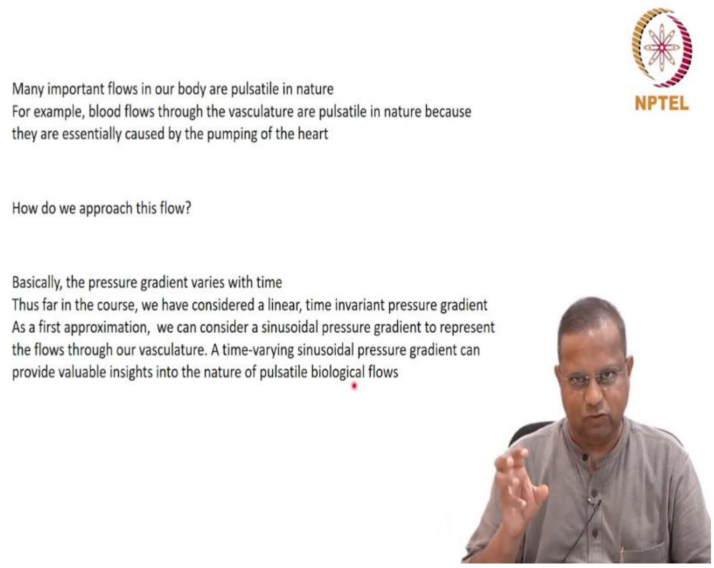
Transport Phenomena in Biological Systems
Prof. G.K. Suraishkumar
Department of Biotechnology Bhupat and Jyoti Mehta School of Biosciences building
Indian Institute of Technology – Madras

Lecture – 35
Pulsatile Flow

Welcome back to this class. Today we will look at pulsatile flow again in some detail of mathematical complexity as done for the unsteady flow I am not going to get too much into the mathematical basis for the various things I am just going to present the results the people who are so interested can go and pick up the details from other books and so on so forth the various aspects of involved functions.

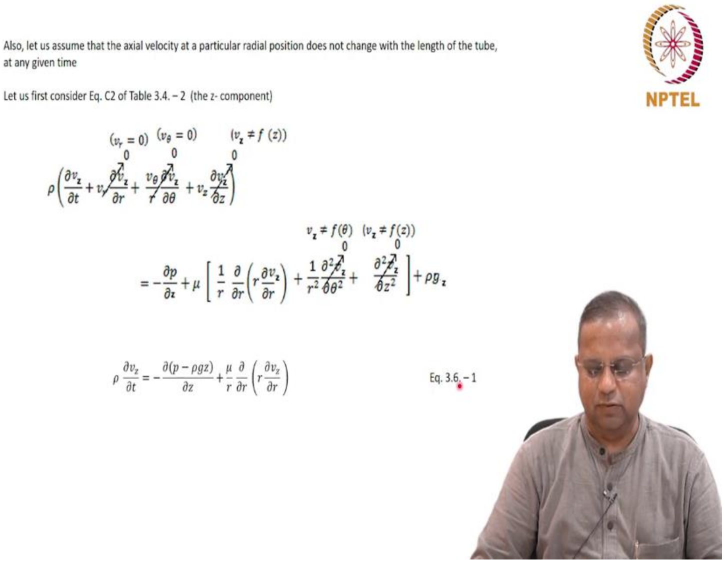
(Refer Slide Time: 00:49)



Let us go for it. Pulsatile flow and many important flows in our body are pulsatile in nature; we know that our heart pumps and therefore, the flow of blood is not a continuous one, but a pulsatile one. Let us see how to handle that. How do we approach this flow? If we look at the pressure gradient in this case, the pressure gradient is varying with time. So instead of having a constant ∆P, you have a time varying ∆T. That is the basic difference here.

So far we considered a linear time invariant pressure gradient. And now we need to consider time variant pressure gradient. As a first approximation, mathematical approximation, we can consider a sinusoidal pressure gradient to represent the flows of our vasculature. A time-varying sinusoidal pressure gradient can provide valuable insights into the nature of pulsatile biological flows, even though the actual variation could be a lot more complex. A first approximation to a time varying pressure drop can be obtained by using a sinusoidal variation of the pressure drop.

