Transport Phenomena in Biological Systems
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Lecture - 42
Friction Facets for Packed Beds

Welcome today we look at the friction factor for packed beds; beds of many different materials are very relevant for biological engineers. We will see some examples when we in the next slide. So we are going to look at the friction factor for packed beds.

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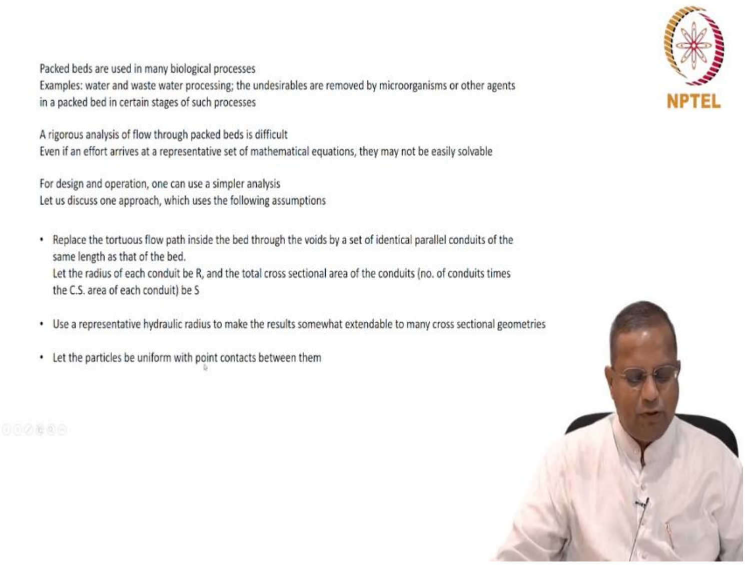

Packed beds are used in many biological processes for example water and waste water processing the undesirables are removed by microorganisms or other agents in a packed bed of soil for example in certain stages of such processes, a rigorous analysis of flow through packed beds is rather difficult right it is flowing through tortuous channels in a bed of bits is soil. So rigorous analysis becomes difficult even if an effort arrives at a representative set of mathematical equations from first principles they may not be easily solved.

This is normal difficulty, however packed beds are widely used and therefore we need some way of having some inside into them some understanding so that we can use it for designing those operations and for operation of this. One can use a simpler analysis for such a thing and this whatever we are going to see now the friction factor approach gives us one such simpler analysis approach. We are going to replace the tortuous flow path inside the beds through the voids by a set of identical parallel conduits of the same length as that of the bed.

So the actual path could be something like this, so we are going to replace all those by path through a series of parallel cylindrical conduits or parallel conduits itself usually cylindrical identical parallel conduits of the same length as that of the bed. Let the radius of each conduit be R, so it is cylindrical and the total cross sectional area of the conduits is number of conduits times the cross sectional area for each conduits.

That will take to be S, we will use a representative of hydraulic radius to make the result somewhat extendable to many cross sectional geometries, you can consider other cross sections also and as long as we use hydraulic radius cross sectional area by wetted perimeter remember the definition we can extend this to other geometries also. Let the particles be uniform with point contacts between them this is an idealize situation for analysis. And we will assume laminar flow in the conduits to begin.

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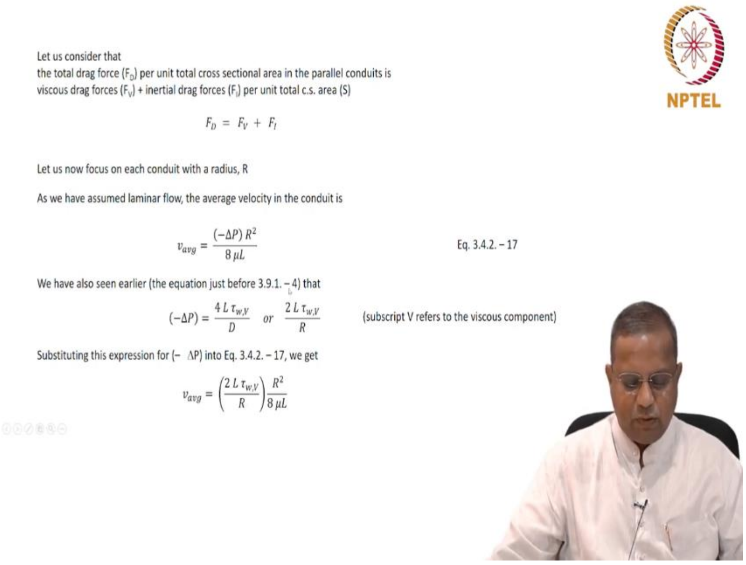

The total drag force per unit cross sectional area in the parallel conduits is the viscous drag force plus inertial track force per total cross section area. Something called the viscous drag forces, something called the inertial track forces. So F_D the total drag force is the sum of the viscous drag forces per unit area plus the inertial drag forces per unit area. Now let us focus on each conduit with radius R we have assumed laminar flow.

