Transport Phenomena in Biological Systems
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Lecture - 32
Non-dimensional Analysis

Welcome, today we are going to do something slightly different from whatever we have been doing so far. So, far we have been looking at momentum flux in a very rigorous fashion in various coordinate systems and all have very many applications we have been looking at some fundamental aspects, which can be applied to pretty much anything under the sun as long as it is applicable. Today we are going to look at something called dimensional, dimensionless numbers and in the context of non-dimensional analysis.

The dimensionless numbers we have already seen the first dimensionless number that we saw, if you recall what is the Reynolds number ($\rho vd / \mu$) you saw that a long time ago or even before that I think we had introduced some non-dimensional variables even in mass flow mass flux situations and so on. There are various advantages to using non dimensional number or 9 numbers are dimensionless numbers the it makes the solution and general enough does not at first go it is you get an overall picture independent of the actual dimensions of this system.

The solution becomes very general all that we have seen. Apart from that there is a certain kind of an analysis that you could do with dimensionless numbers which give good insights and especially for design and operation purposes, when things are not fully understood. So, I am going to show you and analysis based on non dimensional numbers or non dimensional analysis, and at the end of it, you get a certain relationship, which is pretty much the same relationship that we got through an involved understanding of the problem.

So even if the involved understanding is difficult at that point in time, you could do a non dimensional analysis, get to the point of the relationship between the variables, use that relationship to design, whatever you are going to design, engineering typically does not wait for a complete understanding, we did like to use things. And then we many times we do not know why it works, but it works, because you know, intuitively we have figured out that it works this way or to experience.

And then we try to understand the fundamentals at the molecular level and so on and so forth, as to why things happen a certain way even in engineering systems. So, for us, the use aspect is important and this non dimensional analysis significantly helps in getting at relevant expressions that can be used even without fully understanding the underlying mechanisms. I think it is useful to know this in the context of this transport course. So, let me present this to you.

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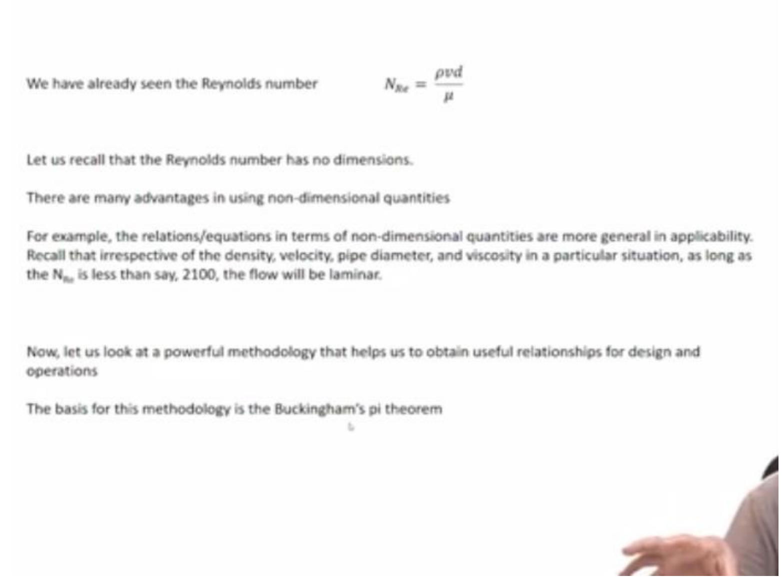

You already seen the Reynolds number ρ vd / μ and it has no dimensions of course, and as I mentioned just now, there are various advantages in using non dimensional quantities. The relationships equations in terms of non-dimensional quantities are more general in applicability. And an example is given here irrespective of the density, the particular density the particular velocity, the particular pipe diameter the particular viscosity in a given situation as long as the Reynolds number.

Which is a combination of those quantities in a certain way to provide non dimensional number is less than 2100 for pipe flow. The flow will be laminar. That is something very powerful that we get. Now, we are going to look at this powerful methodology that dimensioned that helps us to obtain useful relationships for design and operation. And the basis for this methodology is something called the Buckingham's π theorem. That is the mathematical basis. So, there is a certain mathematical basis on which we work on to get to this.

This part is not going to be as rigorous as things were earlier you know, when there things were nicely understood in terms of a certain framework that we had and they therefore we can do rigorously. Most things when we start out are not understood. And thereby we need to figure out ways of getting things done with partial understanding. So slowly in this chapter, we are moving from very rigorous aspects to not so rigorous aspects but very useful aspects for design and operation. So we are kind of covering the entire spectrum.