(Refer Slide Time: 02:10)



Let us assume that the axial velocity at a particular radial position does not change with the length of the tube this is the assumption that we made earlier also otherwise the flow becomes all over the place we cannot even handle it. And this approximation is not too bad and approximation so this is fine. Now, we will consider equation C2 from table 3.4 - 2, which gives us the Z component. So, it is this based on these assumptions of course it is varying with time the pressure drop is varying with time.

Therefore, the velocity will vary with time. So, this term remains there is no v_r here we said still the flows in laminae in this case cylindrical laminae therefore, v_z is not a function of r in any case the v_z is a function of the r there v_r there is no v_r here and therefore, the v_r set to 0. There is v_θ we have considered perpendicular thing therefore, there is no v_r here there is no v_θ here, and we v_z is not a function of z well developed flow let us say and therefore that can be set to 0.

From Eq. C2 of Table 3.4-2 (the z component of the equation of motion)

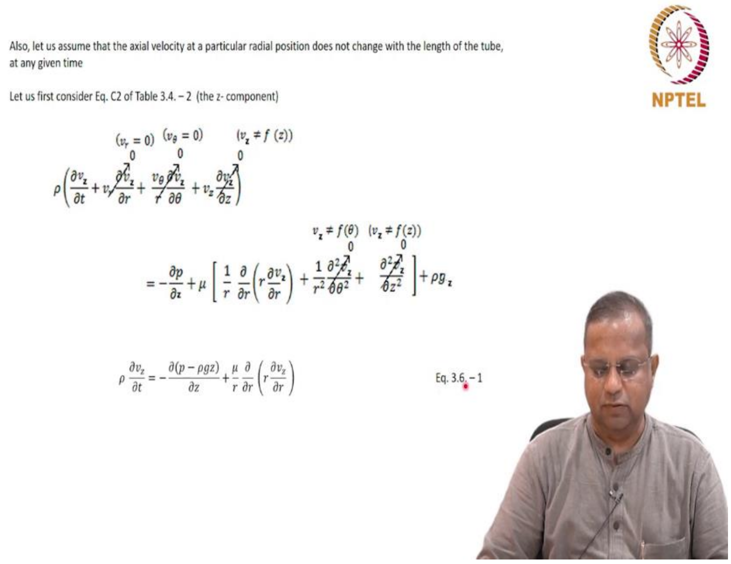
$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\underbrace{\frac{\partial v_z}{\partial r}}_{0\ (v_r=0)}+\frac{v_\theta}{r}\underbrace{\frac{\partial v_z}{\partial \theta}}_{0\ (v_\theta=0)}+v_z\underbrace{\frac{\partial v_z}{\partial z}}_{0\ (v_z\neq f(z))}\right)$$

$$=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{0\ v_z\neq f(\theta)}+\underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{0\ (v_z\neq f(z))}\right]+\rho g_z$$

Note that we have taken $v_z \neq f(z)$ at a particular time. Thus, the remaining terms yield

$$\rho\frac{\partial v_z}{\partial t}=-\frac{\partial(p-\rho g z)}{\partial z}+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \qquad (3.6\text{-}1)$$

(Refer Slide Time: 04:04)



By the same arguments as in Section 3.5 that led to Eq. 3.5-2, we can write

$$\frac{\partial P}{\partial z}=\frac{-\Delta P}{L}+A\sin\omega t \qquad (3.6\text{-}2)$$

where $\frac{-\Delta P}{L}$ is the average pressure gradient; A and ω are the frequency and amplitude, respectively, of the periodic pressure function.

I mentioned is the average pressure gradient, A and ω are the amplitude and frequency respectively it is switched around here; A is amplitude ω is the frequency of the periodic pressure function.

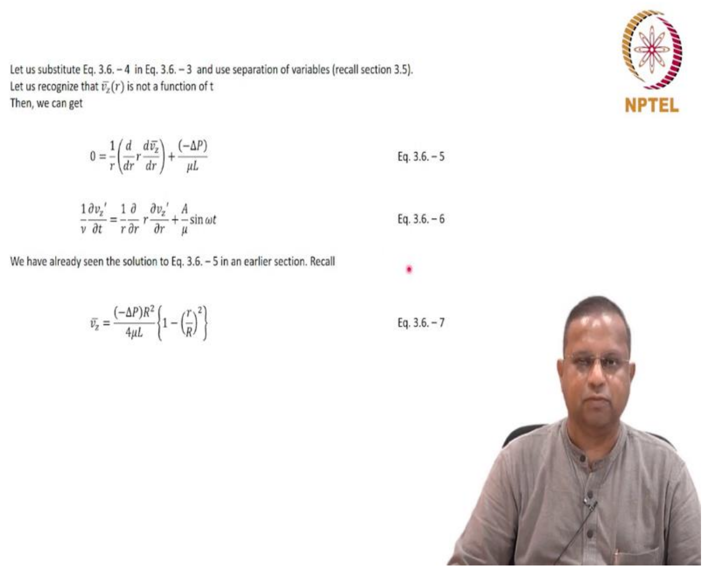
Since $\frac{\mu}{\rho} = \nu$, the equation of motion can be written as

