Transport Phenomena in Biological Systems
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Lecture - 43
Review of Momentum Flux

Welcome back in the previous lecture, we completed the chapter on momentum flux, as we have been doing so far in the course, let us do a quick review of whatever we have done in the chapter on momentum flux, before we move forward, that would help you consolidate the information, the understanding, improve the understanding at this level, understanding at various levels of depth goes on and on for very many years.

So, this is at this level that we are talking about for all that this becomes helpful. And it is also nice to have everything in one place where you can go and refer to once you reach a certain level of understanding, let us begin we looked at started looking at momentum flux a while ago few weeks ago.

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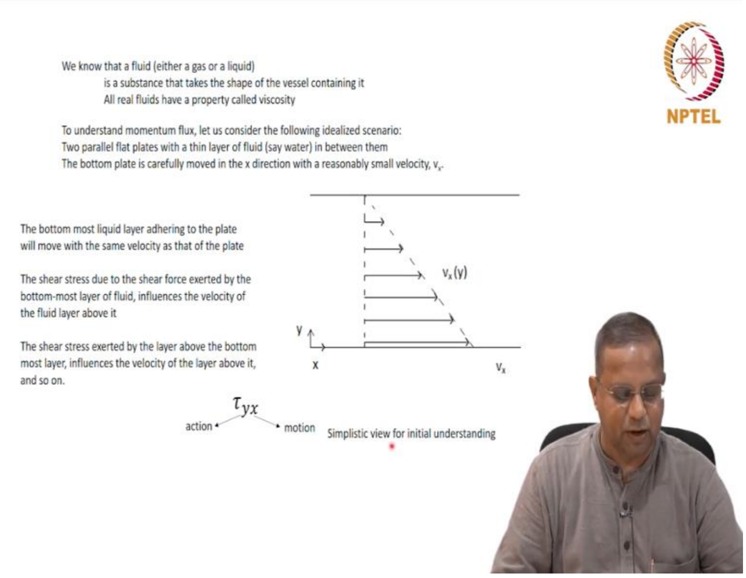

We said that fluids or this becomes highly relevant in a fluid system. Of course, momentum is there even in the particles and so on and so forth that you are very comfortable with so far. We started out by looking at the flow in a thin layer of fluid between 2 flat planes when the bottom plane is moved with a small velocity of v in the positive x direction. Then we talked about the effect of this movement in the perpendicular direction being caused by a shear stress τ_{yx}.

The first subscript is the direction of the gradient, that is y in this case or the direction of action, as I called it, and x is the direction of motion, the direction of velocity. So, the direction of velocity, the direction of velocity gradient and that is what we use to identify the various shear stresses, as we called it. Then, and I also said that this is a simplistic view for an initial understanding, later on we found in a 3 dimensional case where multiple velocity gradients contributed to the same shear stress.

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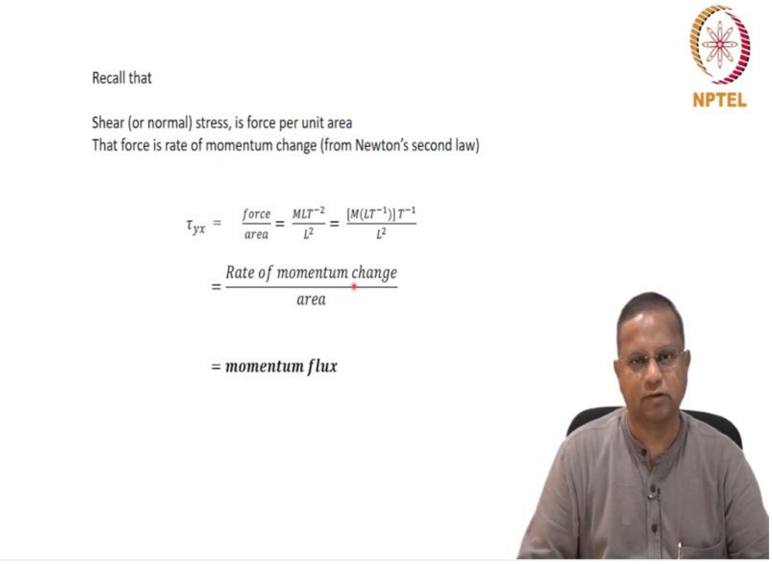

And, we also looked at how a shear stress can be interpreted as momentum flux, the rate of momentum change per area. I had shown it to you from dimensions point of view that would give you an idea therefore, we have brought it in the context of a flux, which has been a common theme across various aspects in this course.

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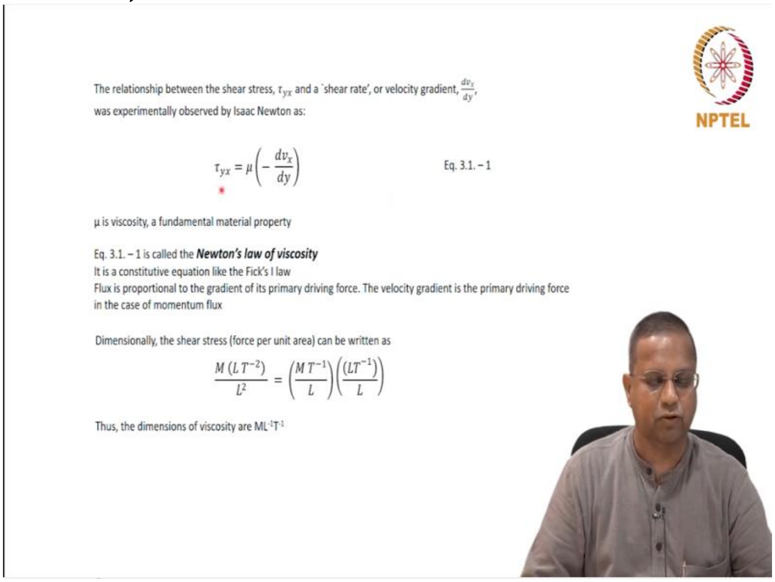

Then we talked about Newtonian fluids, which have this kind of relationship given by Newton based on his experiments, the shear stress is directly proportional to the shear rate or the velocity gradient, the negative of it and the constant of proportionality is called the viscosity. It is also called the Newtonian viscosity at times, the negative you know how that comes about, because we typically take (v_2 - v_1/y_2 - y_1).

However, this would be proportional to v_1 - v_2 therefore, you have a negative thing that is coming in here and viscosity is a fundamental material property. Also the Newton's law of viscosity is a constitutive equation similar to the Fick's first law of motion in mass flux. The also we saw the units of viscosity the dimensions of viscosity are M L^{-1} T^{-1}. I showed you a way to work out such things from the basal relationships when you know the dimensions of the other quantities. Then, we looked at the rheological properties; let me go directly to a more comprehensive figure.

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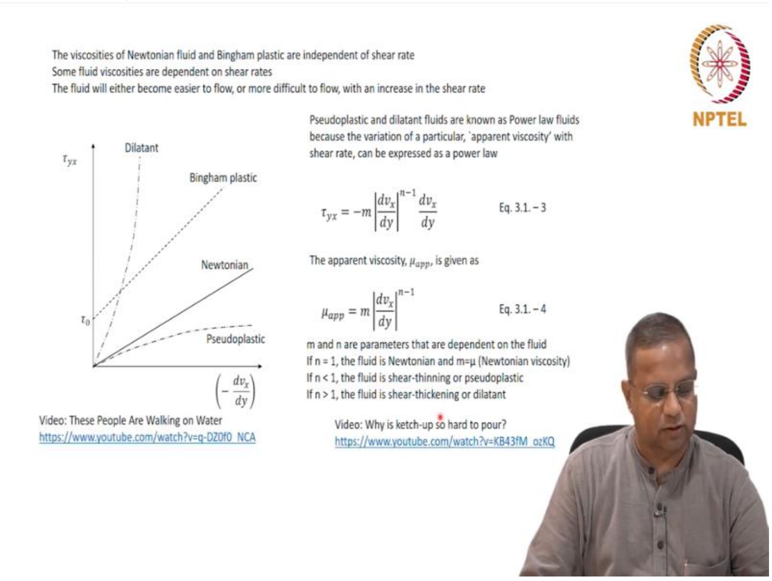

This is the Newtonian fluid which is directly in this the rheological characterization, shear rate versus the shear sorry, shear stress versus the shear rate or the velocity gradient negative of dv_x/ dy. For a Newtonian fluid you have a straight line that passes through the origin the slope is the viscosity for a Bingham plastic which does not move till threshold shear stress is applied tau 0, the behavior is something like this it does not pass through the origin.

