Transport Phenomena in Biological Systems
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Lecture-34
Unsteady State Flow - Continued

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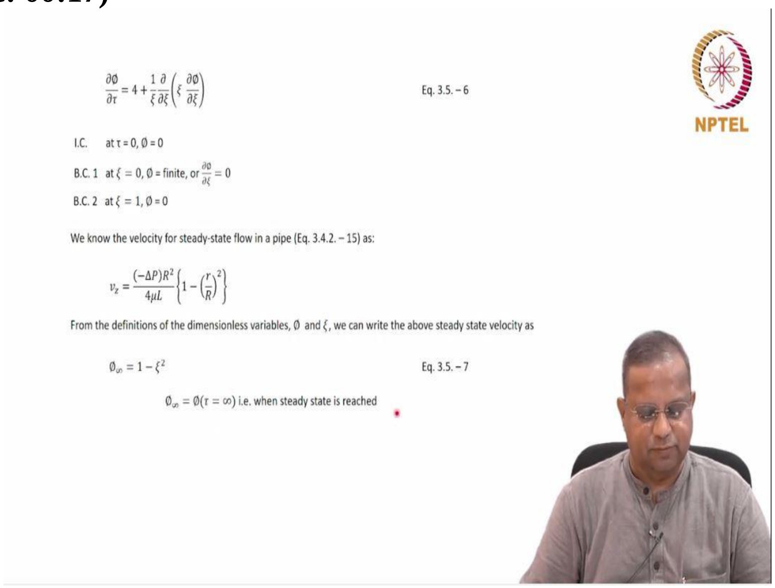

Welcome back, let us continue our discussion with unsteady state situation of fluid flow, very simple situation here we have a fluid in a cylindrical pipe at time t = 0 the flow is started by maybe applying a pressure drop across it. And we are interested in the time it takes from time t = 0 to the time when steady flow is achieved. So, we are looking at the unsteady region and we are trying to get some insights into that.

We arrived at this differential equation in terms of the non dimensional variables that we had defined

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) \qquad (3.5\text{-}6)$$

and the initial condition and boundary conditions were written in terms of the non dimensional variables. We took a break at this point and we said we would continue with the solution here it is tedious and therefore, we are doing it in paths. So, here we know that the velocity for steady state flow in a pipe is something like this.

We already know this from our earlier derivation

For a steady state flow, we can use Eq. 3.4.2-15 to get

$$v_z = \frac{(-\Delta P)R^2}{4\mu L}\left\{1-\left(\frac{r}{R}\right)^2\right\}$$

the parabolic velocity profile that we got for laminar flow in a pipe. So, this is the steady state velocity profile. So, from the definitions of dimensionless variables φ and ξ, we can write the above steady state velocity as some

We can write the relationship in terms of dimensionless variables as

$$\phi_\infty = 1 - \xi^2 \qquad (3.5\text{-}7)$$

where $\phi_\infty = \phi(\tau = \infty)$ i.e. when steady state is reached.

You can see how this works, the remaining actually turned out to be φ and we calling it φ_∞ because it is the ultimate it is reaching steady state at the end of in terms of an infinite time for this particular view

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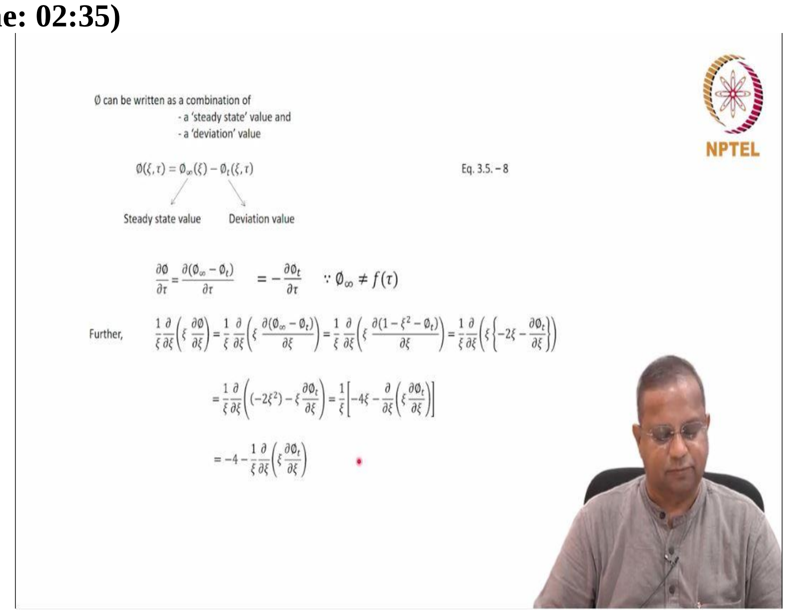