This is somewhere right in the middle. You are going to go do a little bit of back and forth. We are going to do this and then we are going to get back to some rigorous stuff and then get back to a macroscopic view, which is more of not so rigorous analysis, but very useful analysis. So there we have a little bit but, but we get very useful relationships. Anyway, let us do this. Now, this is the right place to do this in this course.

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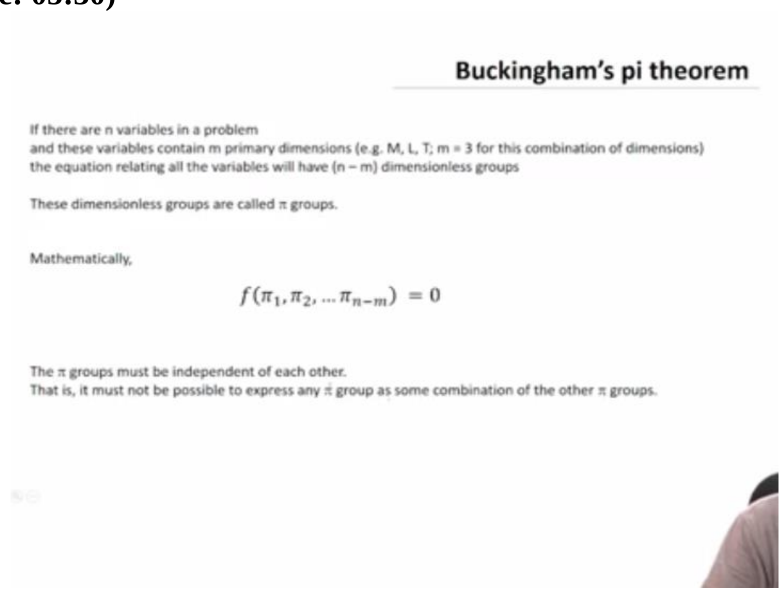

The Buckingham's π theorem formally states this if there are n variables in a problem. And those variables contain m primary dimensions; you know what the dimensions are MLT could be one of the set of primary dimensions. You could have many different combinations, as you know, but let us stick to the simplest one MLT, you could have f and something and sorcery. But let us just stick to the standard once MLT for this, but this formulation is applicable for any m primary dimensions.

And if this is the case, then the equation relating all the variables will have n - m. And there n is a number of variables m is the number of primary dimensions, n - m dimensionless groups. So we are after an equation that contains n - m dimensionless groups that relates the various variables here and that would be helpful for design and operation. These dimensionless groups are called π groups. And mathematically speaking, you would write the same thing as a function of π_1, π_2, and so on till $\pi_{n - m} = 0$. That is a mathematical representation.

The important thing to note here is that the π groups must be independent each other. So, this is the mathematical formulation we are going to base our development on this mathematical formulation, but we are also going to use certain practices that happened to give us easy results, results in an easy fashion. So, this has been proved time and again we may not have a mathematical basis for why that practice provides us with information.

That is what I mean we are going to slightly relaxed the rigor aspect in this part of the course. What it means to say that the π groups must be independent of each other is that it must not be possible to express any π group as a combination of other π groups. It will become clear as we go along. This never ceases once more. It must not be possible to express any π group as a combination of other π groups.

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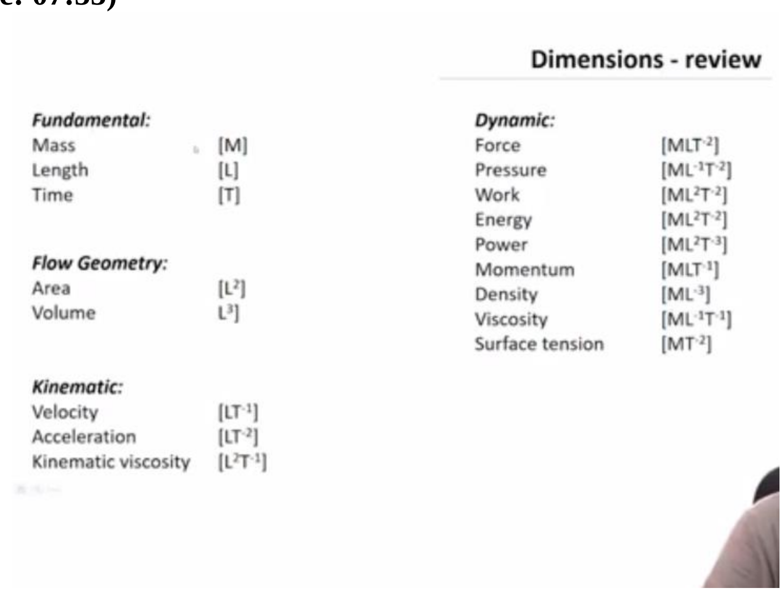

