Transport Phenomena in Biological Systems

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Lecture - 33
Unsteady State Flow

Welcome back in this class, let us start looking at unsteady state flow, we are going back to some rigorous analysis. And then in the last class we did non dimensional analysis which is not very mathematically rigorous and so on so forth but was very helpful useful. But we are going now to rigorous analysis, which based on good understanding of what is happening. Let us do this to get an idea of the unsteady state aspects which are very relevant for our situations, applications to biological systems.

Many different places you have unsteady state flow, or when you have unsteady state flow, you must be able to address that. And once we do this unsteady state flow and and then we get to aspects. No, we will still do we still stick with some reasonable level of rigor we will get to turbulent flow which again we will start hand waving a bit and then we will get to macroscopic aspects .

So, as we already seen, whenever you have unsteady state, then you have the time derivative coming along with it that additional derivative, the partial differential equation becomes mathematically rigorous, mathematically time consuming or tedious to address but let us do that because there are some advantages and numerical in a analytical solution. I will just provide some thoughts on a numerical solution also.

If you are looking at numerical route, you need a lot of grounding, you need to pick up a lot of information on the numerical aspects of numerical analysis, numerical simulations, aspects and so on so forth, which probably you have not done as a part of an undergraduate course? So you need that. And not just that even there, there are a lot of it is not very straightforward. There are challenges that are very particular to the numerical methods themselves.

It is not that you just use a numerical method, you will get the solution much simpler, there are very many challenges that you need to overcome there also. So, those 2 approaches, they are essential to be able to solve and to use fundamental knowledge for better and better design, analysis and operation. With that, let us get into the details of unsteady flow.

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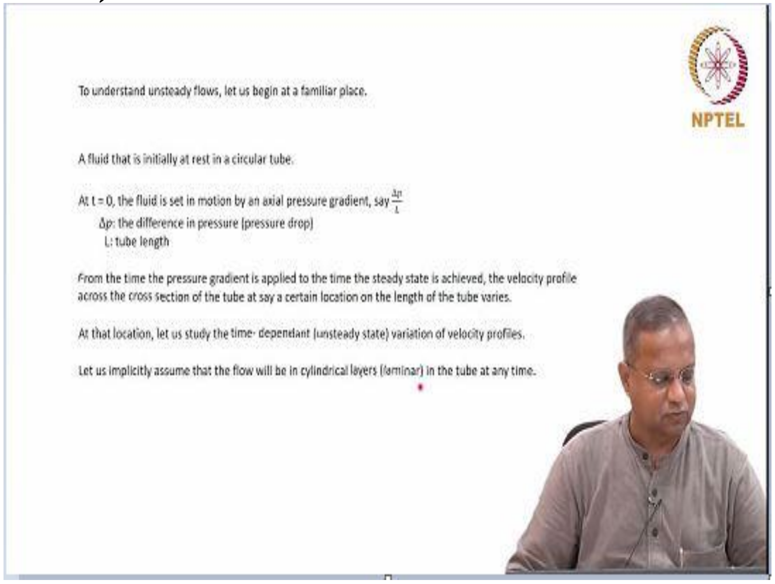

The, to understand this, we are going to start at a very familiar place, which is we got a, consider a fluid in a circular tube that is initially addressed. And then at time t = 0, the fluid is set in motion by an axial pressure gradient, there is a pressure gradient that is imposed the Δp or a certain length and that pressure gradient begins the flow pressure drop begins the flow L is a tube length and from the time the pressure gradient is applied to the time the steady state is achieved.

The steady state is what the properties at a particular point on this case the velocity at a particular point in the flow does not change with time that will happen or a certain after certain distance or a certain time and so on. But, we are interested in the part or at the time in the initial time from the point where the pressure gradient is applied, this flow starts moving to the time when the steady state energy start. The velocity profile across a cross section of the tube at a certain location on the length of the tube will vary that is what we are looking at.

At that location, let us study the time dependent or the unsteady state variation of the velocity profile. If you look at a particular radius particular cross section that is, as soon as the fluid is set into motion the velocity profile would be very different and we are trying to predict how that velocity profile would be the unsteady velocity profile after a certain while if you do not consider the entrance and exit lengths, then after while the in majority of the pipe length it is going to be a steady state.