$$\frac{1}{\nu}\frac{\partial v_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{(-\Delta P)}{\mu L} + \frac{A}{\mu}\sin\omega t \qquad (3.6\text{-}3)$$

We can guess that the solution for v_z consists of a steady state part (average value) and a periodic part (fluctuating value) corresponding to the average and fluctuating pressure gradients, i.e.

$$v_z\ (r,\ t) = \bar{v}_z(r) + v_z'\ (r,\ t) \qquad (3.6\text{-}4)$$

(Refer Slide Time: 05:52)



Substituting Eq. 3.6-4 in Eq. 3.6-3, and using separation of variables as in the Section 3.5 with the recognition that $\bar{v}_z(r)$ is not a function of t, gives two equations

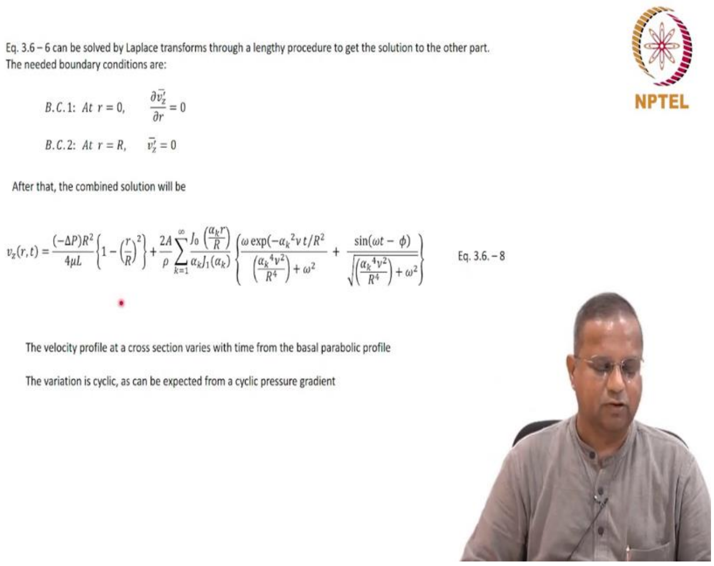
$$0 = \frac{1}{r}\left(\frac{d}{dr}r\frac{d\bar{v}_z}{dr}\right) + \frac{(-\Delta P)}{\mu L} \qquad (3.6\text{-}5)$$

$$\frac{1}{\nu}\frac{\partial v_z'}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_z'}{\partial r} + \frac{A}{\mu}\sin\omega t \qquad (3.6\text{-}6)$$

The solution of Eq. 3.6-5, as seen in an earlier section is

$$\bar{v}_z = \frac{(-\Delta P)R^2}{4\mu L}\left\{1-\left(\frac{r}{R}\right)^2\right\} \qquad (3.6\text{-}7)$$

(Refer Slide Time: 07:43)



Equation 3.6-6 can be solved by Laplace transforms through a lengthy procedure, with the boundary conditions as

$$\text{BC 1: At } r = 0, \quad \frac{\partial \overline{v_z'}}{\partial r} = 0$$