And therefore the average velocity in the conduit is given by the laminar flow expression we have rich average velocity in a laminar flow is

Now, let us focus our attention on each conduit with radius R, for a while. From Eq. 3.4.2-17 the average velocity in the conduit is

$$v_{\text{avg}} = \frac{(-\Delta P)R^2}{8\mu L} \qquad (3.4.2\text{-}17)$$

V is just added to emphasize the viscous component or w is good enough to watch hear but V is added to emphasize the viscous component.

And this is normally found in the formulations this is there the book, so we will follow the same thing here. Now substituting this expression for -Δ P into the previous expression and this 3.4.2 -17,

$$v_{z,\text{avg}} = \frac{(-\Delta P)\times R^2}{2\mu L\times 4} = \frac{(-\Delta P)R^2}{8\mu L} = \frac{1}{2}(v_{z,\max}) \qquad (3.4.2\text{-}17)$$

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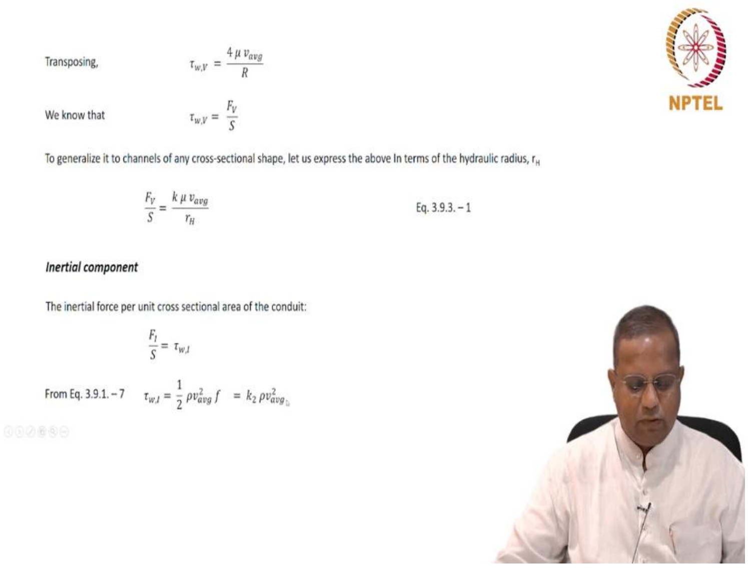

We also know from the equation just before Eq. 3.9.1-4 that

$$(-\Delta P) = \frac{4L\tau_{w,V}}{D} \quad \text{or} \quad \frac{2L\tau_{w,V}}{R}$$

The subscript V refers to the viscous component. Substituting the above expression for $(-\Delta P)$ into Eq. 3.4.2-17, we get

$$v_{\text{avg}} = \left(\frac{2L\tau_{w,V}}{R}\right)\frac{R^2}{8\mu L}$$

Transposing, we get

$$\tau_{w,V} = \frac{4\mu v_{\text{avg}}}{R} \text{ and we know that } \tau_{w,V} = \frac{F_V}{S}$$

And we already know that the shear stress is the viscous force divided by the surface area. And to generalize it to channels of any cross section we are going to express it in terms the hydraulic radius cross sectional area by wetted perimeter.

In terms of the hydraulic radius, r_H (to generalise it to channels of any cross-sectional shape)

$$\frac{F_V}{S} = \frac{k\mu v_{\text{avg}}}{r_H} \qquad (3.9.3\text{-}1)$$

Now, let us look at the inertial component. The inertial force per unit cross-sectional area of the conduit

$$\frac{F_I}{S} = \tau_{w,I}$$

From Eq. 3.9.1-7, we get

$$\tau_{w,I} = \frac{1}{2}\rho v_{\text{avg}}^2 f = k_2 \rho v_{\text{avg}}^2 \qquad (3.9.3\text{-}2)$$