Now, this is the standard trick that is done. This φ can be written as a combination of a steady state value and a deviation when nothing prevents us from doing it. They will ask the non dimensional velocity at the nip point is some value plus or minus a certain deviation standard approach that is taken. So, let us do it this way. So φ of ξ, τ as a function of ξ and τ, which are non-dimensional distance non-dimensional time is φ_∞ and ξ there is no longer depends on τ - φ at a certain t which depends on ξ and τ.

ϕ can be written in terms of a 'steady state' value and a 'deviation' value i.e.

$$\phi(\xi, \tau) = \phi_\infty(\xi) - \phi_t(\xi, \tau) \qquad (3.5\text{-}8)$$

So a steady value and deviation value here it is given as a negative for the negative positive whatever it is, that does not matter, typically taken as negative for the foundation steady state value the deviation we will call this equation 3.5 - 8. Therefore

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial(\phi_\infty - \phi_t)}{\partial \tau}$$

$$= -\frac{\partial \phi_t}{\partial \tau} \quad \because \phi_\infty \neq f(\tau)$$

So, this is pretty much a constant as far as this is concerned. Therefore, we have taken something do not worry about it we have brought it in terms of a fluctuation variable

Also

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial(\phi_\infty - \phi_t)}{\partial \xi}\right)$$

$$= \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial(1 - \xi^2 - \phi_t)}{\partial \xi}\right)$$

$$= \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \left\{-2\xi - \frac{\partial \phi_t}{\partial \xi}\right\}\right)$$

$$= \frac{1}{\xi}\frac{\partial}{\partial \xi}\left((-2\xi^2) - \xi\frac{\partial \phi_t}{\partial \xi}\right)$$

$$= \frac{1}{\xi}\left[-4\xi - \frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi_t}{\partial \xi}\right)\right]$$

$$= -4 - \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi_t}{\partial \xi}\right)$$

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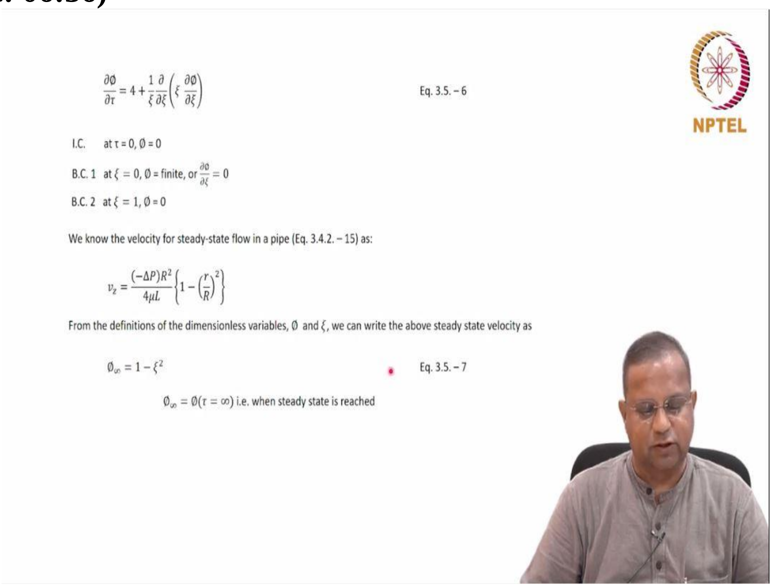

Substituting the above in Eq. 3.5-6, we get

$$-\frac{\partial \phi_t}{\partial \tau} = 4 - 4 - \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi_t}{\partial \xi}\right)$$

We have written it in terms of a steady state variable and a deviation variable. Now we are rewriting this equation in terms of the deviation variable. And that whatever remains the steady state variable inside the derivative it is going to it was it dropped out.