Then we also saw a pseudo plastic fluid which is a shear thinning fluid as the velocity gradient increases, the viscosity decreases or it becomes easier to let us say stir the pseudo plastic fluid and this is the dilatant fluid is just the opposite, where the viscosity increases with shear rate;

here the slope increases with shear rate therefore, the viscosity increases with shear rate and it becomes harder to cause the velocity or to move around or to even stir the fluid.

An example of this is your paint the paint that you use, you stir faster it becomes easier to stir. This an example of this is quicksand the faster you move, the more difficult it gets to move, this is pseudo plastic this is dilatant then we looked at the various expressions the Bingham plastic was a shear, a threshold shear stress plus at Newtonian part that we saw for the pseudo plastic and the dilatant you could use a power law, where the shear stress is given as a certain $m(dv_x/ dy)^{n-1}$.

It is velocity gradient dependent viscosity and you take the minus with this if as we normally do $-dv_x/ dy$. Here if this term is called the apparent viscosity, $m (dv_x/ dy)^{n - 1}$, m and n are parameters dependent on the fluid. If n = 1, this 1 disappears and therefore, you have a Newtonian fluid and m of course becomes Newtonian viscosity. If n is less than 1, the fluid is shear thinning or pseudo plastic.

And n less than 1 this is this becomes a negative the index here the exponent here becomes negative and therefore, the fluid pseudo plastic fluid is shear thinning. If n is greater than 1, the exponent is greater than 1 and therefore, the fluid the shear thickening or dilatant.

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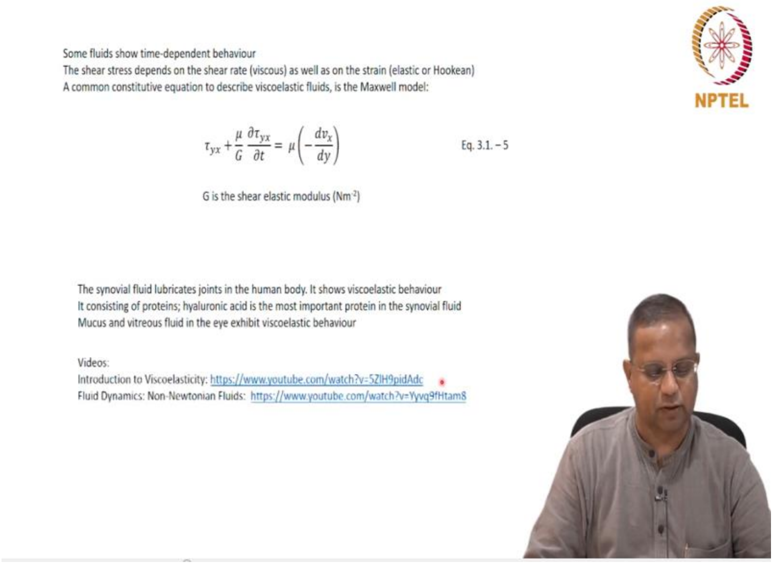

Then I give you a few videos and we also said there are other types of fluids that are biologically relevant, which show a time dependent behavior, those are called viscoelastic fluids, and this is the equation that governs viscoelastic fluids. Examples are synovial fluid, the hyaluronic acid in the synovial fluid, hyaluronic acid in other parts of the body, the vitreous humor and so on and so forth. They are all good examples of solutions which show this behavior.

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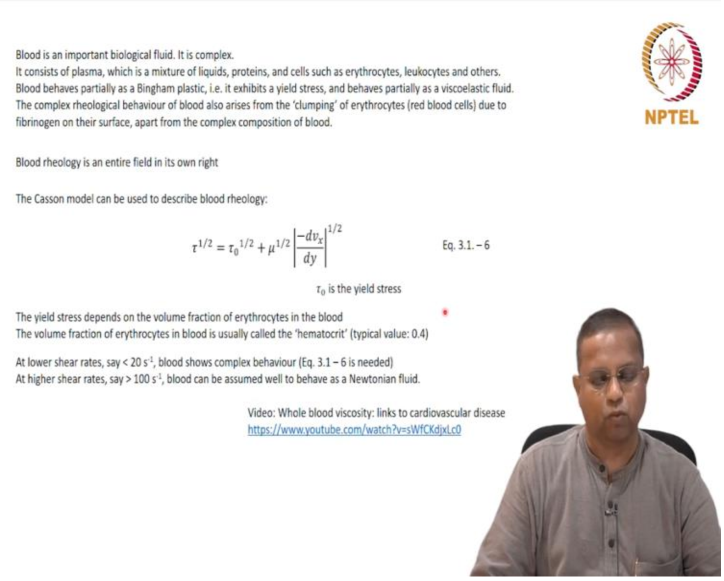

Some videos and then we briefly looked at blood, very complex fluid. Typically, you could use the Casson model for a first approximation. Of course, blood rheology is a discipline in its own, you have books written on blood rheology, some link between blood viscosity and cardiovascular disease is given in this video nice video if you could watch that.

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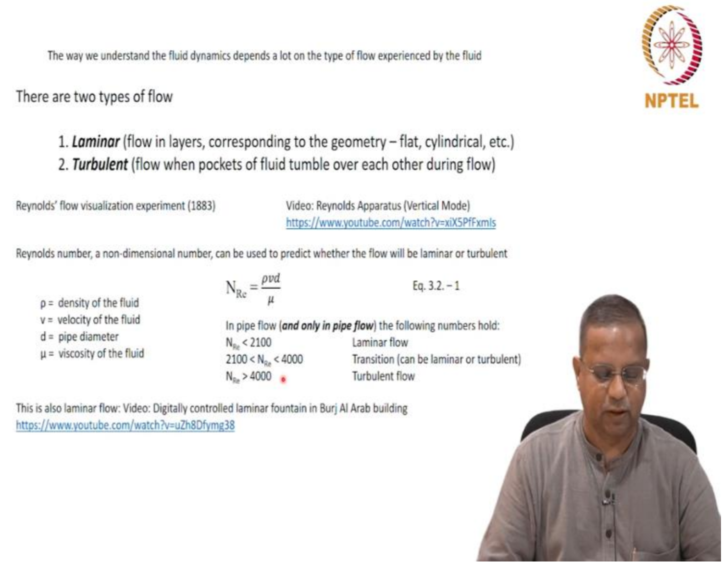

And then we looked at types of flows. There are 2 major types of flows one is laminar where the flow occurs in layers depending on the geometry, either flat layers, cylindrical layers, one cylinder, seconds and the third cylinder and so on and so forth, one inside the other in layers and they flow happens without intermixing of the layers or the other type is the turbulent flow where pockets of fluid tumble over each other when the flow occurs.

And then we saw that the Reynolds number gives us a means by which we could predict whether the flow is going to be laminar or turbulent, if the laminar number is sorry if the Reynolds number is less than a certain value in any situation, then the flow is laminar if it is greater than a certain value then it is turbulent. Those values for the case of tube flow and only for the case of tube flow turned out to be 2100 less than 2100 it is laminar and greater than 4000 it is turbulent in between we do not really know.

Then that was a long introductory chapter, but that was a lot of fun information, therefore, it would not have been tiring. Then we started getting into the analysis of these systems. From the point of view of momentum flux, as we mentioned earlier, there are 2 major approaches to the analysis. One is a shell balance approach, the other one is the conservation equation approach. The shell in the shell balance approach you write momentum balances over a thin shell representative of the geometry of the system and then you could integrate that to get the overall picture.

