frac{\Delta P\, R^2}{4\mu L}$$

Thus,

$$v_z = \frac{\Delta P}{4\mu L}(r^2 - R^2) = \frac{(-\Delta P)\, R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \quad \text{Eq. 3.4.2. – 15}$$

Thus the velocity profile is parabolic across the diameter

Note that $\Delta P = P_L - P_0$; typically, for the flow to occur, $P_L < P_0$

At r = 0 we know that, okay let me spend a little bit of time on this. If we have some boundary conditions, we can solve this. We need a boundary condition for $\frac{dv_z}{dr}$ in terms of r to be able to solve this directly or we have to wait for another integration and then substitute if the boundary conditions are appropriate. Now we have a relevant boundary condition directly and that is this. We have a cylindrical geometry here.

The conditions at the center must be the same irrespective of the radius that it choose to traverse in to reach the center from the surface, right. So tube consider certain cross section whether you go through this radius, whether you go through this radius, whether you go through this radius, whether you come from that radius everything should, all converge to the same condition at the center.

You have already seen one such situation earlier when we did mass flux for a spherical system that for such a thing to be physically valid, the only way it will be physically valid is that you either have a maxima or minima at the center or $\frac{dv_z}{dr}$ must equal 0 at the center. If you substitute this boundary condition, this condition there boundary condition, then C2 would turn out to be 0, you can substitute and check.

Upon integration, we get

$$r\frac{dv_z}{dr} = \frac{\Delta P}{L}\frac{r^2}{2\mu} + C_2$$

At $r = 0$, C_2 must be equal to 0.
Therefore

$$\frac{dv_z}{dr} = \frac{\Delta P\, r}{2\mu\, L} \qquad (3.4.2\text{-}13)$$

Integrating this, we get

$$v_z = \frac{\Delta P\, r^2}{4\mu\, L} + C_3 \qquad (3.4.2\text{-}14)$$

Now, using the BC that at $r = R$, $v_z = 0$ ('no slip boundary condition')

$$C_3 = -\frac{\Delta P\, R^2}{4\mu\, L}$$

Thus

$$v_z = \frac{\Delta P}{4\mu L}(r^2 - R^2) = \frac{(-\Delta P)R^2}{4\mu L}\left[1-\left(\frac{r}{R}\right)^2\right] \qquad (3.4.2\text{-}15)$$

The wall is stationary. Therefore the layer of liquid closest to the wall needs to be stationary for it to not slip okay.

No-slip boundary condition that is a very common situation practically speaking also. So the layer that is closest to the wall sticks to the wall. So, the velocity of that layer is the same as the velocity of the wall. The velocity of the wall is 0 and therefore v_z at r = R = 0 and this is the no-slip boundary condition that we have already seen in mass flow and the earlier situation when we started when we did shell balances for momentum.

$P_0 - P_L$ makes better sense, initial minus final, but ΔP is defined as final minus initial and therefore we would like to write it in terms of - ΔP which is kind of a natural way of saying things. Let us call this equation 3.4.2. – 15. This gives us the velocity profile across the radius in the pipe okay that is what we are looking for, we are looking for the velocity profile. We have arrived at the velocity profile just by using the equation of motion.

And what is the shape of this velocity profile? What I would like you to do is just go to your spreadsheet, MS excel or something like that and take some constant here ($-\Delta PR^2/4\mu L$), they are all constants for a given case, just substitute various values of R and see how v_z varies okay and plot that and see. You will see that the velocity profile is parabolic, right.

This is one half of the radius, this is the other half of the radius, it will turn out to be something like this and it will be parabolic. I would like you to do that, that is the reason I am not showing it explicitly here for you, do that and you will find a nice good parabolic profile here, maybe later I will show that to you. Here, here itself it is there, but I would like you to do this. So, this is the velocity profile here okay.

So this is the velocity scale $v_z = 0$ is this, velocity varies in this direction, at the wall it a 0, it reaches a maximum in a parabolic fashion that is what it is. So, this is the typical parabolic velocity distribution in laminar flow of a Newtonian fluid, good to remember this. And PL – P0 is ΔP and the flow to occur PL needs to be greater, sorry PL needs to be less than P0, – Δ P needs to be positive okay.