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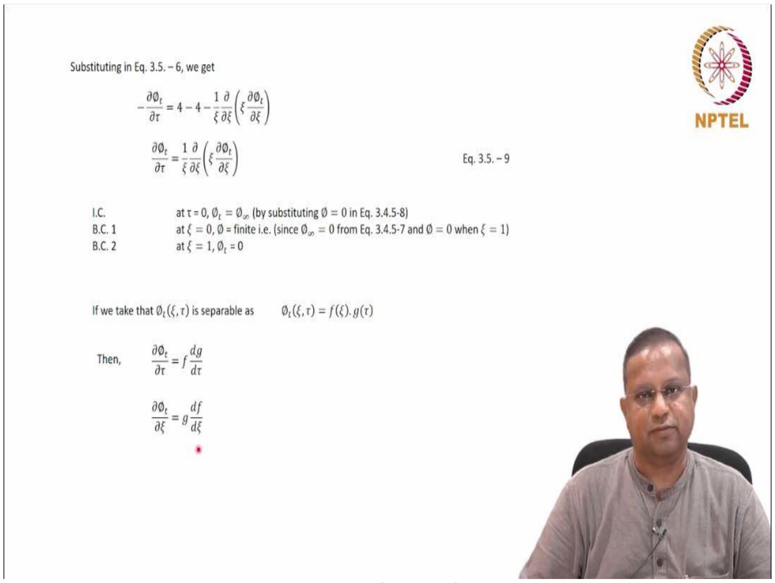

$$\frac{\partial \phi_t}{\partial \tau} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_t}{\partial \xi}\right) \tag{3.5-9}$$

Now, the initial and boundary conditions are

IC: At $\tau = 0$, $\phi_t = \phi_\infty$ (by substituting $\phi = 0$ in Eq. 3.5-8)
BC 1: At $\xi = 0$, ϕ = finite i.e. (since $\phi_\infty = 0$ from Eq. 3.5-7 and $\phi = 0$ when $\xi = 1$)
BC 2: At $\xi = 1$, $\phi_t = 0$

We will call this equation 3.5 - 9 you have successfully reduced that to this form

If we assume that $\phi_t(\xi, \tau)$ is separable as

$$\phi_t(\xi, \tau) = f(\xi) . g(\tau)$$

then

$$\frac{\partial \phi_t}{\partial \tau} = f\frac{dg}{d\tau} \quad \text{and} \quad \frac{\partial \phi_t}{\partial \xi} = g \times \frac{df}{d\xi}$$

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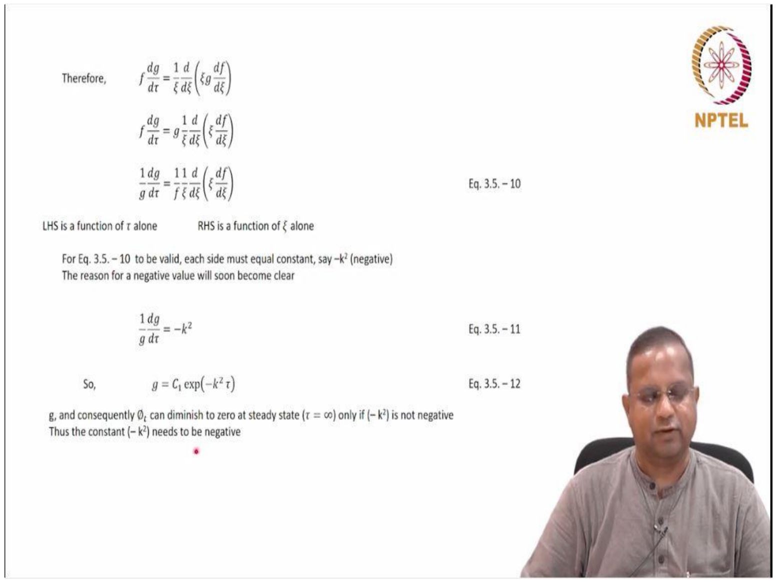