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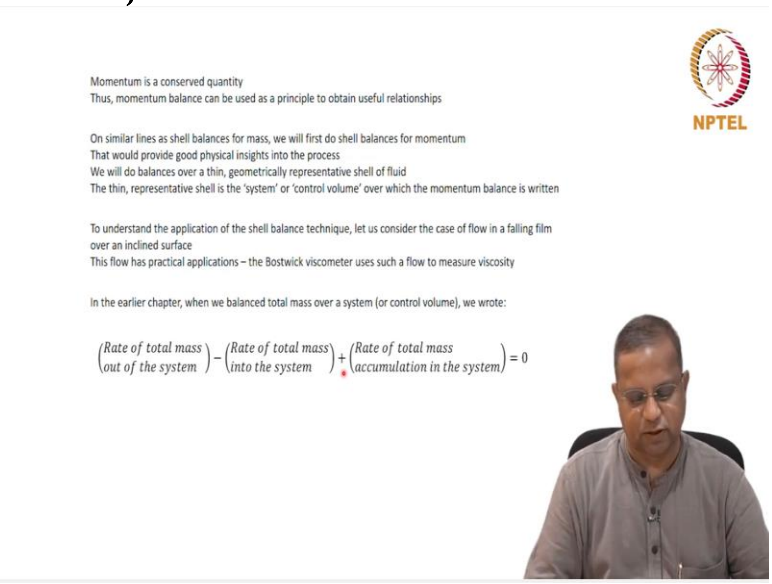

So, we had used the flow over an inclined flat plane to illustrate the shell momentum balance. We had written from by extension of mass balance, mass is a quantity momentum is also a quantity. Only thing is that the momentum balance strictly speaking is Newton's second law of motion, where the rate of change of momentum is directly proportional to the flux acting on the control volume there.

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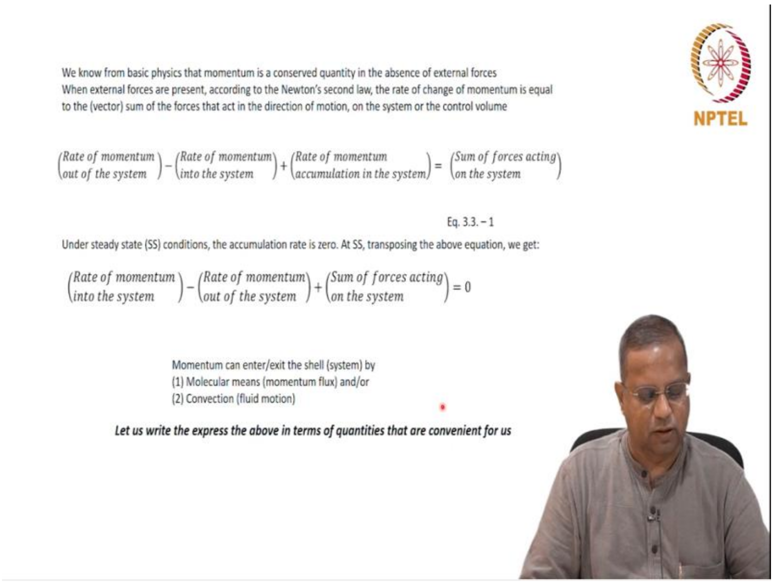

Therefore, we wrote our balance equation to take that into account the sum of forces and the rate of momentum accumulation. And then we started looking at a steady state case if it is not steady state, then you will have to take this into account also. So for a steady state case the momentum balanced turned out to be rate of momentum into the system minus rate of momentum out of the system, plus the sum of forces acting on the system equals 0.

Then we said that the momentum can enter or exit the shell in our case of a thin layer of fluid over an inclined flat plane. The shell was a thin cuboid, a very thin cuboid at pretty much the center of say the liquid layer that is flowing. And we looked at how momentum could enter momentum could exit because these are these 2 terms. And then we looked at some of forces that are acting on that particular system and equated those, that is all we did; this is the simple thing we did.

We expressed these aspects, the momentum aspects and the force aspects in terms of the quantities that we know. And that is what made the equation look a little daunting. The basic principle is just the same. Also, I would like to remind you that this course is not a mathematics course, we will be using mathematics and whatever mathematics we need, we will pick up and use it here, you will not be tested on your mathematical progress and so on and so forth in this course.

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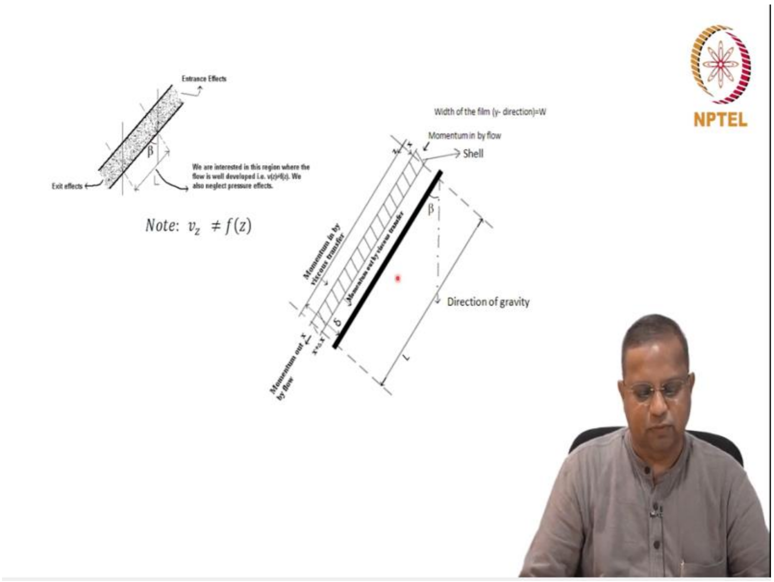

And when we went about doing that, we of course looked at the situation where the flow is well developed, the velocity at a particular in this case it was x a particular thickness of the layer that remains the same irrespective of the length over which it travels. In other words, we are looking at the region where the flow is well developed.

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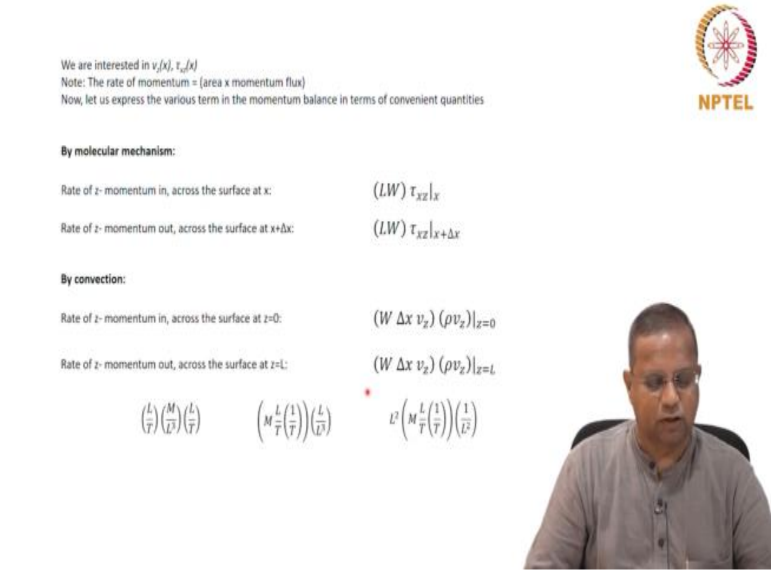

We looked at this and then we did the balance by writing in detail, what terms contribute to the entry and exit of momentum by molecular mechanism and by convection.