That is good enough as we go along these significance of that statement will become clearer we need dimensions and we need the relationship between dimensions and quantities. So, let us review the relationship between dimensions and quantities. You already know this from high school. So, let us just review it quickly. You could view these some of these as fundamentals such as mass is M, the dimension of mass is M, the dimension of length L, the dimension of time is T

Mass	[M]
Length	[L]
Time	[T]
Flow Geometry	
Area	$[L^2]$
Volume	$[L^3]$
Kinematic	
Velocity	$[LT^{-1}]$
Acceleration	$[LT^{-2}]$
Kinematic viscosity	$[L^2T^{-1}]$
Dynamic	
Force	$[MLT^{-2}]$
Pressure	$[ML^{-1}T^{-2}]$
Work	$[ML^2T^{-2}]$
Energy	$[ML^2T^{-2}]$
Power	$[ML^2T^{-3}]$
Momentum	$[MLT^{-1}]$
Density	$[ML^{-3}]$
Viscosity	$[ML^{-1}T^{-1}]$
Surface tension	$[MT^{-2}]$

And that is it is good to keep this in mind or you can make a copy of this. Table keep it next to you, you can refer to this whenever you want. The emphasis on the course in this course will not be on memorization.

That is never the case, especially when you have these complex equations. There is no point in memorizing those, you need to understand you need to go from one point to another in a seamless fashion that is more important than memorization. Memorization of course, it is a good skill on its own. It is just that it may not be very relevant for this particular some aspects of this particular course memory is certainly relevant anywhere.

You need basic memory otherwise you do not even remember each other's names. You know, basically speaking you on would not even remember basic things. So I am not saying memory is not important. It is important. Only thing is that memorization at the cost of understanding is a certainty and no in this course.

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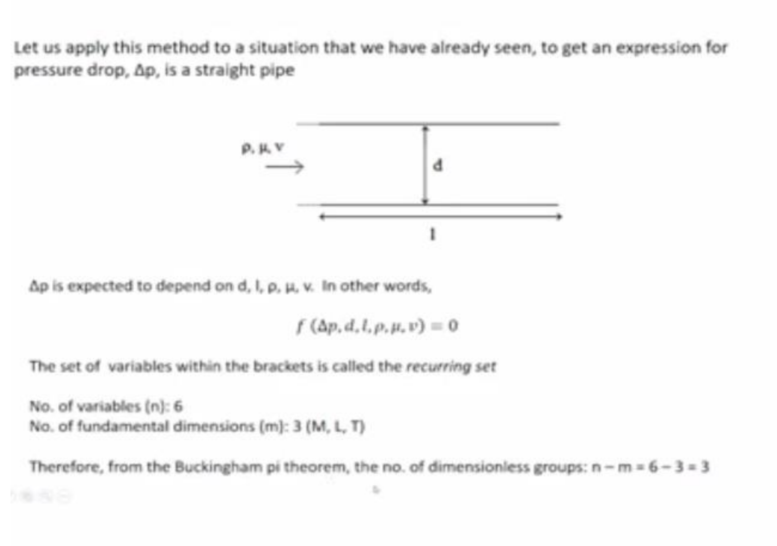

Now let us get back to this method. let us apply this added to a situation that we already seen as I mentioned earlier to get an expression for the pressure drop the Δ p in a straight pipe you already have an expression for it, let us see how we get to pretty much the same expression, but just the dimensional considerations no flow nothing just the dimensions. So, this is a situation here we have let us say a horizontal pipe of diameter d, the fluid of density ρ and viscosity μ is flowing with the velocity of v through the pipe of length l.

So, this is what we have here. And the pressure drop you can say is expected to depend on these properties, even without too much insight, you can say that you can expect it to depend on these properties or these quantities, these parameters or in other words, mathematically speaking a function of all these quantities here

Let us apply this method to a situation that we have already seen – to get an expression for pressure drop Δp in a straight pipe (Fig. 3.4.4-1).

Δp is expected to depend on d, l, ρ, μ, v. In other words

$$f(\Delta p, d, l, \rho, \mu, v) = 0$$

The set of variables within the brackets is the recurring set.

Number of variables (n): 6
Number of fundamental dimensions (m): 3 (M, L, T)

And the number of variables here as 6, and number of fundamental dimensions is 3. And from practice, so Buckingham π theorem itself, the number of dimensionless groups is n - m, therefore, we expect a relationship between 3 dimensionless groups for this case that is what we get from the Buckingham π theorem, mathematical theorem.