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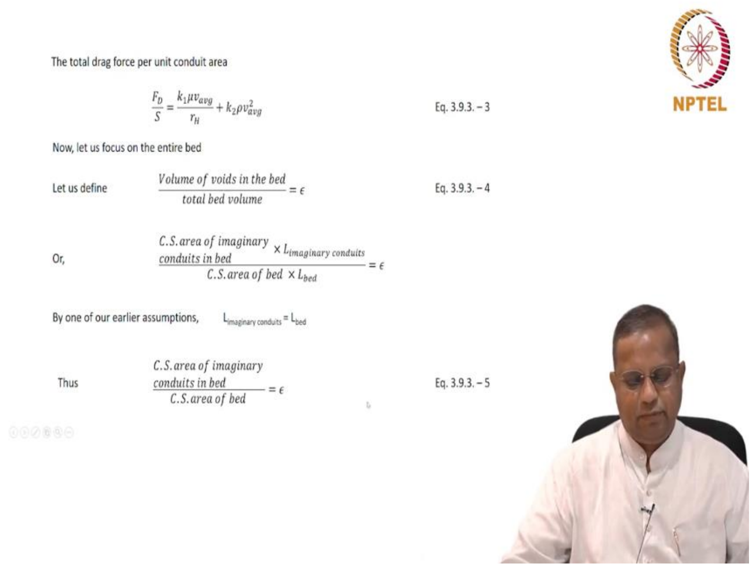

Thus, the total drag force per unit conduit area according to the summative consideration of the viscous and inertial components, is

$$\frac{F_D}{S} = \frac{k_1 \mu v_{\text{avg}}}{r_H} + k_2 \rho v_{\text{avg}}^2 \tag{3.9.3-3}$$

Now, let us focus on the entire bed. Let us define

$$\frac{\text{Volume of voids in the bed}}{\text{Total bed volume}} = \in \tag{3.9.3-4}$$

In other words

$$\frac{\text{CS area of imaginary conduits in bed} \times L_{\text{imaginary conduits}}}{\text{CS area of bed} \times L_{\text{bed}}} = \in$$

So cross section area the imaginary condition the bed times the length of the imaginary conduits, so this gives we are just equating the volumes here. So this the total volume of these straight pipes straight conduits must equal the volume of voids in the bed that is essentially what this says the cross section area of the imaginary conduits and bed times the length of the imaginary conduits that gives you the volume of the imaginary conduits that must equal the volume of beds.

And CS area of bed times the length of the bed is nothing but area of the bed length to the bed is nothing but the total bed volume that must be here epsilon void fraction this is the void fraction porosity void fraction by one of our earlier assumptions the length of the imaginary conduit is the same as the length of the bed. So this on the scan can be canceled. Therefore the cross sectional area of the imaginary conduits in the bed divided by the cross sectional area of the bed is your epsilon equation 3.9.3 - 5.

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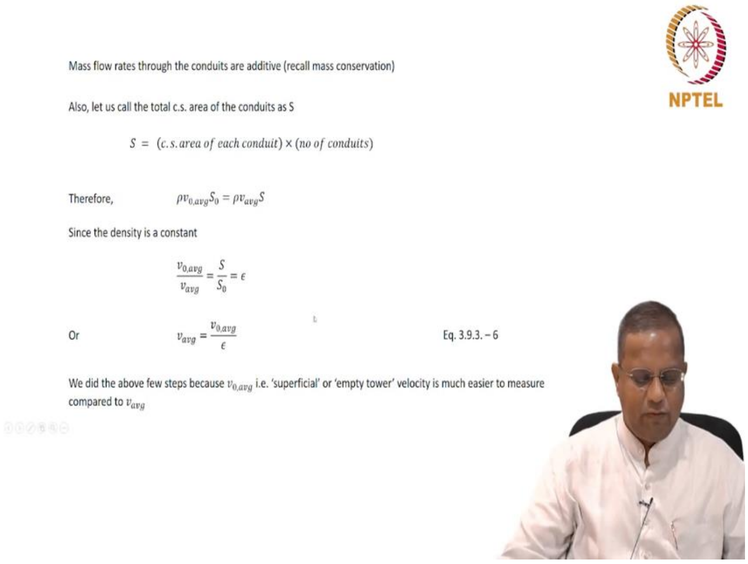

The mass flow rates through the conduits are all additive masses are scalar the mass flow rates are all additive also the total cross sectional area of the conduits we are going to call the total area of the conduits as S. So the S is nothing but the cross sectional area of each conduit times the number of conduits. Therefore $\rho\, v_{0,avg}\, S_0 = \rho v_{,\, avg}\, S$. So mass times the velocity times the area has to be the same because we are considering 2 different situations here. Since density is a constant we can cancel out those this is nothing but mass conservation nothing else; mass rate here must mass rate equal here mass by unit volume and so on so forth you can work out the units turn out to be mass rate.