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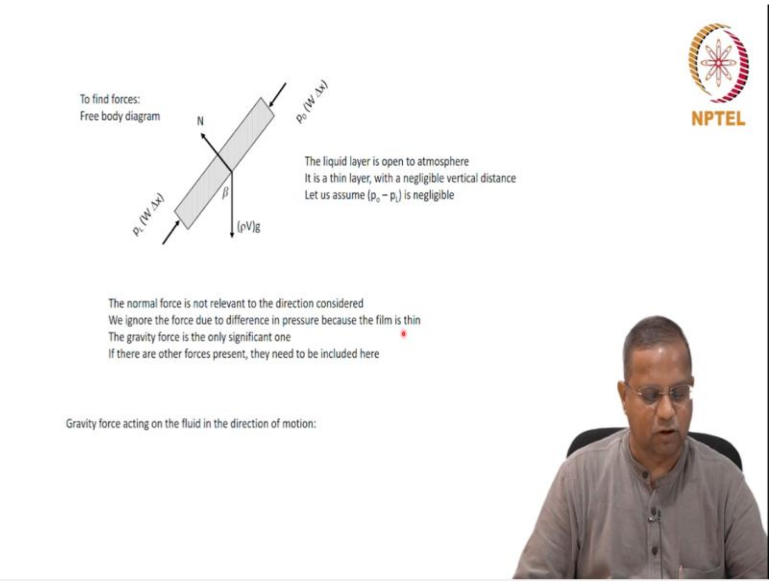

Then we put all these terms to this is a free body diagram, which shows that you have essentially, you know, a free body diagram, you cut out a part of the thing that you want to analyze, look at the forces that are going to act on it as though it is going to stand separately and that is a standard means of analysis. So, you have a pressure force that is acting on this side, a pressure force that is acting on this side, a normal force that acts and gravitational force that acts right down.

It is at an angle of beta. The plane is at an angle of beta to the vertical. Then since the film is thin, we argued that we could ignore the difference between p_0 and p_L, because the upper part is anyway open to atmosphere that is going to be the same the lower part of course, the thickness, since the thickness is very very thin, very very small, we could ignore any changes and therefore, this force cancels with this force, this force is irrelevant in the direction of motion. Therefore, only a component of the gravitation force comes into being on that we talked about.

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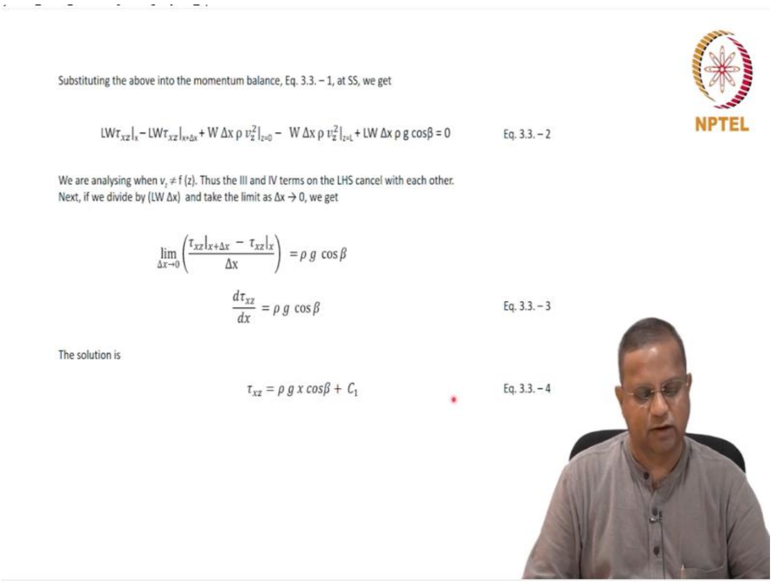

And then we did a balance, we simplified the terms, we wrote it in terms of a derivative, then we could get an expression for the shear stress profile. Once you got an expression for this shear stress profile, which is this we said that you need a relationship between the shear stress and the velocity to get the velocity profile. Typically, these are aspects of interest, a shear stress profile, the velocity profile for various different situations. So for a case of Newtonian fluid, it is quite simple. You have the Newton's law of viscosity, which relates the shear stress to the velocity gradient, and that is what we used.

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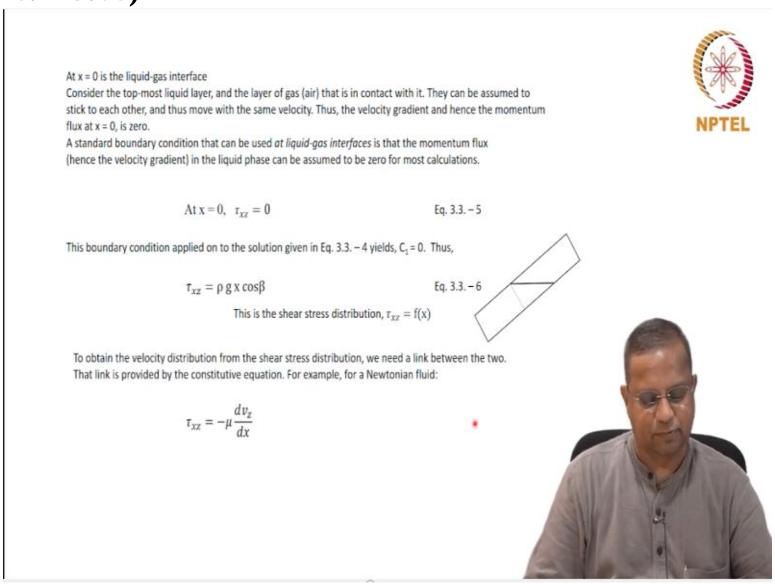

Before that I think this is the shear stress profile. This is the linear profile in this case, 0 here and maximum at the base of the wall, the plane over which the flow happens.

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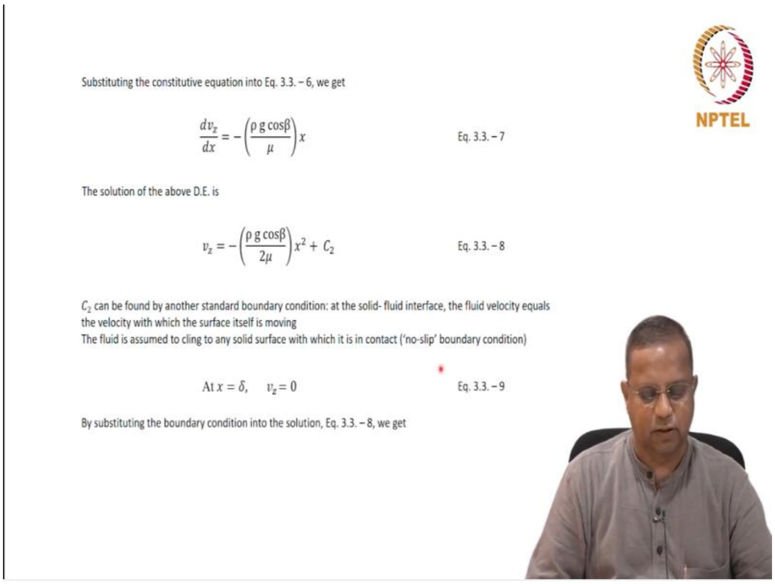

Then, we did we substituted the Newton's law of viscosity to get the velocity profile. We also talked about 2 important boundary conditions. At the liquid gas interface, the shear rate equals shear stress equals 0 because the velocity gradient is 0, the layer closest layer of gas closest to the liquid gets pulled along with the liquid in the with the same velocity and therefore the velocity gradient there is 0 and therefore, the shear stress is 0 that is one boundary condition.

And then the other boundary condition was on the other edge the liquid solid interface. The no slip boundary condition says that the layer closest to the solid surface clings on to the solid surface and does not move if the arc moves the same velocity as that of the solid surface. In this case, the solid surface does not move and therefore, the velocity of the layer closest to the wall happens to be 0. So, that is a no slip boundary condition.