Therefore

$$f\frac{dg}{d\tau} = \frac{1}{\xi}\frac{d}{d\xi}\left(\xi g\frac{df}{d\xi}\right)$$

$$f\frac{dg}{d\tau} = g\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right)$$

$$\frac{1}{g}\frac{dg}{d\tau} = \frac{1}{f}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) \qquad (3.5\text{-}10)$$

Since the LHS is a function of τ alone and the RHS is a function of ξ alone,

We have finally attained that situation that is a case both have to be equal to the same constant. And we are going to call that constant as $-k^2$. There is a reason for this will become apparent in the next few steps. But just be with me here,

$$\frac{1}{g}\frac{dg}{d\tau} = -k^2 \qquad (3.5\text{-}11)$$

This implies

$$g = C_1 \exp(-k^2\tau) \qquad (3.5\text{-}12)$$

If ($-k^2$) is not negative, then g, and consequently ϕ_t cannot diminish to zero at steady state ($\tau = \infty$); thus the constant ($-k^2$) needs to be negative.

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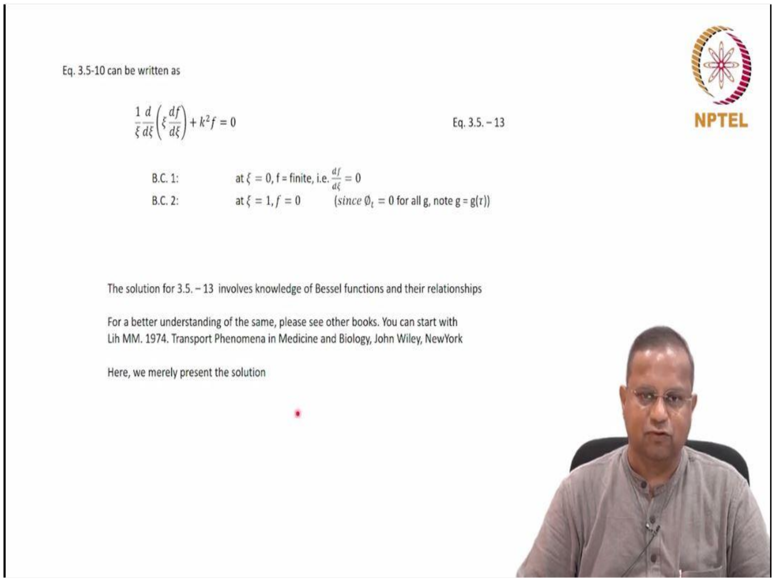

Equation 3.5-10 can be written as

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right)+k^2 f=0 \qquad (3.5\text{-}13)$$

The boundary conditions are given below

BC 1: At $\xi = 0$, f = finite, i.e. $\dfrac{df}{d\xi} = 0$

BC 2: At $\xi = 1$, $f = 0$ (since $\phi_r = 0$ for all g, note that $g = g(\tau)$)

Now, the solution of this needs a knowledge of Bessel functions, which you would have seen in your math course, I am not going to go into detail Bessel function, if I get into that from a very preliminary point of view that is an entire course by itself.

So I am not going to do that this happens to arise in this situation. Therefore, I am just going to draw from mathematical relationships of Bessel functions, just in the context of this solution to give you the search. That is what we going to do. If you want to better understand it, you could read books like this Lih MM 1974 transport phenomena in medicine and biology. The nice book this gives you a lot of detail.

You could read this book to get more insights about Bessel functions, or you could look at books on Bessel functions to know more about Bessel functions, the whole entire field there Bessel functions. Here we are going to merely present the solution.