By mass conservation, since the mass flow rates through the conduits are additive, and S = total number of conduits × cross-sectional area of each conduit.

$$\rho v_{0,avg} S_0 = \rho v_{avg} S$$

Since the density is a constant

$$\frac{v_{0,avg}}{v_{avg}} = \frac{S}{S_0} = \in$$

or

$$v_{avg} = \frac{v_{0,avg}}{\in} \qquad (3.9.3\text{-}6)$$

$v_{0,avg}$ i.e. 'superficial' or 'empty tower' velocity is much easier to measure compared to v_{avg}.

We did the above few steps because $v_{0\ average}$ the superficial velocity before it reaches the bed the velocity of the flow rate before it reaches the bed or the empty tower velocity is much easier to measure compared to the velocity in each conduits each of these imaginary conduits or conceptual conduits that we have looked at. It is a lot more difficult to measure that they are conceptual anyway.

Whereas this is this can be measured you know the cross sectional area by the wetted perimeter especially when before it reaches the bed is very easy to measure. Sorry cross sectional area by bedded perimeter is your hydraulic radius here we are looking at volumetric flow rate divided by the area that will give you the velocity.

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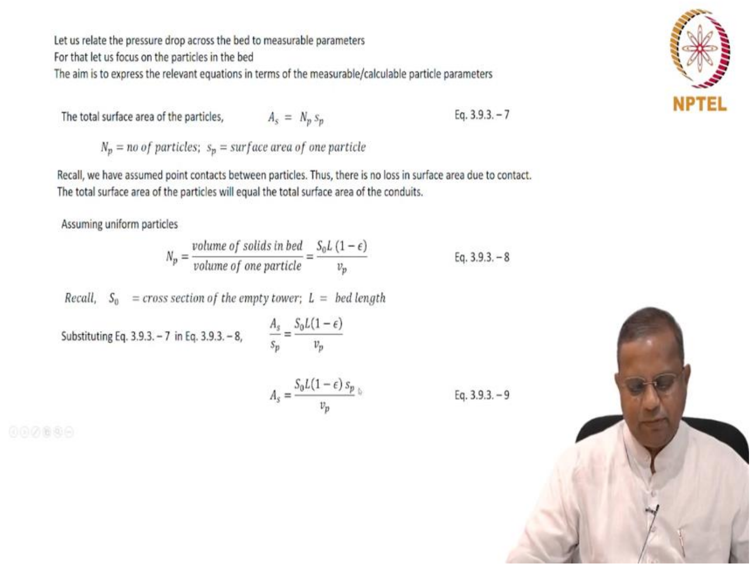

Let us relate the pressure drop across the bed to measurable parameters. And let us focus on the particles in the bed. The aim is to express the relevant equations in terms of measurable and calculable particle parameters. That is what we are trying to do here remember we are looking at the bed we are approaching the bed in some way with our view so that we can get some useful things for design and operation.

Now, let us relate the pressure drop across the bed to measurable parameters. To do that let us focus on the particles in the bed for a while. The aim is to express the relevant equations in terms of the measurable/calculable particle parameters.

The total surface area of the particles is A_S

$$A_s = N_p\ s_p \qquad (3.9.3\text{-}7)$$

where N_p is total number of particles in the bed and s_p is area of one particle.

Set here recall we have assumed point contacts between particles.

There is no loss in surface area due to contact the total surface area of particles will equal the total surface area the conduits. And if you assume uniform particles then the number of particles is nothing but the volume of solids in the bed divided by the volume of a particle. So volume of solids in the bed is nothing but S_0 L; S_0 is the surface area length this will give you the volume. Volume you subtract the void volume which is epsilon times the total volume from the total volume you get the volume of solids.

Assuming uniform particles

$$N_p \text{ is also} = \frac{\text{Volume of solids in bed}}{\text{Volume of one particle}} = \frac{S_0 L(1-\in)}{v_p} \qquad (3.9.3\text{-}8)$$

where S_0 is cross-section of empty tower and L is bed length.