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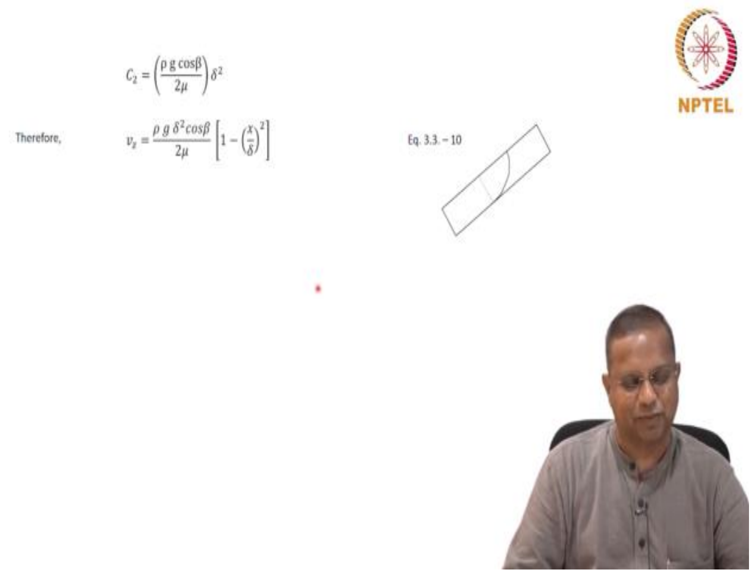

And then using those we got an expression for the velocity gradient. Once you get an expression, you could just put in numbers and sketch it, and therefore, I have shown it to you in some cases, I have asked you to go and sketch. So, that you pick up how to go about sketching, how to visualize flow velocity, flow profiles, shear stress profiles and so on and so forth.

So in this case, I showed it to you as a free sketch it is a nice parabolic curve. Imagine this to be a parabolic curve. It is a function of x square and then we looked at where the maximum velocity occurs? What is the expression for it? What is the average velocity?

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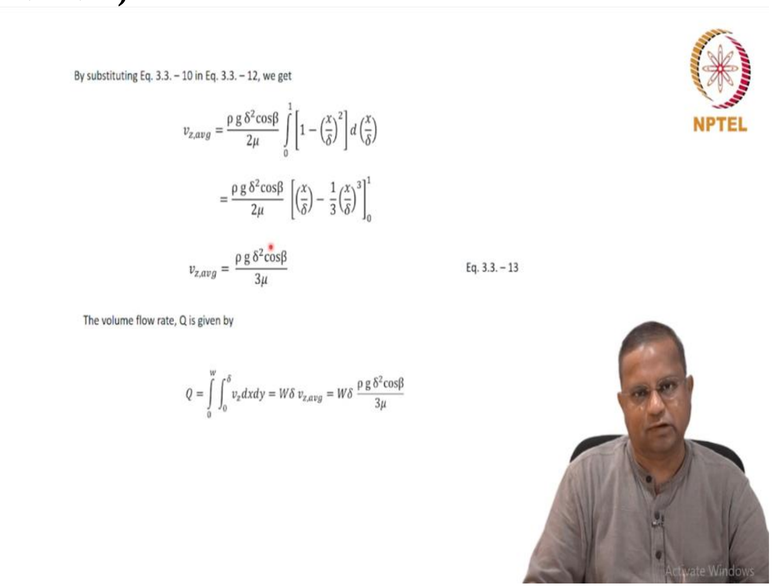

And what is the average flow rate? Or the flow rate which is nothing but the area times the average velocity. So, all this we did by shell balances or this was an application of shell balances to momentum or shell momentum balances to get insights into a flow. It also has practical applications as in a Bostwick viscometer essentially use this kind of a flow to calculate the viscosity.

Then we looked at the other approach, which is the conservation equation approach. This being a review this might well be long but it does not matter because you have already seen all these things. You are just looking at it in one go. So I do not, I hope it is not too tiring if it is tiring, just stop it for a while and then come back and take a look at it.

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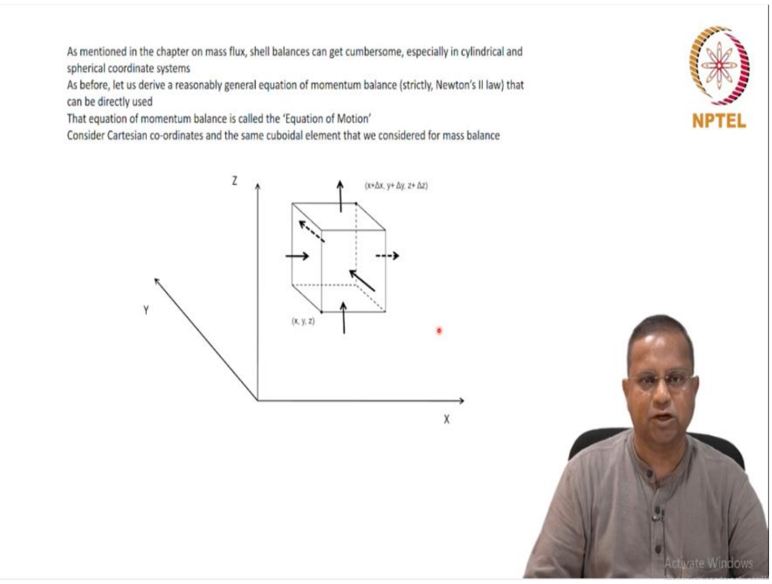

So, the equation of motion we actually derived for a rectangular Cartesian coordinate control volume system. So, once you have this, you could always move from this coordinate system to the other coordinate systems using the methodology that has been given or detailed in your appendix, the first appendix of the book textbook. So we considered this cuboid of dimensions Δx, Δy, Δz, and then we looked at the same things.

The ways by which the momentum could enter and exit, the ways by the accumulation of momentum in the system; earlier, we were looking at a steady state case, we are clear that we are looking at a steady state case well developed flow and all that. This is a general expression that we are trying to derive here and therefore, we consider the accumulation term. And then of course, the forces that act on this at least 2 major forces that act on this were considered the pressure force and the gravitational force.

If there are additional forces, then you need to add it at that stage and rederive the entire expression, it will turn out to be an additive feature. So, if once you develop that view, then you can go to the final equation and start adding, but till then, you need to come here and then add the forces at this initial step of derivation.

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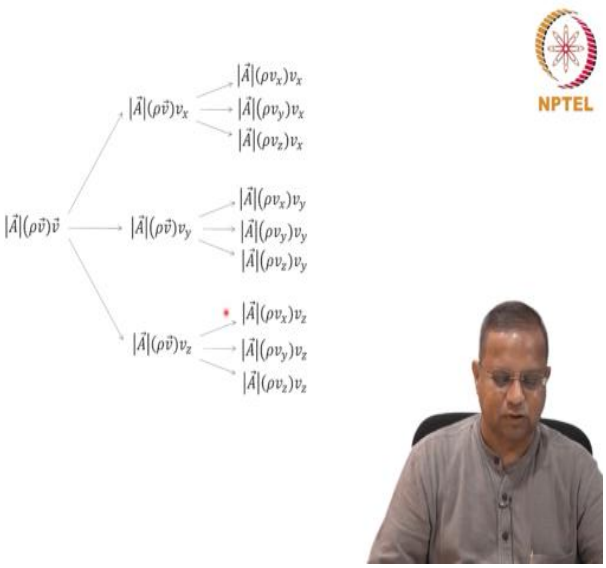

So we looked at the various expressions as a part of this, we needed to understand a second order tensor and a dyadic product all those new things ρ_{vv} is the dyadic, v v was a dyadic product.

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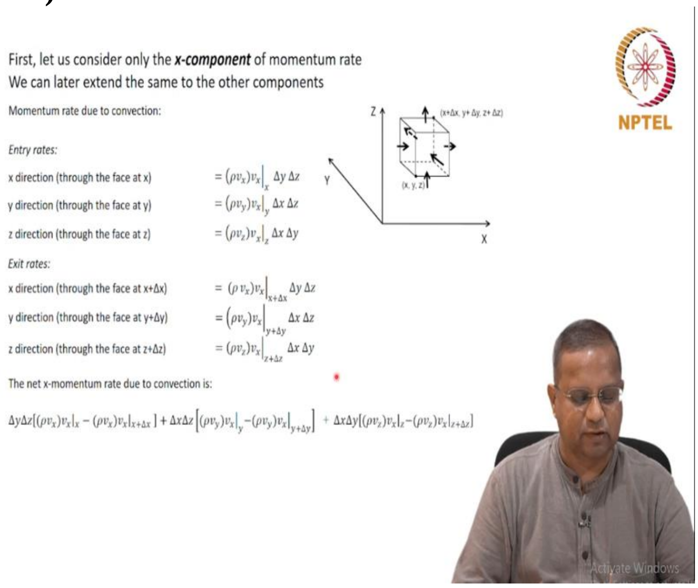

And then we looked only at one component of the momentum because each component has 3 different parts to it. And because it is a vector, therefore, there are 3 different parts to it. And therefore, there are 3 components therefore there are a total of 9 parts. So instead of dealing with all the 9 parts together, we looked at only the x component and then extended that to the y and z components. So these were the entry rates, exit rates that we got this is the net x momentum.