So, we are interested in the intervening time. Let us implicitly assume that the flow will be in cylindrical layers and the tube at any time this assumption we are making some of these assumptions through intuition we will make or have been made. And they have been found to work. In other words, those assumptions are not bad assumptions. So here we are going to implicitly assume that the flow will be in cylindrical layers, it is always in laminar flow.

So always in layers only thing is that it is not a flow that will give you a parabolic velocity profile when it is unsteady state. That is the only situation here we are not considering the intermixing of layers and so on so forth.

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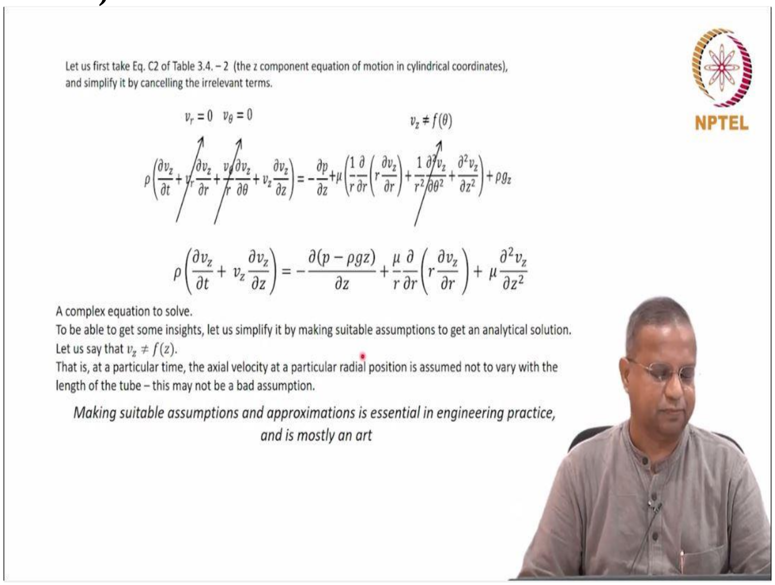

So, if this is the case, then we can take equation C2 it is still laminar flow. So, we can take C2 still flow in layers, so we can take C2 and C2 gives you this z component of equation in the cylindrical coordinates and that is this equation. If it cancel out the terms the first term which we were canceling, pretty much till now cannot be canceled because we said the velocity in z direction is the z direction let us same varies with time.

$$\rho\left(\frac{\partial v_z}{\partial t} + \underbrace{v_r \frac{\partial v_z}{\partial r}}_{v_r = 0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{v_\theta = 0} + v_z \frac{\partial v_z}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{v_z \neq f(\theta)} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

$$\rho\frac{\partial v_z}{\partial t} + v_z\frac{\partial v_z}{\partial z} = -\frac{\partial(p - \rho g z)}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \mu\frac{\partial^2 v_z}{\partial z^2}$$

So, this is what we have as the governing equation for unsteady state flow when the fluid starts boiling in a cylindrical pipe horizontal pipe.

This is a very complex equation to solve. If you recall from your math differential equations course, the partial differential equation and this is going to be difficult. So what is normally done is with some in sight, some assumptions are made and of course, those assumptions are tested. We are going to see whatever assumptions have been made and found as good assumptions. So, this is the first time we are looking at this and this is already been done.

So many times from a long time ago, and so we are going to use that knowledge and we are going to look at that knowledge here. So, the some of the suitable assumptions are to get an analytical solution is v_z is not a function of z people made this assumption and found this to be reasonable. In other words v_z does not depend on the length does not depend on the distance along the flow axis. That is an assumption by that this is the basic equation, if you can find some means to solve it work.

But this is difficult and therefore, even with these assumptions, it is quite lengthy, quite tedious. But let us do this v_z equals is not a function of z. Therefore, all those $v_z \frac{dv_z}{dz}$dropped out this is going to dropout this going to drop out. Of course the others we cannot do much about. So, if you do that, some explanation of this that is at a particular time the axial velocity at a particular radial vary radial position is not assumed to vary with the length of the tube, then this may not be a bad assumption.