Substituting Eq. 3.9.3-7 in Eq. 3.9.3-8, we get

$$\frac{A_s}{s_p} = \frac{S_0 L(1-\in)}{v_p}$$

$$A_s = \frac{S_0 L(1-\in)s_p}{v_p} \qquad (3.9.3\text{-}9)$$

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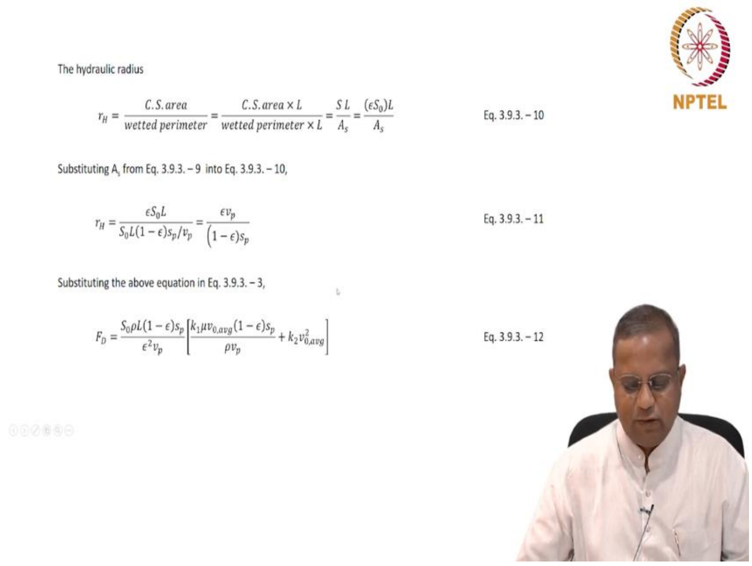

$$r_H = \frac{\text{CS area}}{\text{Wetted perimeter}} = \frac{\text{CS area} \times L}{\text{Wetted perimeter} \times L} = \frac{S\,L}{A_s} = \frac{(\in S_0)L}{A_s} \qquad (3.9.3\text{-}10)$$

In the equation above, since we have assumed point contacts between particles, and hence there is no loss in surface area due to contact, the total surface area of the particles will equal the total surface area of the conduits.

Substituting A_s from Eq. 3.9.3-9 into Eq. 3.9.3-10, we get

$$r_H = \frac{\in S_0 L}{S_0 L(1-\in)s_p / v_p} = \frac{\in v_p}{(1-\in)s_p} \qquad (3.9.3\text{-}11)$$

Substituting the above equation in Eq. 3.9.3-3, we get

$$F_D = \frac{S_0 \rho L(1-\in)s_p}{\in^2 v_p}\left[\frac{k_1 \mu v_{0,\text{avg}}(1-\in)s_p}{\rho v_p} + k_2 v_{0,\text{avg}}^2\right] \qquad (3.9.3\text{-}12)$$

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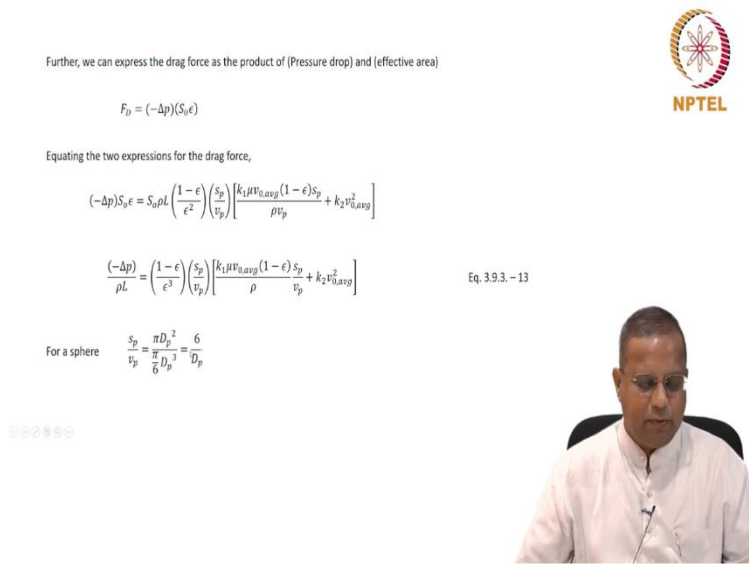

And we can express the drag force as a product of the pressure drop times effective area.

We can also express the drag force as the product of (pressure drop) and (effective area), i.e.