Therefore, from the Buckingham pi theorem, the number of dimensionless groups: $n - m = 6 - 3 = 3$.

Also, from experience, it is known that the recurring set must contain 3 (the same number as the number of dimensionless groups) variables that cannot themselves be formed into a dimensionless group. Thus, l and d cannot be chosen together since (l/d) is dimensionless. Δp, ρ and v cannot be chosen together since $(\Delta p/\rho v^2)$ is dimensionless. Therefore, let us choose d, v and ρ.

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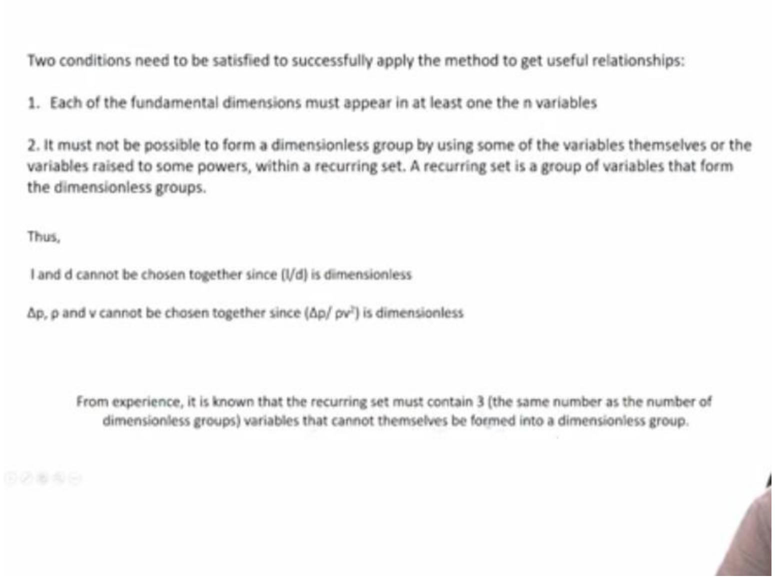

The dimensions are

$$
\begin{aligned}
d &= [\mathrm{L}] \\
v &= [\mathrm{LT}^{-1}] \\
\rho &= [\mathrm{ML}^{-3}]
\end{aligned}
$$

Let us rewrite the dimensions in terms of the chosen variables.

$$
\begin{aligned}
[\mathrm{L}] &= d \\
[\mathrm{M}] &= \rho d^3 \\
[\mathrm{T}] &= d v^{-1}
\end{aligned}
$$

Now let us take the remaining variables, Δp, l and μ, in turn. First

$$\Delta p = [ML^{-1}T^{-2}]$$

Therefore

$$\Delta p[M^{-1}LT^{2}] \text{ is dimensionless}$$

Thus

$$\pi_1 = \Delta p \ (\rho d^3)^{-1}(d) \ (d^{-1}v)^2$$
$$= \Delta p/\rho v^2$$

Now, let us consider the length, l

$$l = [L]$$

Therefore

$$l[L]^{-1} \text{ is dimensionless}$$

Thus

$$\pi_2 = l/d$$

Now, finally, let us consider μ

$$\mu = [ML^{-1}T^{-1}]$$

Therefore

$$\mu \ [M^{-1}LT] \text{ is dimensionless}$$

Now, to apply this method 2 conditions need to be satisfied that is been found from practice. Do not ask me for a basis for this, works. The first one is that each of these of the fundamental dimensions must appear in at least one of the n variables that we must make sure. And then it must not be possible to form a dimensioned as group by using some of the variables themselves or the variables raised to some powers within a recurring set you know what a recurring set is the set of those variables that we are considering 6 in this case.

So, let me read this again it must not be possible to form a dimensionless group by using some of the variables themselves are variables raised to some powers within the recurring set. And we have already seen what a recurring set is. Therefore, we cannot choose l and d together and from that recurring set of 6 variables, because if you just do l and d l and d becomes dimensionless. That is cannot happen. It must not be possible to form a dimensionless group by using some of the variables themselves are variables raised to some powers.

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Therefore, let us choose d, v and ρ

The dimensions are:
$d = [L]$
$v = [LT^{-1}]$
$\rho = [ML^{-3}]$

Let us rewrite the dimensions in terms of the chosen variables

$[L] = d$
$[M] = \rho d^3$
$[T] = dv^{-1}$

Now let us take the remaining variables, Δp, l and μ, in turn.