(Refer Slide Time: 11:35)

The maximum velocity occurs at r = 0 (from Eq. 3.4.2. – 15), i.e., at the centreline (axis) of the tube

$$v_{z,max} = \frac{(-\Delta P)R^2}{4\mu L} \qquad \text{Eq. 3.4.2. – 16}$$

The average velocity across the cross section

$$v_{z,av} = \frac{\int_0^{2\pi}\int_0^R v_z\, r\, dr\, d\theta}{\int_0^{2\pi}\int_0^R r\, dr\, d\theta}$$

$$= \frac{\int_0^{2\pi}\int_0^R \frac{(-\Delta P)R^2}{4\mu L}\left\{1-\left(\frac{r}{R}\right)^2\right\} r\, dr\, d\theta}{\frac{R^2}{2}\times 2\pi}$$

$$= \frac{(-\Delta P)R^2}{\pi R^2 \times 4\mu L}\left[\int_0^{2\pi}\int_0^R r\, dr\, d\theta - \int_0^{2\pi}\int_0^R \frac{r^2}{R^2} r\, dr\, d\theta\right]$$

Okay, now let us draw a little more insights as we did in the earlier cases. The maximum velocity is quite easy to see occurs here okay. What is this r = 0, r increases in this direction and this direction, therefore r = 0, at r = 0 you get the maximum velocity.

The maximum velocity occurs at $r = 0$ (from Eq. 3.4.2-15), i.e., at the centre line (axis) of the tube.

$$v_{z,\max} = \frac{(-\Delta P)R^2}{4\mu L} \qquad (3.4.2\text{-}16)$$

The average velocity across the cross-section

$$v_{z,\text{avg}} = \frac{\int_0^{2\pi}\int_0^R v_z r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta}$$

$$= \frac{\int_0^{2\pi}\int_0^R \frac{(-\Delta P)R^2}{4\mu L}\left\{1-\left(\frac{r}{R}\right)^2\right\} r dr d\theta}{\frac{R^2}{2}\times 2\pi}$$

Nice to know we have an expression by which we can predict the maximum velocity in a laminar flow in a pipe. Equation 3.4.2. – 16. It is also good to know the average velocity across the cross section and since there is variation across the cross section, we need to take each velocity and then average it over the entire cross section we get the average velocity. So, you know the way to do that.

(Refer Slide Time: 13:56)

$$= \frac{(-\Delta P)}{4\pi\mu L}\left\{\left[\frac{R^2}{2}\times 2\pi - \frac{r^4}{4R^2}\right]_0^R 2\pi\right\}$$

$$= \frac{(-\Delta P)}{4\pi\mu L}\left(\frac{R^2}{2}-\frac{R^2}{4}\right)2\pi$$

$$= \frac{(-\Delta P)}{4\pi\mu L}\left(\frac{R^2}{2}-\frac{R^2}{4}\right)2\pi$$

$$v_{z,avg} = \frac{(-\Delta P)\times R^2}{2\mu L\times 4} = \frac{(-\Delta P)R^2}{8\mu L} = \frac{1}{2}(v_{z,max}) \qquad \text{Eq. 3.4.2.} - 17$$

The volumetric flow rate, Q = area x $v_{z,av}$

$$Q = \frac{\pi R^2\times(-\Delta P)R^2}{8\mu L} = \frac{\pi}{8\mu L}R^4(-\Delta P) \qquad \text{Eq. 3.4.2.} - 18$$

Hagen- Poiseuille (pronounced as Pwah- zoo- yuh) equation

Note: Q $\alpha\,(-\Delta P)$

$\alpha\, R^4$

If the radius is doubled at the same ($-\Delta P$), the volumetric flow rate increases 16-fold

And that would turn out to be you integrate that and substitute the limits here, you would get –

$$= \frac{(-\Delta P)R^2}{\pi R^2 \times 4\mu L}\left[\int_0^{2\pi}\int_0^R r dr d\theta - \int_0^{2\pi}\int_0^R \frac{r^2}{R^2} r dr d\theta\right]$$

$$= \frac{(-\Delta P)}{4\pi\mu L}\left\{\left[\frac{R^2}{2}\times 2\pi - \frac{r^4}{4R^2}\right]_0^R 2\pi\right\}$$

$$= \frac{(-\Delta P)}{4\pi\mu L}\left(\frac{R^2}{2} - \frac{R^2}{4}\right)2\pi$$

$$v_{z,\text{avg}} = \frac{(-\Delta P)\times R^2}{2\mu L \times 4} = \frac{(-\Delta P)R^2}{8\mu L} = \frac{1}{2}(v_{z,\max}) \qquad (3.4.2\text{-}17)$$

The average velocity is half the maximum velocity in laminar flow in a cylindrical pipe. Equation 3.4.2. – 17. You could also find out the volumetric flow rate which is nothing but the area times the average velocity. We already have an expression for the average velocity, so just substitute it here, Q is area is πR^2 the cross-sectional area circular.