This is been found to be a decent assumption. That is why we are doing this and making suitable assumptions and approximations as ascension in engineering practice, and is mostly an art. We do not expect people being exposed to this for the first time that you in this course, to be able to do this. This comes with a lot of experience in the field and a lot of insights and so.

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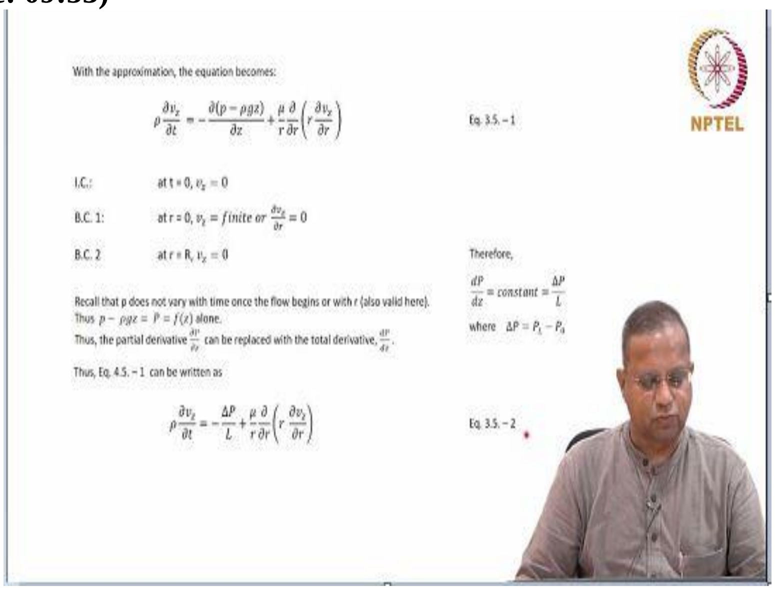

With this approximation, the equation becomes this

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial (p - \rho g z)}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) \qquad (3.5\text{-}1)$$

IC: At $t = 0$, $v_z = 0$

BC 1: At $r = 0$, v_z = finite or $\frac{\partial v_z}{\partial z} = 0$

BC 2: At $r = R$, $v_z = 0$

Since p does not vary with time once the flow begins or with r (as seen earlier, and which is also valid here), $p - \rho g z = P = f(z)$ alone. Thus, the partial derivative $\frac{\partial P}{\partial z}$ can be replaced with the total derivative, $\frac{dP}{dz}$.

Therefore

$$\frac{dP}{dz} = \text{Constant} = \frac{\Delta P}{L}$$

where

$$\Delta P = P_L - P_0$$

3.5 - 1 and the initial condition, the see how we are doing in terms of time and in terms of tedium, I think we will do this in 2 different classes that might be better at t = 0, this is the initial condition $v_z = 0$ that we know. And we need 2 boundary conditions second order here. So 2 boundary conditions that give us space relationships. So there has to be radial symmetry. Therefore, whichever are your travels by you need to get to the same situation at the center? And that is possible mathematically only if there is a minimum or a maximum at the center of velocity in this case. We said in this case and therefore, if there was a minima or maxima the derivative with respect to radius has to be 0 at the center point.

The pressure does not vary with time once the flow begins is no reason for it to vary. And across the cross section. We have found that to be a constant earlier that was for city state flow we are trying to use that inside here. And that seems to be valued here. That is what is being found. So, all these assumptions are actually turned out to be good assumptions. Therefore, p - ρ v_z, which is P is a function of z alone, which is this.

Thus, Eq. 3.5-1 can be written as

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\Delta P}{L} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) \qquad (3.5\text{-}2)$$