$$F_D = (-\Delta p)(S_0\in)$$

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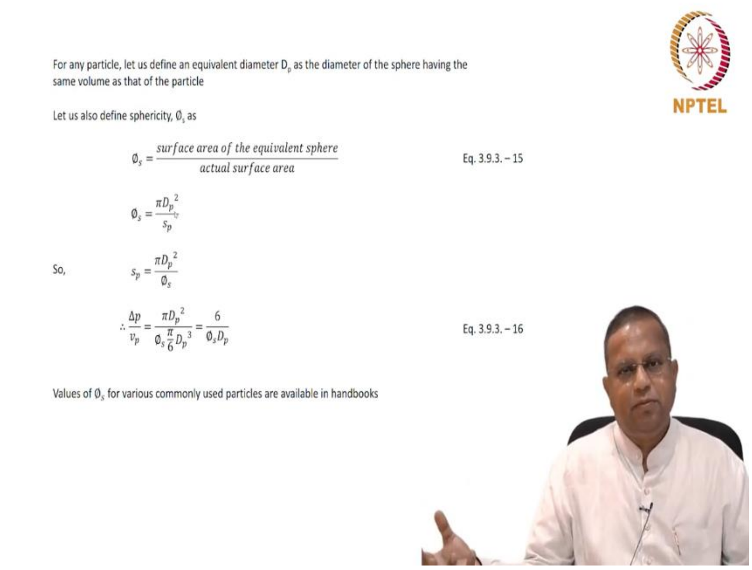

Equating the two expressions for the drag force, we get

$$(-\Delta p)\, S_0 \in = S_0 \rho L \left(\frac{1-\in}{\in^2}\right)\left(\frac{s_p}{v_p}\right)\left[\frac{k_1 \mu v_{0,\mathrm{avg}}(1-\in) s_p}{\rho v_p} + k_2 v_{0,\mathrm{avg}}^2\right]$$

$$\frac{(-\Delta p)}{\rho L} = \left(\frac{1-\in}{\in^3}\right)\left(\frac{s_p}{v_p}\right)\left[\frac{k_1 \mu v_{0,\mathrm{avg}}(1-\in)}{\rho}\frac{s_p}{v_p} + k_2 v_{0,\mathrm{avg}}^2\right] \qquad (3.9.3\text{-}13)$$

For a sphere

$$\frac{s_p}{v_p} = \frac{\pi D_p^2}{\frac{\pi}{6} D_p^3} = \frac{6}{D_p} \qquad (3.9.3\text{-}14)$$

And so for any particle if we define an equal diameter there is no guarantee that the particles are all spheres. So let us define an equivalent diameter and the same length as that of the equal radius; so equivalent diameter is the diameter of the sphere having the same volume as that of the particle and sphericity as the surface area of the equivalent sphere divided by the actual surface area. So this gives us some link between the idealized sphere and the actual particle.

So we take a certain type of sphericity and we can substitute that and the analysis that we have done for spherical particles were kind of extended to other particle shapes also by the use of this sphericity.

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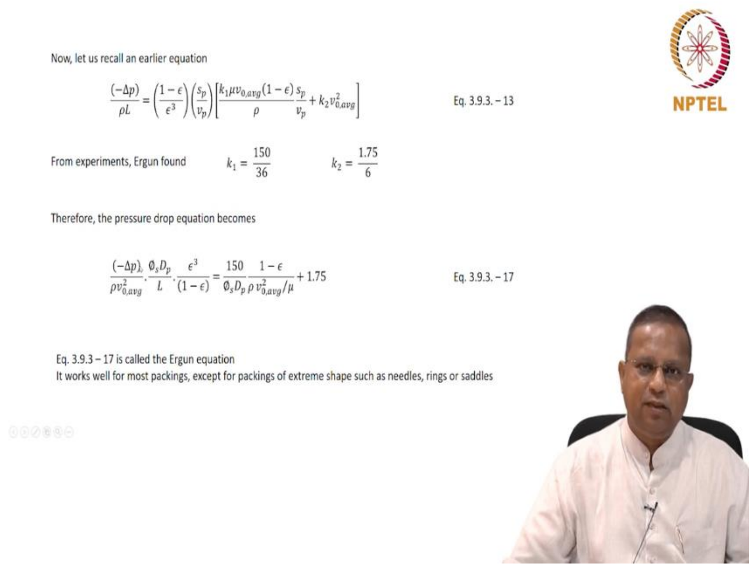

Now let us recall an earlier equation which is this we have already seen this; this comes from the drag force coming from 2 components and so on so forth 3.9.3 - 13 from experiments Ergun scientist he found that k_1 turns out to be 150 / 36 over a wide range of experimentation k_2 turns out to be 1.75 / 6 and therefore for a packed bed the pressure drop becomes pressure drop is one of the important design parameters.

Values of ϕ_s for various commonly used particles are available in handbooks.