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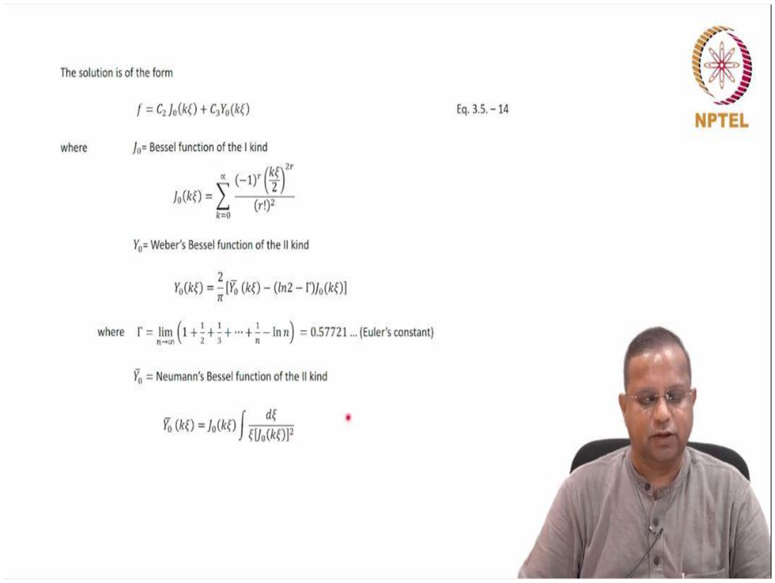

The solution for Eq. 3.5-13 requires knowledge of Bessel functions and their relationships. The student is directed to other appropriate books (e.g. Lih 1974) for a better understanding of the same. Here, we merely present the solution.

The solution is of the form

$$f = c_2 J_0(k\xi) + c_3 Y_0(k\xi) \tag{3.5-14}$$

where J_0 is a Bessel function of the I kind

so this is the Bessel function of the first thing and Y_0 is the Weber's Bessel function of the second kind.

$$J_0(k\xi) = \sum_{k=0}^{\alpha} \frac{(-1)^r \left(\frac{k\xi}{2}\right)^{2r}}{(r!)^2}$$

Y_0 is a Weber's Bessel function of the II kind

$$Y_0(k\xi) = \frac{2}{\pi}\left[\overline{Y}_0(k\xi) - (\ln 2 - \Gamma)J_0(k\xi)\right]$$

where $\Gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = 0.57721\ldots$ (Euler's constant),

and $\overline{Y}_0$ is a Neumann's Bessel function of the II kind.

$$\overline{Y}_0(k\xi) = J_0(k\xi)\int \frac{d\xi}{\xi[J_0(k\xi)]^2}$$

$C_3 = 0$ (from BC 1; otherwise the term would not be finite since $Y_0(0) = -\infty$)

$C_2\ J_0\ (k) = 0$ (from BC 2).

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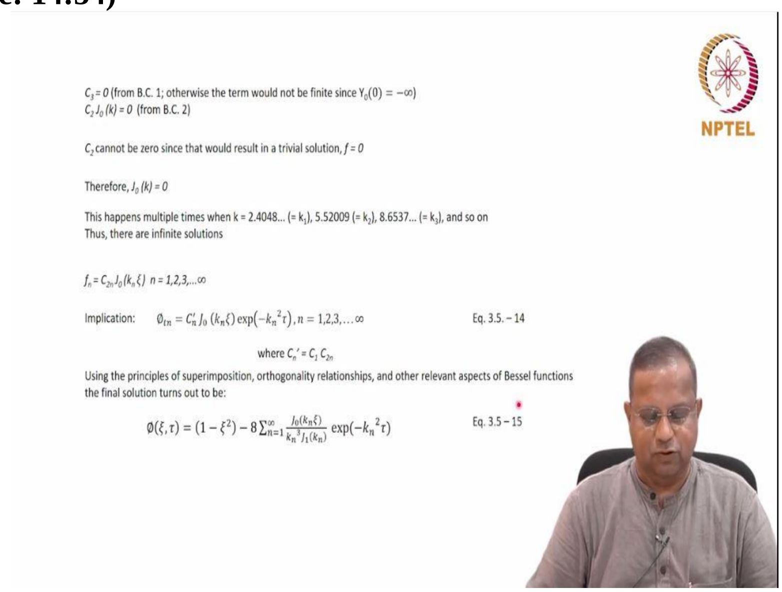

Be just taking this at face value going this fall in this in terms of what it turns out to be, I will not expect you to solve this in any aspect that requires solutions in this course, but this is what it involves, you need to know that now C 3 = 0 from boundary condition 1, otherwise the term would not be finite, and therefore, C 2 the Bessel function of k of the first kind of k = 0, and C 2 cannot be 0 since that would result in a trivial solution f = 0.

$$J_0\ (k) = 0$$

This happens multiple times when k = 2.4048... (= k_1), 5.52009 (= k_2), 8.6537... (= k_3), and so on.