$\Delta p = [ML^{-1}T^{-2}]$

Therefore, $\Delta p[M^{-1}LT^{2}]$ is dimensionless

Thus, $\pi_1 = \Delta p\ (\rho d^3)^{-1}(d)\ (dv^{-1})^{2} \quad = \Delta p/\rho v^{2}$

Therefore

$$\mu\ [M^{-1}LT]\ \text{is dimensionless}$$

Thus

$$\pi_3 = \mu(\rho d^3)^{-1}(d)\ (dv^{-1})$$

$$= \frac{\mu}{\rho v d} \text{ or } 1/N_{\text{Re}}$$

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Now, let us consider the length, l

$l = [L]$

Therefore, $l[L]^{-1}$ is dimensionless

Thus, $\pi_2 = l/d$

considering μ $\mu = [ML^{-1}T^{-1}]$

$\mu\ [M^{-1}LT]$ is dimensionless

Thus, $\pi_3 = \mu(\rho d^3)^{-1}(d)\ (dv^{-1}) = \frac{\mu}{\rho v d}$ or $1/N_{Re}$

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From the Buckingham's pi theorem

$$\frac{\Delta p}{\rho v^2} = f\left(\frac{1}{d}, \frac{1}{N_{\mathrm{Re}}}\right)$$

or

$$\frac{\Delta p}{\rho v^2} = k\left(\frac{1}{d}\right)^a \left(\frac{1}{N_{\mathrm{Re}}}\right)^b$$

Thus, from a mere dimensional analysis, we know the form of the relationship between the relevant variables. Let us see the validity of what we have got by comparing the above relation to what we already know. We had seen earlier, in pipe flow, the volumetric flow rate

So there is some functionality, functionality it is general k, and you are raising it to a certain power as long as you find the powers as a property and k appropriately it should work, there is nothing wrong there is no reason why it should not work. So, this is a relationship between these variables and notice by a mere dimensional or dimensional analysis as it is called involving non dimensional numbers, we know the form of relationship between the relevant variables and that is powerful.

So even if you do not understand the process, you can do one such thing for design an operation which relates these variables in for the system that you are interested in. And now, let us check the validity we have this relationship from dimensional analysis.

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We had seen earlier that in pipe flow that the volumetric flow rate

$$Q = \frac{\pi \, \Delta p}{8\mu l} r^4$$

or, $$Av = \frac{\pi \Delta p}{8\mu l} \left(\frac{d}{2}\right)^4$$

$$(\pi d^2/4)v = \frac{\pi \Delta p}{28\mu l}\left(\frac{d}{2}\right)^4$$

$$\Delta p = (vd^{2}\ \mu l \times 32)/(d^{4-2}) = 32(\left(\frac{\mu v}{d}\right)\left(\frac{l}{d}\right)$$

$$\frac{\Delta p}{\rho v^2} = 32\left(\frac{\mu}{\rho v d}\right)\left(\frac{l}{d}\right) = 32\left(\frac{1}{N_{Re}}\right)\left(\frac{l}{d}\right)$$

$$\frac{\Delta p}{\rho v^2} = f\left(\frac{1}{N_{Re}}, \frac{l}{d}\right) \text{ and } a = b = 1;\ k = 32$$

$$Q = \frac{\pi \Delta p}{8 \mu l} r^4$$

or

$$Av = \frac{\pi \Delta p}{8 \mu l} \left(\frac{d}{2} \right)^4$$

$$(\cancel{\pi} d^2 / 4) v = \frac{\cancel{\pi} \Delta p}{2\cancel{8} \mu l} \left(\frac{d}{2} \right)^4$$

$$\Delta p = (v \cancel{d}^{\cancel{2}} \mu l \times 32) / (d^{\cancel{4}\,2}) = 32 \left(\frac{\mu v}{d} \right) \left(\frac{l}{d} \right)$$

$$\frac{\Delta p}{\rho v^2} = 32 \left(\frac{\mu}{\rho v d} \right) \left(\frac{l}{d} \right)$$

$$= 32 \left(\frac{1}{N_{\text{Re}}} \right) \left(\frac{l}{d} \right)$$

$$\frac{\Delta p}{\rho v^2} = f \left(\frac{1}{N_{\text{Re}}}, \frac{l}{d} \right) \text{ and } a = b = 1;\ k = 32$$

Q is Hagen poiseuille equation or flow rate nothing but area time's velocity, a constant area or area times velocities flow rate anyway,

So just by a dimensional analysis, we get a relationship, which is actually vary, the form of the relationship is the same, which is a very powerful thing to know, then you can do a couple of experiments to know the constants involved.

You can get out for a couple from a couple of experiments by fitting the data to this, and then that you are all set to go for design of all related aspects. So that is the power of dimensional analysis involving non dimensional numbers. I am sure this will become useful at some point in time. I think that is all I have for this lecture. Let us stop here and continue when we get back. See you in the next class.