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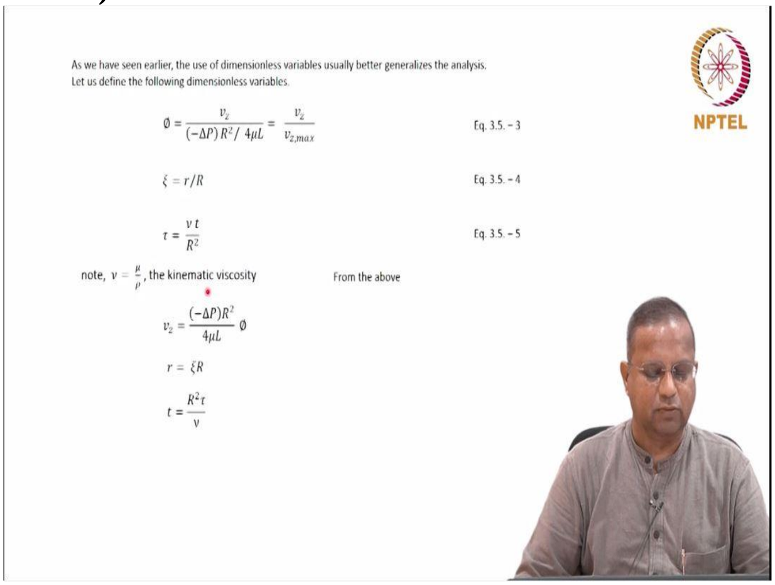

And let us use dimensionless variables which will give us a generalize solution. And to do that, let us define the following dimensionless variables.

Use of dimensionless variables usually simplifies analysis. Let us define the following dimensionless variables.

$$\phi = \frac{v_z}{(-\Delta P)R^2/4\mu L} = \frac{v_z}{v_{z,\max}} \qquad (3.5\text{-}3)$$

$$\xi = r/R \qquad (3.5\text{-}4)$$

$$\tau = \frac{\nu t}{R^2} \qquad (3.5\text{-}5)$$

where $\nu = \frac{\mu}{\rho}$, the kinematic viscosity.

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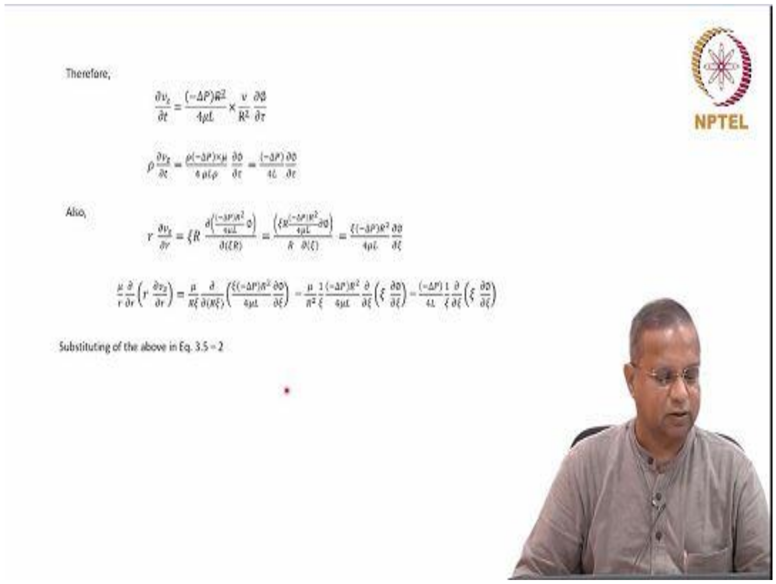

So, we are going term by term, we have taken care of one term and finally, this is the equation in terms of the non dimensional variables the same and of course, along with reasonable assumptions as was later found after they check the assumptions and

From the above definitions

$$v_z = \frac{(-\Delta P)R^2}{4\mu L}\phi$$

$$r = \xi R$$

$$t = \frac{R^2 \tau}{\nu}$$

Thus

$$\frac{\partial v_z}{\partial t} = \frac{(-\Delta P)\cancel{R^2}}{4\mu L} \times \frac{\nu}{\cancel{R^2}} \frac{\partial \phi}{\partial \tau}$$

$$\rho \frac{\partial v_z}{\partial t} = \frac{\cancel{\rho}(-\Delta P) \times \cancel{\mu}}{4\cancel{\mu} L \cancel{\rho}} \frac{\partial \phi}{\partial \tau} = \frac{(-\Delta P)}{4L} \frac{\partial \phi}{\partial \tau}$$