Thus, there are infinite solutions

$$f_n = C_{2n}\ J_0\ (k_n\xi)\ n = 1,\ 2,\ 3,\ldots\ \infty$$

This implies that

$$\phi_{tn} = C_n' J_0(k_n\xi)\ \exp\ (-\ k_n^2\ \tau),\ n = 1,\ 2,\ 3,\ldots\ \infty$$

where $C_n' = C_1 C_{2n}$.

Using the principles of superimposition, orthogonality relationships, and other relevant aspects of Bessel functions, the final solution is

$$\phi(\xi,\tau) = (1-\xi^2) - 8\sum_{n=1}^{\infty}\frac{J_0(k_n\xi)}{k_n^3 J_1(k_n)}\exp(-k_n^2\tau) \qquad (3.5\text{-}15)$$

A representative plot of ϕ versus ξ for various values of τ is given in Fig. 3.5-1.

We are not going to none of the solutions would need you to manipulate on these aspects, but this is the way the solution comes you need to know at least the basis of this solution and the complexity of the solution 3.5 - 15.

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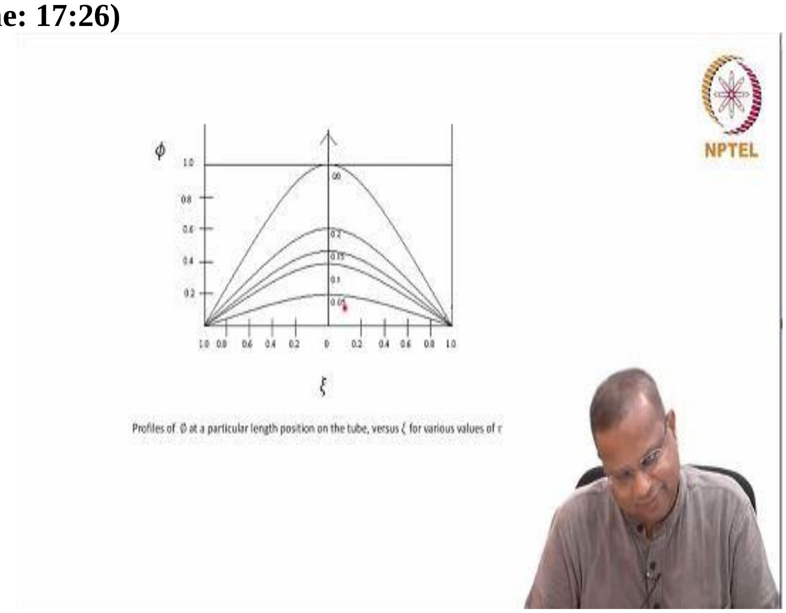

If you plot the solution here you have φ as a function of ξ that is what we are looking for. If you plot that, φ as a function of ξ, this is valid for any radius any viscosity any density and so on and so forth. At various τ's, this is 0.05 it will be something like this at 0.1 τ will be something like this 0.15 it will be like this 0.2 reveal like this and so on. Till τ reaches ∞, you get to your parabolic velocity profile, which is the steady state velocity profile n valued n laminar flow well develop laminar flow in a cylindrical pipe.

So, that is what I have for you here, unsteady state, I wanted to present something so that you have an idea as to what it is, do not worry about the mathematical complexity you will not be expected to do manipulations and the Bessel function demon in this course. But you need to know what a Bessel function is. And for those of you who are comfortable with it, please go ahead please get more insights into it and so on and so forth. Please do them.

Let us stop here for this class. We are out of time, out of patience suppose we lot of rigorous aspects, some of which we had to do blind because of the level of information that is needed. There is a lot more then a basic manipulation level, let us stop here. Let us meet to the next class. The next class also promises to be math full.

I am going to let you figure out by drawing what the problems are going to be and so on. But you need to know this all these are very relevant, the unsteady aspects happen in a lot of disease situations and so on. So in the arteries in it know about it, see in the next class, bye.