$$\text{BC 2: At } r = R, \quad \overline{v_z'} = 0$$

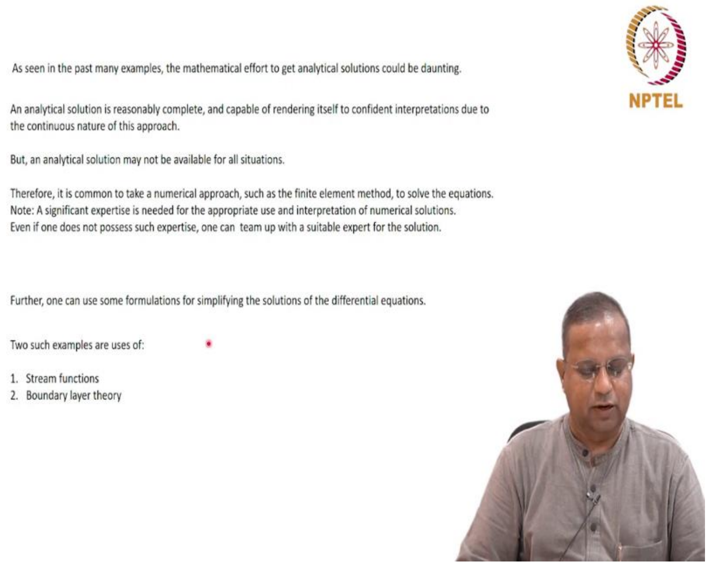
Basal function of the second kind of function of ω and so on and so forth a complex solution. What I would like you to do is just take some constant values for this c, use various values of r in an excel sheet, then plot the excel sheet for various r s at a given time. And see what the kind of variation is, what is the kind of profile that you get for v_z as a function of t for a particular t, v_z is a function of r for a particular t, then change t and then you get another curve.

Then change t then you get another curve and see how the velocity profile evolves in pulsatile flow with time. The velocity profile as a variation of v_z with r how that evolves with time in pulsatile flow, the solution is this equation 3.6 - 8 and how it varies from the basal parabolic profile is what I would like you to find out the variation cyclic, as you would expect, because the pressure variation the pressure drop variation is cyclic sine wave and therefore, the variation velocity will also be cyclic.

$$v_z(r,t)=\frac{(-\Delta P)R^2}{4\mu L}\left\{1-\left(\frac{r}{R}\right)^2\right\}$$
$$+\frac{2A}{\rho}\sum_{k=1}^{\infty}\frac{J_0\left(\frac{\alpha_k r}{R}\right)}{\alpha_k J_1(\alpha_k)}\left\{\frac{\omega\exp(-\alpha_k^2 \nu t/R^2)}{\left(\frac{\alpha_k^4 \nu^2}{R^4}\right)+\omega^2}+\frac{\sin(\omega t-\phi)}{\sqrt{\left(\frac{\alpha_k^4 \nu^2}{R^4}\right)+\omega^2}}\right\} \tag{3.6-8}$$

Thus, the velocity profile at a cross-section varies with time from the basal parabolic profile. The variation is cyclic, as can be expected from a cyclic pressure gradient.

(Refer Slide Time: 09:57)



Now, let me present a few thoughts and leave you at that mathematical effort to get analytical solutions can be daunting. However, we prefer analytical solutions because it is reasonably complete although with assumptions, with those assumptions can be verified. If they are not appropriate, then you need to go back and rework the solutions with appropriate assumptions. And they can render themselves to confident interpretations due to the continuous nature of analytical solutions.

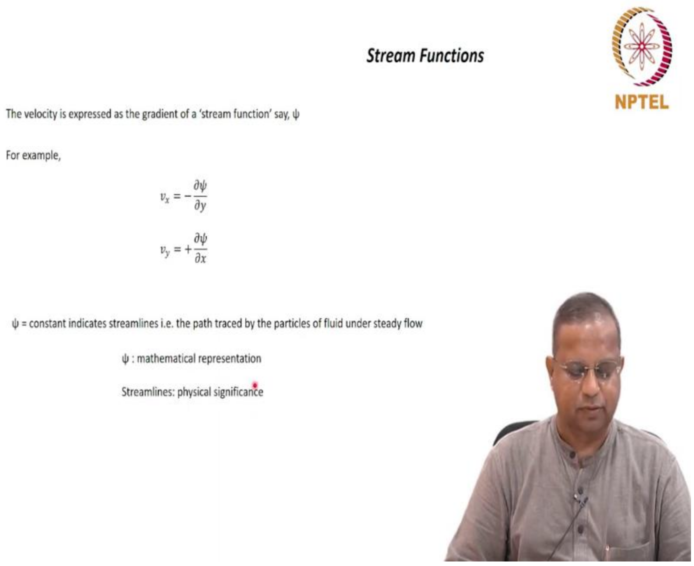
In numerical solutions, you do not have this continuous nature. So, we are always limited by that. But if you do not get an analytical solution at all, what do you need to get to numerical solutions. Analytical solutions may not be available for all situations even this was quite daunting. Therefore, it is common to take a numerical approach such as a finite element method to solve the equations.