Ergun correlated experimental data and found that

$$k_1 = \frac{150}{36} \quad \text{and} \quad k_2 = \frac{1.75}{6}$$

Thus, the pressure drop equation can be written as

$$\frac{(-\Delta p)}{\rho v_{0,\mathrm{avg}}^2} \cdot \frac{\phi_s D_p}{L} \cdot \frac{\epsilon^3}{(1-\epsilon)} = \frac{150}{\phi_s D_p} \frac{1-\epsilon}{\rho v_{0,\mathrm{avg}}^2 / \mu} + 1.75 \qquad (3.9.3\text{-}17)$$

Equation 3.9.3-17 is called the Ergun equation. The above equation works well for most packings – except for packings of extreme shape such as needles, rings or saddles.

So that we obtained by using the friction factor approach for pack beds. This is highly useful in the design of pack beds and it works on most packings except packings of extreme shapes such as needles, rings or saddles. It is not so reliable or we get an estimate out of this and then we need to use a huge what can I say fudge factor or a safety factor and then design that is what we do.

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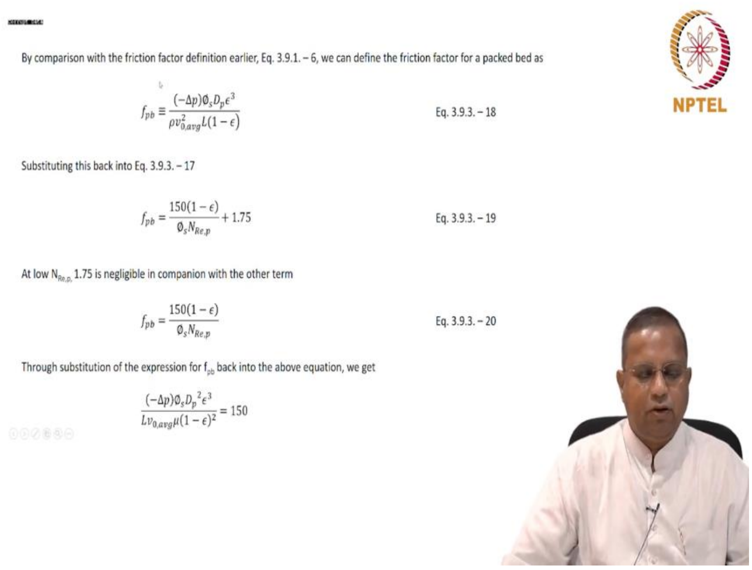

So by comparison to the friction factor defined earlier we can define a friction factor for a packed beds

By comparison with the friction factor defined earlier, Eq. 3.9.1-6, we can define the friction factor for a packed bed as

$$f_{pb} \equiv \frac{(-\Delta p)\phi_s D_p \in^3}{\rho v_{0,\text{avg}}^2 L(1-\in)} \qquad (3.9.3\text{-}18)$$

Substituting this back into Eq. 3.9.3-17, we get

$$f_{pb} = \frac{150(1-\in)}{\phi_s N_{\text{Re},p}} + 1.75 \qquad (3.9.3\text{-}19)$$

At low $N_{\text{Re},p}$, 1.75 is negligible in comparison with the other term. Thus, at low $N_{\text{Re},p}$

$$f_{pb} = \frac{150(1-\in)}{\phi_s N_{\text{Re},p}} \qquad (3.9.3\text{-}20)$$

This implies that (through substitution of the expression for f_{pb} back into the above equation)

$$\frac{(-\Delta p)\phi_s D_p^2 \in^3}{L v_{0,\text{avg}} \mu (1-\in)^2} = 150$$

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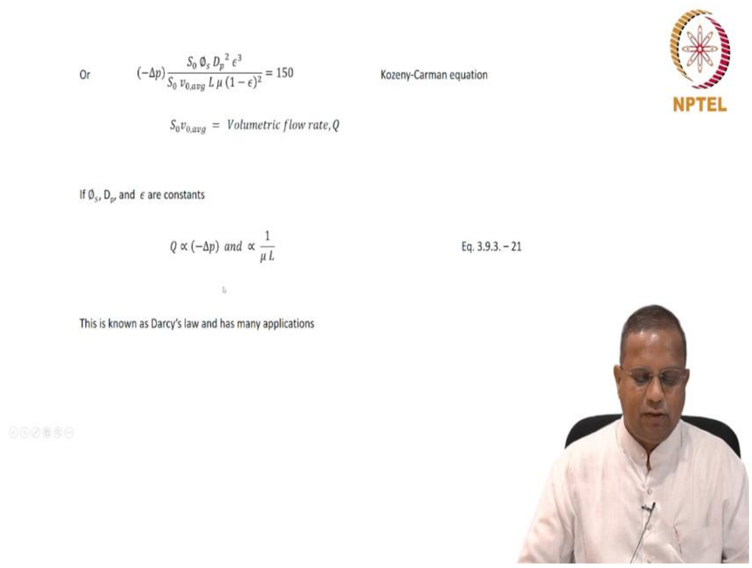