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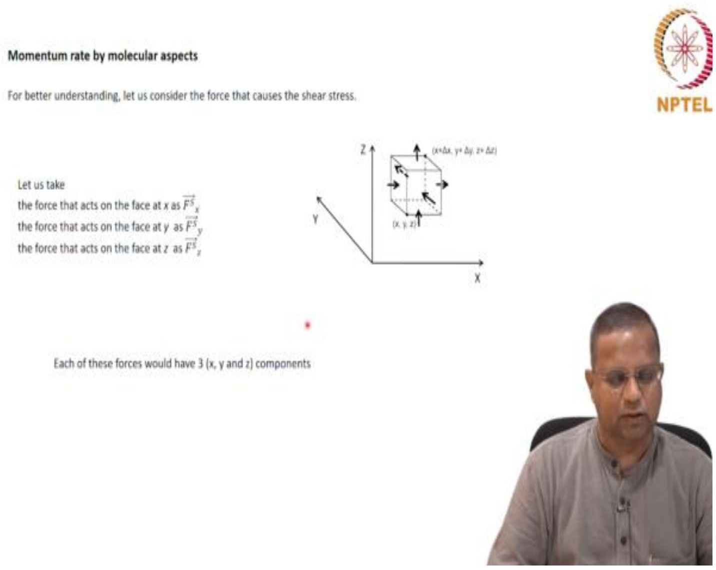

Then we looked at molecular aspects we went through understanding the force that causes the shear stress and how that has so many different components.

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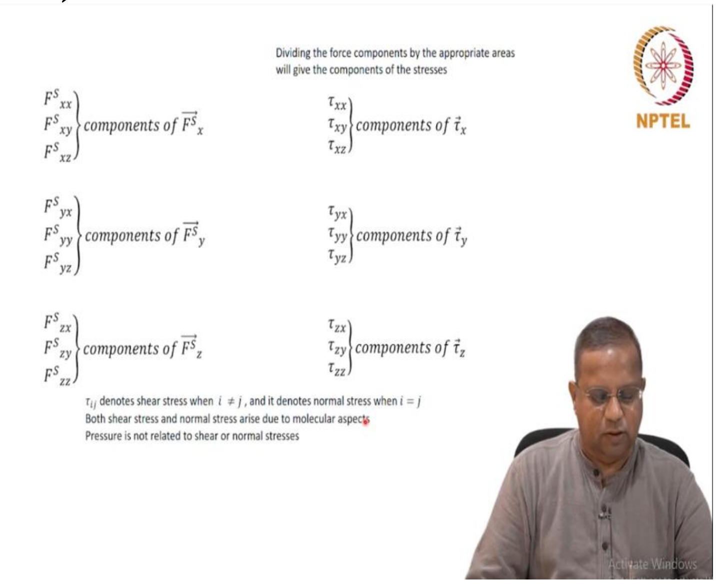

And you divide each by the area you get so many different stresses. Some of them are shear stresses, some of them are normal stresses and they also cautioned you that the normal stress is different from pressure.

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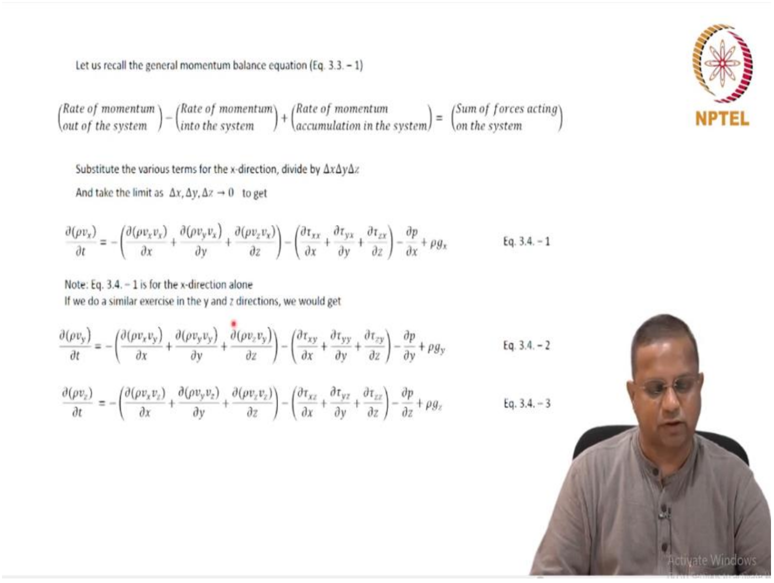

Then we went through the derivation we put everything together, including the forces of fluid pressure and gravity alone here, then we wrote our differential equation we just substituted all those terms we divided throughout by Δx, Δy, Δz took the limits as Δx tends to 0, Δy tends to 0, Δz tends to 0, to finally arrive at this expression, which is the x momentum balance.

And then we realize that this was the product of our product we could you know simplify this further. We went ahead before that, the other directions. This is 1 set of equations we wanted to simplify this further for our own use.

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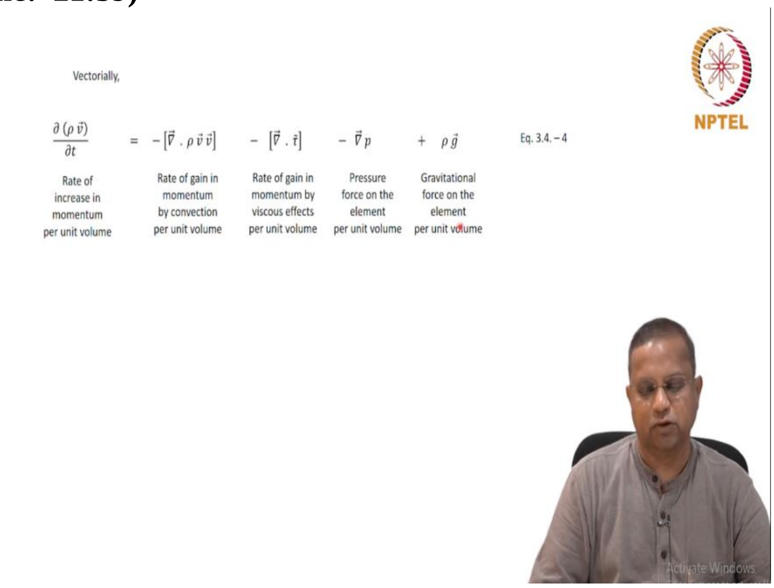

This is a vectorial representation of that equation. So, the left hand side is rate of increase in momentum per unit volume, this is rate of gain and momentum by convection per unit volume, this is rate of gain and momentum by viscous effects per unit volume, this is pressure force on the element per unit volume, this is gravitational force on the element per unit volume.

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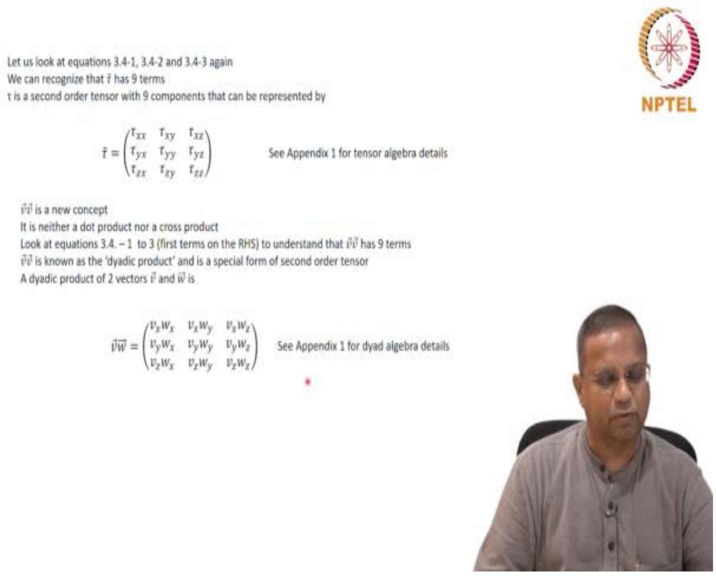

Then, we looked at the tensors because we are looking at that for the first time on the dyadic product.