A significant expertise is needed for appropriate use and interpretation of numerical solutions. It is not easy either, but at least it gives you some solution. Even if one does not pursue such expertise, one can always team up with a suitable expert in numerical solutions for this solution. That is why people work in teams, you come up with something, you may not be an expert in numerical solutions, but you have a colleague who is an expert.

So you go and talk to that person and get that person's expertise as to how to interpret these numbers and whether it makes sense at all to begin with and then how do you get deeper insights into it? And then both these expertise is together is what solves a bigger problem. That is why people working things. Further one can use some formulations for simplifying the solutions of differential equations. These are old formulations.

Let me just present them these are well established formulations. I am just going to tell you the basis of the formulation nothing else and just 2 examples of those. The first example is that of use of stream functions, what are called stream functions. The second is use of something called a boundary layer theory.

(Refer Slide Time: 12:10)

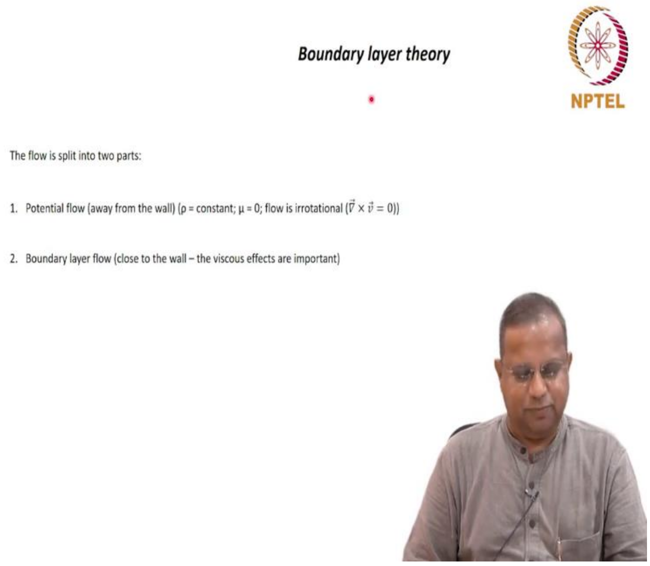


Let me explain what these are stream functions or something like this. The velocity is expressed as a gradient of certain potential or a certain stream function ψ. So, if it is a 2-dimensional case v_x is represented as a negative of the y derivative of ψ the stream function as a potential. So,

v_x is - $\frac{\partial\psi}{\partial y}$and v_y is expressed as + $\frac{\partial\psi}{\partial x}$. So this formulation significantly reduces the mathematical tedium and ψ indicates streamlines the path trace by particles of fluid under steady flow.

So it has a physical relevance. ψ is not conceptual one, it also has a physical relevance, it is still conceptual. Streamlines are conceptual, but these are constant amount streamlines.

(Refer Slide Time: 13:16)



The next thing is boundary layer flow. Boundary Layer theory assumes or splits the flow into 2 regions. For example, if you have a flow through a pipe, that is a bad example; flow over a flat plate, there is a certain region that is affected by the flat plate, certain region and the flow that is affected by the plate. And there is a huge region that is not affected by the plate. So you split those and then solve them separately and then put them together.

That is all the boundary layer theory allows you to do. It is a lot more complex, when there is an effect of the plate on the flow on the boundary layer effects is coming. So, this to tell you the essence of it is essentially the flow is split into 2 parts, potential flow away from the wall where you can assume density to be a constant, viscosity is actually 0 and the flows is rotational flows a rotation mathematically it is expressed as $\nabla . v = 0$.

That is how rotation flows defined and a boundary layer flow is close to the wall here the viscous effects are important the viscous effects at all the ones that bring in the second derivatives of velocities with respect to space, and all those complications are addressed close to the wall. So, this is normally done stream function approach boundary layer approach is normally taken.

When you come across these the literature you would not be lost you know what this is and then you can get into the formulation is the deeper formulation of mathematics and so on so forth and understand that better if you need to understand. So, in this course, we have given you just a peek into that this the basis of that you can develop on that for your own needs as you go along. I think we need to stop here for this class let us continue with something else in the next class. I think we are going to look at turbulent flow. Next class, we will do that when we come back again soon. See you.