The volumetric flow rate, Q = Area × $v_{z,\text{avg}}$. Thus

$$Q = \frac{\pi R^2 \times (-\Delta P)R^2}{8\mu L} = \frac{\pi}{8\mu L}R^4(-\Delta P) \qquad (3.4.2\text{-}18)$$

Thus

$$Q \quad \alpha \ (-\ \Delta P)$$
$$\alpha \ R^4$$

If the radius is doubled at the same $(-\ \Delta P)$, the volumetric flow rate increases 16-fold.

So, let us look at this equation a little closely. Volumetric rate is inversely proportional to the length and so on and so forth, this is a constant for a given pipe let us keep it that way. It is directly proportional to the pressure drop, so higher the pressure drop higher will be the volumetric flow rate and it is proportional to the radius power 4 okay.

Which means if you double the radius the flow rate is going to increase 16 fold okay it is a good insight to get. If the radius is doubled, the volumetric at the same –Δ P, the volumetric flow rate increases 16. This equation flow rate 3.4.2-18 is a very famous equation, it is called the Hagen-Poiseuille equation and this is widely used in design and operation itself and sometimes even for analysis you can use this.

This directly gives you the variation of the flow rate, flow rate is how much reaches at in a particular time, very important design parameter. So this one tells you that it is directly proportional to the pressure drop, so you need to create that much pressure drop and more the pressure drop will create more the flow rate you can expect. Not just that if you double the radius, you are going to get 16 times the flow rate okay.

So, the flow rate is going to change 16 fold, increase of 15-16 fold. So, these are good insights I think that is all, okay we still have the shear stress profile. We have we have just seen the velocity profile, we have seen some nice things that can come out of it, now let us see the shear stress profile.

(Refer Slide Time: 17:28)

To derive an expression for the shear stress profile, let us consider Eq. C1 of Table 3.4. – 2

To visualize $\tau_{\theta z}$:

- note that the first subscript, θ, refers to the direction of the velocity gradient
- the second subscript, z, refers to the direction of the stress or the force

If v_z is different at different θ, then $\tau_{\theta z}$, could arise

But, that is not the case here, in laminar flow

A similar visualization would provide $\tau_{zz} \neq f(z)$, since v_z does not vary with z, for this well developed flow

$(v_r = 0)$ $(v_\theta = 0)$ $(v_z \neq f(z))$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

$\tau_{\theta z} \neq f(\theta)$ $\tau_{zz} \neq (f(z))$

$$= -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right] + \rho g_z \qquad \text{Eq. 3.4.2.} - 19$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(r\,\tau_{rz})\right) = -\frac{\partial p}{\partial z} + \rho g_z \qquad \text{Eq. 3.4.2.} - 20$$

To derive an expression for the shear stress profile, all we need to do is go to equation number 1 which is a more complete equation of rather C1 of table 3.4 – 2, please look at table C1 first. Now, let me formally tell you how to visualize $\tau_{\theta z}$ okay. I gave you some idea earlier that may not have gotten across completely, so let me spend some time to visualize this $\tau_{\theta z}$ and so on and so forth.

$$\rho\left(\overset{(SS)}{\cancelto{0}{\frac{\partial v_z}{\partial t}}} + v_r \overset{(v_r = 0)}{\cancelto{0}{\frac{\partial v_z}{\partial r}}} + \overset{(v_\theta = 0)}{\cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}} + v_z \overset{(v_z \neq f(z))}{\cancelto{0}{\frac{\partial v_z}{\partial z}}}\right)$$

$$= -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) + \overset{\tau_{\theta z} \neq f(\theta)}{\cancelto{0}{\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}}} + \overset{\tau_{zz} \neq (f(z))}{\cancelto{0}{\frac{\partial \tau_{zz}}{\partial z}}}\right] + \rho g_z \qquad (3.4.2\text{-}19)$$

The first subscript refers to the direction of the velocity gradient, you know the direction of action direction of motion right or the direction of the velocity gradient and direction of the velocity itself. θ is the direction of the velocity gradient, z is the direction of the velocity of the stress or the force let us say and if v_z is different at different θ, only then could $\tau_{\theta z}$ be relevant at all okay. If v_z is the same at different θ s, then $\tau_{\theta z}$ would be 0.

So, this is a nice way of trying to figure out whether certain terms will exist or whether that certain terms will drop out okay. So look at the variation of the velocity in the direction of force and see whether a gradient exists at all. A gradient will exist only if the velocity is different at different points of that second variable. That is not the case in laminar flow, therefore $\tau_{\theta z}$ is equal to 0. Similarly check with it v_z is different at different z's.

In this case v_z is a well-developed flow, v_z is the same at all z, therefore τ_{zz} is not a function of z and v_z does not vary with z and therefore this term goes to 0 okay. So, this, this kind of a thing you can use to take care of or figure out the relevant terms for the shear stresses, let us do that here. Equations C1 is this, let me not read out the terms, you can read it out from your table that you made a copy of.