This is called the Kozeny Carman equation which is also useful. S_0 $v_{0 \text{ average}}$ is the volumetric flow rate and f is sphericity the diameter of the particle and the void fraction are constants the flow rate is directly proportional to the pressure drop and inversely proportional to the product of the radius the product of the viscosity times the length of the bed, so this is what it turns out to be $S_0\ v_{0\ average}$ is the flow rate we are taking it to the other side.

$$(-\Delta p)\frac{S_0 \phi_s D_p^2 \in^3}{S_0 v_{0,\text{avg}} L \mu (1-\in)^2} = 150$$

where $S_0 v_{0,\text{avg}}$ is volumetric flow rate, Q. The above equation is called the Kozeny-Carman equation.

If ϕ_s, D_p, and $\in$ are constants

$$Q \propto (-\Delta p) \text{ and } \propto \frac{1}{\mu L} \qquad (3.9.3\text{-}21)$$

This is known as Darcy's law and has many applications.

This expression is important this relationship is important. The flow rate is directly proportional to the pressure drop and inversely proportional to the viscosity times the length is called the Darcy's law and it is very, very wide applications.

Darcy's law is used in research significantly in a fruit the application is only limited by imagination as long as you have a have something that can be viewed as a packed bed even fruits then you know the motion of the fluid through the fruit and so on so forth then you can apply to Darcy's law there flow rate is directly proportional to the pressure drop inversely proportional to μ times L.

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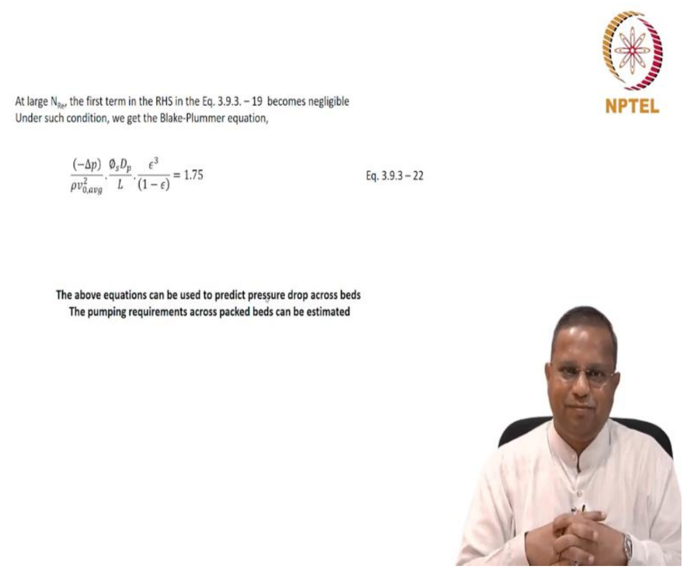

At large Reynolds numbers the first term in the right hand side of 3.9.3 - 19 becomes negligible what is that 3.9.3 - 19 at large Reynolds numbers you have Reynolds number in the denominator therefore this term becomes negligible compared to 1.75. And therefore the friction factor of the packed bed at large Reynolds number equals 1.75 which is called the Blake plummer equation this expanded you get equal to 1.75, 3.9.3 - 22.

At large N_{Re}, the first term in the RHS in Eq. 3.9.3-19 becomes negligible. Under such condition, we get the Blake-Plummer equation, i.e.

$$\frac{(-\Delta p)}{\rho v_{0,avg}^2} \cdot \frac{\phi_s D_p}{L} \cdot \frac{\in^3}{(1-\in)} = 1.75 \qquad (3.9.3\text{-}22)$$

The above equations can be used to predict pressure drop across beds. Hence, the pumping requirements across packed beds can be estimated.

And the above equations can be used to predict pressure drop across beds pressure drop is a very important design parameter. And we have seen a way by which we can get a measure of the pressure drop by using the various measurable parameters. The pumping requirements across pack beds can be estimated once you know the pressure and so we have seen the application of the friction factor approach to various different situations to a piping network consisting of straight pipes.

And pipe fittings to a contraction stenosis in a biological situation for motion relative motion of a solid and a liquid through a liquid relative motion between a solid and a liquid and now pack beds, pack beds are very widely used. We will stop here for this class. When we come back we will review the momentum flux chapter large chapter momentum flux chapter and then we will move forward. See you then. Bye.