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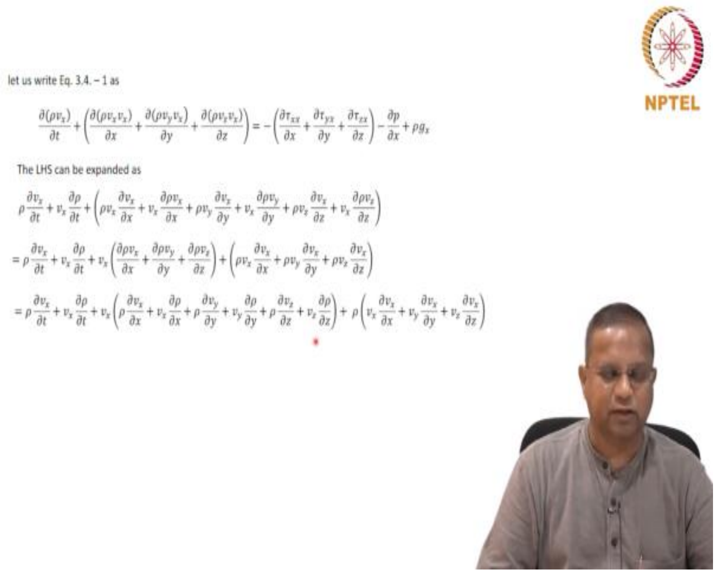

Then this is what we did, we went and expanded this because this is a function of 3 variables. We took 2 at a time expanded since till we came down to individual variables because that helped us simplify the equation quite a bit.

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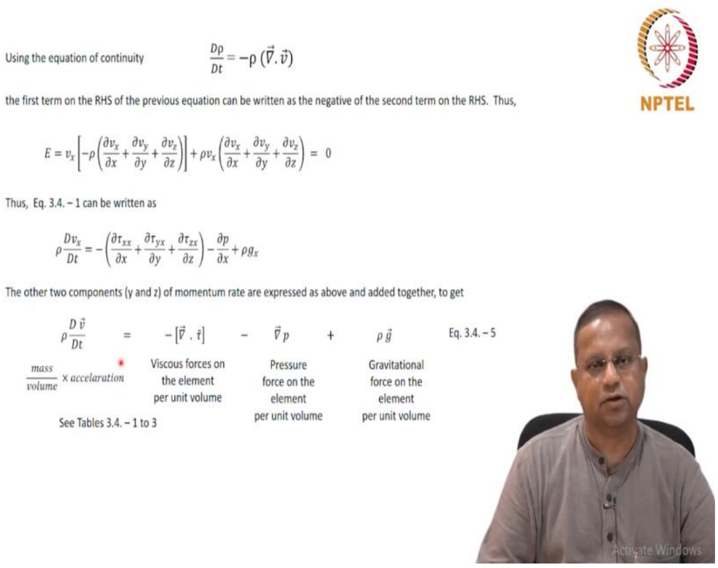

The other two components (y and z) of momentum rate can be similarly expressed and added together, to get a 3-D representation

$$\rho\frac{D\vec{v}}{Dt} = -[\vec{\nabla}.\tilde{\tau}] - \vec{\nabla}p + \rho\vec{g} \tag{3.4-5}$$

$\frac{\text{Mass}}{\text{Volume}}\times$ Acceleration	Viscous forces on the element per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume

So, this is what we got and we this is mass divided by volume times acceleration that is this term. This is the viscous forces on the element per unit volume, or in other words this is force mass into acceleration per unit volume, this is viscous forces on the element per unit volume, this is pressure force on the element per unit volume this is gravitational force on the element per unit volume.

So this equation, the various in terms of its individual components is what is given in table 3.4 - 1 to 3. I had asked you to make a copy of this, as well as the other 3 tables that were given, which I showed you briefly, please make a copy of this, because we had be using this equation over and over again, as you have already seen, probably we need to do this review in 2 stages. It is quite a lot of information. So might as well, we let us do for a couple more minutes and then take a break and then come back and do the remaining part of the review.

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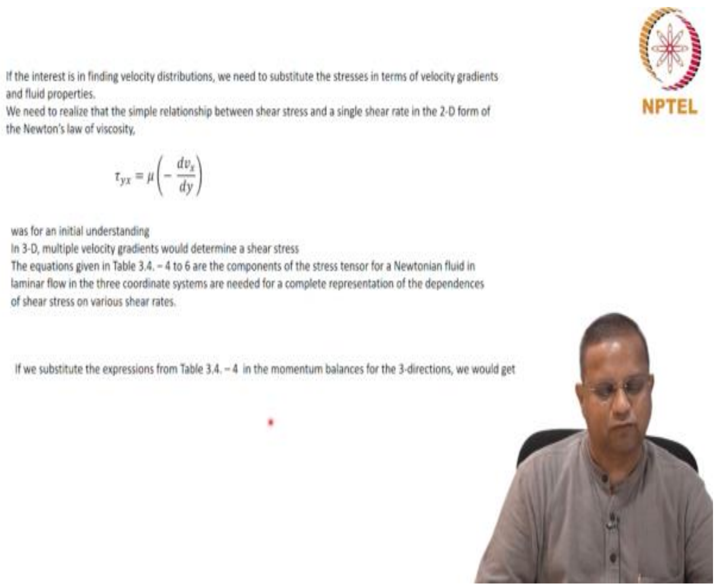

Then if the interest is in finding the velocities then you need the relationship between the shear stress and the velocity gradients and the tables 3.4 - 4 to 6 give you that in the 3 coordinate systems, we are interested in only the rectangular Cartesian coordinate system here for this derivation.

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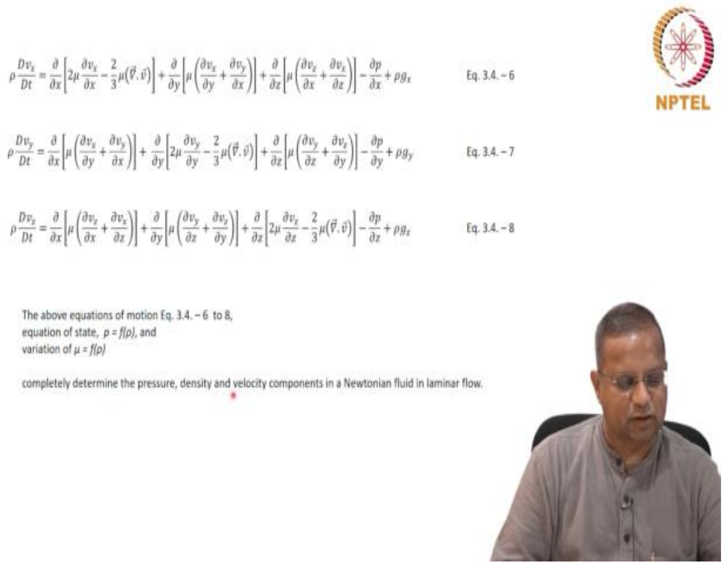