The first term drops out because we are looking at a steady state analysis, this term there is no derivative that goes to 0. There is no v_r therefore that goes to 0. There is no v_θ that goes to 0. There is v_z but v_z is not a function of z, well-developed flow, therefore this term goes to 0, $\frac{\partial P}{\partial z}$ exists of course, (1/r) $\frac{\partial}{\partial z}$ $(r\tau_{rz})$ okay. Because of the motion in the z direction, the velocity is going to vary in the r direction okay, yeah it does vary, the velocity at different r's are different, therefore this term will remain.

The z velocity at different θ are the same, therefore this term goes to 0. The different z velocities at various z's are the same therefore that term goes to 0 and of course g_z remains very relevant. So what remains here in equation 3.4.2 -19

The terms that remain yield

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(r\tau_{rz})\right) = -\frac{\partial p}{\partial z} + \rho g_z \qquad (3.4.2\text{-}20)$$

We will call this equation 3.4.2. – 20.

(Refer Slide Time: 21:34)

Let us define $P = p - \rho g_z z$, and note that $g_z = g$
Then, we can write Eq. 3.4.2. – 20 as

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(r\tau_{rz})\right) = -\frac{\partial P}{\partial z} \qquad \text{Eq. 3.4.2.} - 21$$

Using the same argument as for solving equation 3.4.2. – 7, the solution is

$$\tau_{rz} = -\frac{\Delta P\, r}{2L} + C' \qquad \text{Eq. 3.4.2.} - 22$$

B.C.: $\tau_{rz} = 0$ at r = 0

Therefore $$\tau_{rz} = \left(-\frac{\Delta P}{2L}\right) r \qquad \text{Eq. 3.4.2.} - 23$$

If we define $P = p - \rho g_z z$, with the recognition that $g_z = g$, we can write the above equation as

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(r\tau_{rz})\right) = -\frac{\partial P}{\partial z} \qquad (3.4.2\text{-}21)$$

We reuse the same argument for solving the equation. This is a function of z alone, this is a function of r alone, therefore we can convert it to total derivatives. Since this is a function of r alone and the right hand side is a function of z alone, the only solution possible is that both are equal to the same constant.

Using the same argument that we used for solving Eq. 3.4.2-7, the solution becomes

$$\tau_{rz} = -\frac{\Delta P\, r}{2L} + C' \qquad (3.4.2\text{-}22)$$

B.C.: $\tau_{rz} = 0$ at $r = 0$.

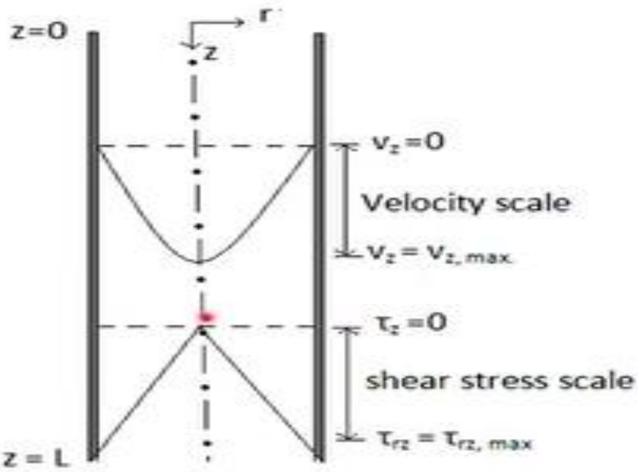
Thus

$$\tau_{rz} = \left(-\frac{\Delta P}{2L}\right) r \qquad (3.4.2\text{-}23)$$

The linear profile for τ_{rz} is shown in Fig. 3.4.2-1.

So we have a shear stress profile directly, some constant for a given case for a given Δ P and for a given length, it varies linearly with r, when r = 0 τ_{rz} = 0, when r = R the radius τ_{rz} = the maximum, the wall shear stress okay.

(Refer Slide Time: 23:36)



So that is what is shown here. This is the τ scale. If you can imagine this as the τ scale in the cylindrical pipe placed vertically, at the center the τ_{rz} is 0 and it linearly increases to its maximum value at the walls. Earlier we had shown a parabolic velocity profile, this is a velocity scale here and now we have shown the shear stress variation. Remember this, these solutions are useful in a wide variety of cases, by repeated use this will become a part of you.

It is also good to remember that you get a parabolic velocity distribution in laminar flow and the shear stress distribution is linear with a 0 value at the center. I think that is all I have for this class. We have done quite a bit again in this class, therefore we need to move on to the next thing, I think we will do capillary flow next and that let us begin in the next class. See you then.