Further

$$r \frac{\partial v_z}{\partial r} = \xi R \frac{\partial \left(\frac{(-\Delta P)R^2}{4\mu L}\phi \right)}{\partial(\xi R)} = \frac{\left(\xi R \frac{(-\Delta P)R^2}{4\mu L} \partial \phi \right)}{R \quad \partial(\xi)} = \frac{\xi(-\Delta P)R^2}{4\mu L} \frac{\partial \phi}{\partial \xi}$$

$$\frac{\mu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) = \frac{\mu}{R\xi} \frac{\partial}{\partial (R\xi)} \left(\frac{\xi(-\Delta P)R^2}{4\mu L} \frac{\partial \phi}{\partial \xi} \right)$$

$$= \frac{\mu}{R^2} \frac{1}{\xi} \frac{(-\Delta P)R^2}{4\mu L} \frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi} \right)$$

$$= \frac{(-\Delta P)}{4L} \frac{1}{\xi} \frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi} \right)$$

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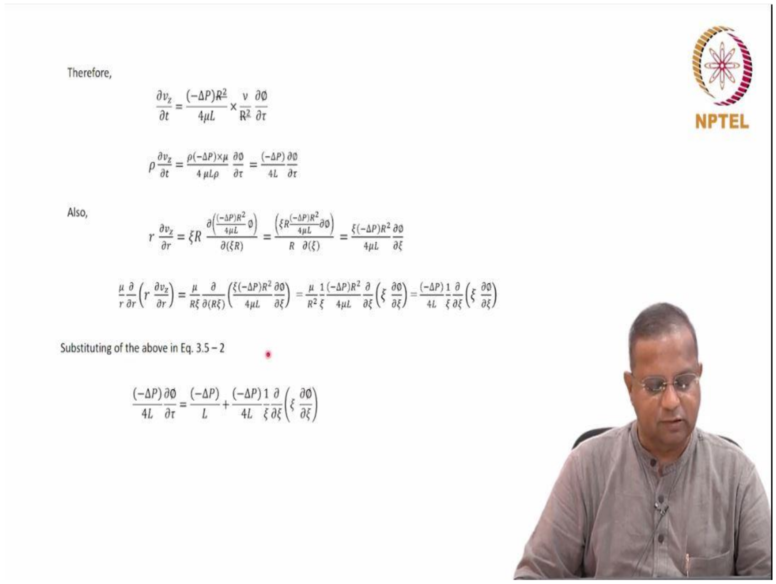

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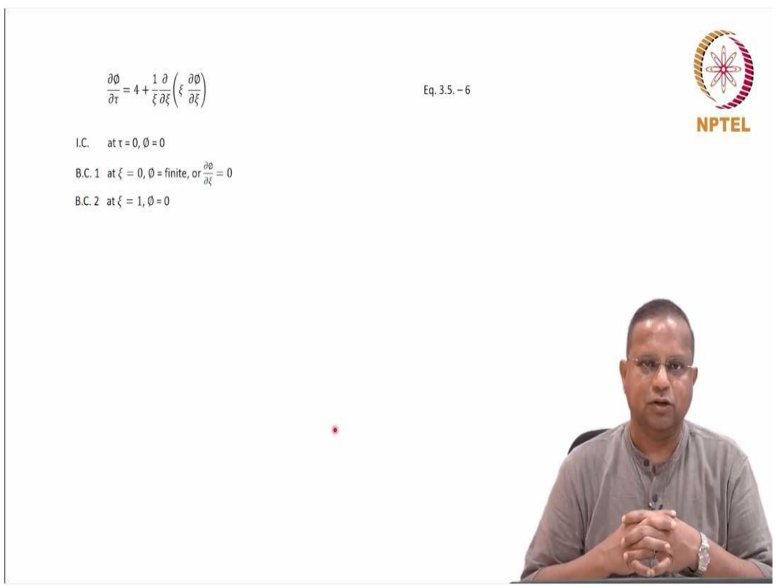

Through substitution of the above expressions in Eq. 3.5-2 we get

$$\frac{(-\Delta P)}{4L}\frac{\partial \phi}{\partial \tau} = \frac{(-\Delta P)}{L} + \frac{(-\Delta P)}{4L}\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi}{\partial \xi}\right)$$

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi}{\partial \xi}\right) \qquad (3.5\text{-}6)$$

Now, I think we have been at this for some time. It is a tedious process. So let us do it in bits and pieces, which are manageable. Let us stop this lecture here. And then when we come back, I will continue with these solutions for the unsteady state flow. See you.