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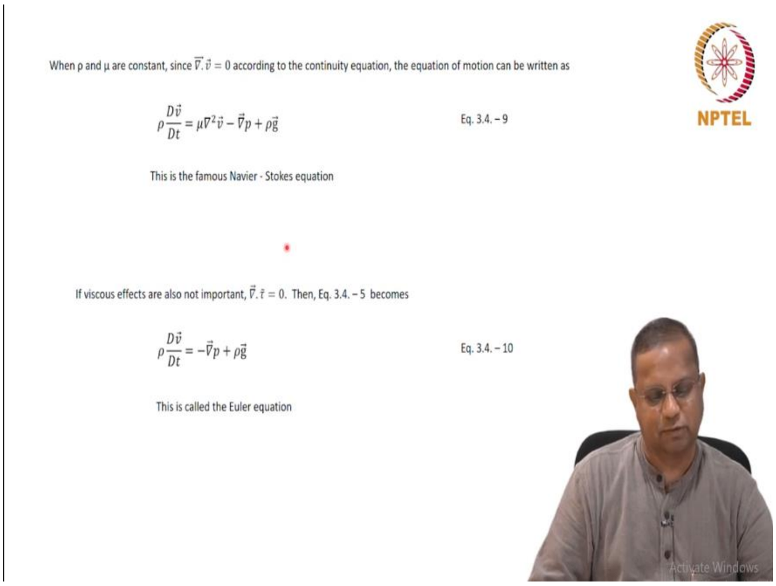

And therefore, we took those expressions substitute it we got these. And then we could simplify the equation in terms of the velocities, pressure, and density and the viscosity alone. And this is valid of course, for Newtonian fluid because this relationship between the shear stress and the shear rate, the table 3.4 - 4 to 6 are valid only for a Newtonian fluid. So, if we are considering a Newtonian fluid and at constant ρ and μ, then we found that this equation can be used and these are the second equations in the tables 1 to 3, 3.4 - 1 to 3.4 - 3.

If viscous effects are not important, $\vec{\nabla} . \tilde{\tau} = 0$. Then, Eq. 3.4-5 becomes

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \rho \vec{g} \qquad (3.4\text{-}10)$$

Equation 3.4-10 is called the Euler equation.

and this is the famous Navier - stokes equation, then the viscous effects are not important this term goes out, then you get only these 2 terms remaining on the right hand side this is called the Euler equation and so on. So, these equations are actually special cases of the complete momentum balance expression.

Then, we started looking at the applications of the equation of motion to different situations. Let me look at 1 application and close today. And in the second part of the review, I will consider whatever we have done post that the once we had the equation of motion then I showed you the tables. I am not going to show you the tables here, the some applications.

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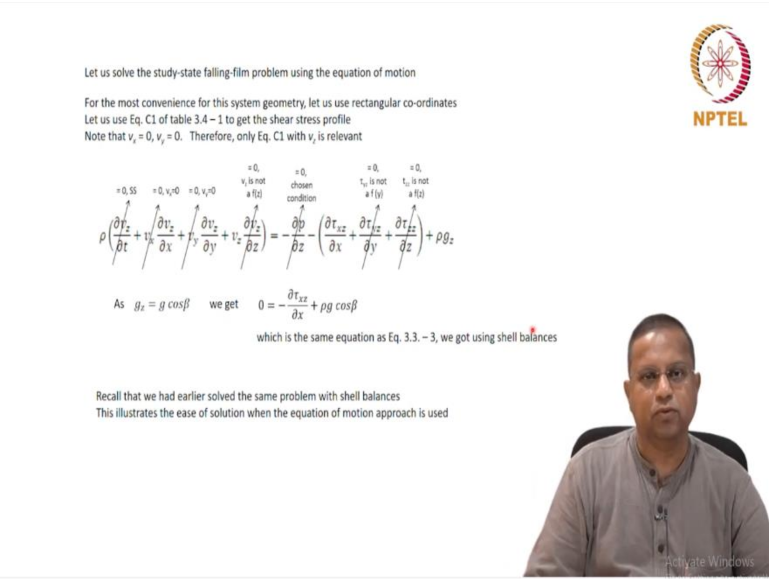

So, when we applied the equation of motion to the same case of a thin falling film over in an inclined surface that we did earlier using shell balances in 1 step we could get the answer. So, that is what I showed this is the equation corresponding to the rectangular Cartesian coordinate system which the geometry of the system satisfies. And therefore, we took this equation cancelled out the terms that are irrelevant.

For example, the first time closer because we were looking at steady state, there is bulk velocity in this case, there is no velocity in the x direction bulk velocity in the x direction there is no bulk velocity in the y direction, of course, there is a bulk velocity in the z direction. However, because the flow is well developed the z velocity is not a function of the distance along the z direction, and therefore, this term goes to 0.

This is the chosen condition that the pressure does not change and therefore, by that approximation, this term goes to 0. This of course remains this term goes to 0 because τ_{yz} is not a function of y. For this, I said later that you could use that here also. So, you look at the direction of the velocity and then look at the direction of the velocity gradient, if there is a possibility of a velocity gradient here because of a velocity in a perpendicular direction then this term becomes relevant.

If you look at yz here is z is the direction of the velocity, which is this and the velocity gradient is in the y direction, there is no velocity gradient in the y direction here, and therefore, this goes to 0. Similarly, because of the velocity in the z direction, the second subscript here, is there a velocity gradient in the z direction, there is no meaning also there so, this in this case, there is no meaning. Therefore, this term goes to 0 and we got this expression.

Since

$$g_z = g \cos\beta$$

we get

$$0 = -\frac{\partial \tau_{xz}}{\partial x} + \rho g \cos\beta$$

which is the same equation as Eq. 3.3-3.

which is the same equation that we got by using shell balances. Of course, we got a lot of insights into the physical aspects of the system using shell balances, but it can become cumbersome especially in cylindrical and spherical coordinate systems and therefore, use of this is preferred.

Of course, there are limitations to the use of this when there is a change in let us say the cross section area these approximations cannot be used because we had cancelled the areas throughout that was one of the main approximations of that was one of the main assumptions made when we derived this although it is not very apparent. So, this equation will not be applicable, when you have flow through a cone and so on so forth.

Conical section this is not applicable so, the first equation is of course, that the first equation is general. It is only when we derived the Newtonian fluid constant ρ and μ we made this assumption.

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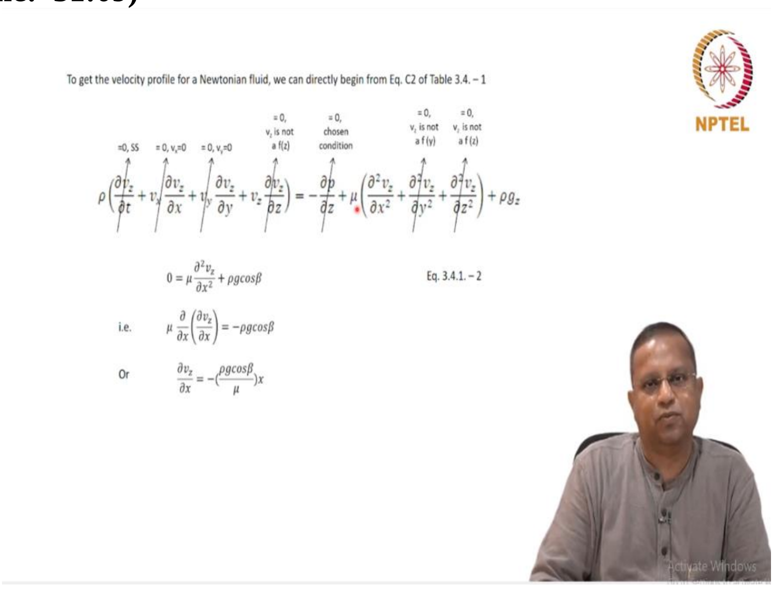

So, check that as long as you are cancelling the areas it will not be applicable so, check whether when the first one is applicable. So then we looked at how to get the velocity gradient here again, 1 step you take the velocity equation cancel out the terms, and then we got the same equation as the earlier case for getting the velocity as a function of x or this one was the expression, the same expression that we got earlier.

So it is much easier to use the conservation equation approach is what we said, we have been at it for quite a while now, maybe about half an hour or so, it even the review would get tiring. So let us do the review in steps or stages. So this is part 1 of the review of momentum balances. It is a long chapter of course when we meet next we will do the second part see you again.