Transport Phenomena in Biological Systems

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Lecture - 44
Review of Momentum Flux - Continued

Welcome to the second part of the review of momentum balances, we realize that the review itself is just too long and there is no point in having more than about half an hour at a time. Even though we go for little long periods, you could always stop whenever you want and then come back. Whenever you feel tired, you should do that. That is one of the nice things about this mode of learning.

I think you should make full use of it you should start looking at it with a fresh clear mind. Let us continue hope we go a little faster, another half an hour, maybe we must be able to get done. Otherwise we will do a part 3 of this video. We in the earlier part we reviewed whatever we did till this point, which was the shell balances, shell momentum balances, and the derivation of the equation of motion, application of the equation of motion, to simplify the analysis that was done using shell balances or you could get to insights much quicker.

If you use the equations of motion, subject to certain constraints, you should know where to apply it properly. Let us start using the equation of motion to get insights into processes aspects that are fundamental to pretty much all aspects of biological engineering. The first one is flow through a cylindrical pipe. We started out with this vertical pipe I hope you figured out why we looked at a vertical pipe instead of a horizontal pipe I had let you figure that out.

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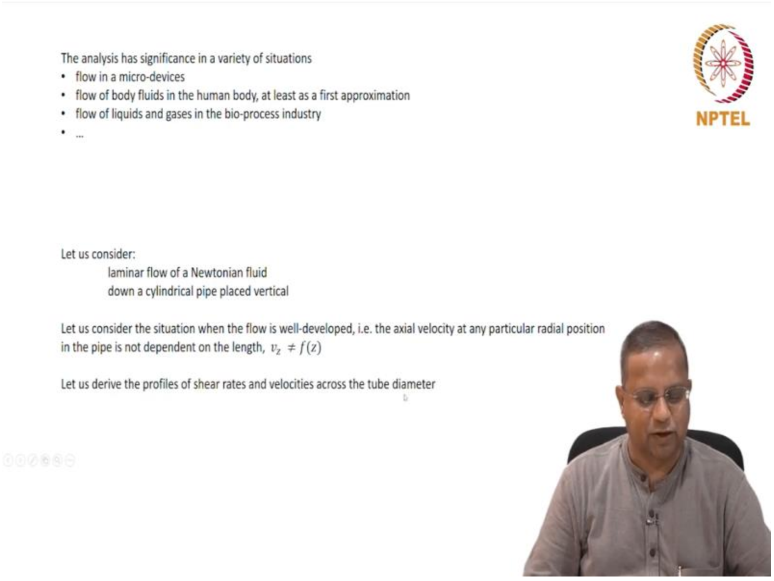

The flow of course, has flow through a pipe has various different applications, flow and micro devices flow in inside the human body flow of gases and fluids in the bioprocess industry and so on so forth. And we said we will consider a laminar flow of a Newtonian fluid and down a cylindrical pipe placed vertically.

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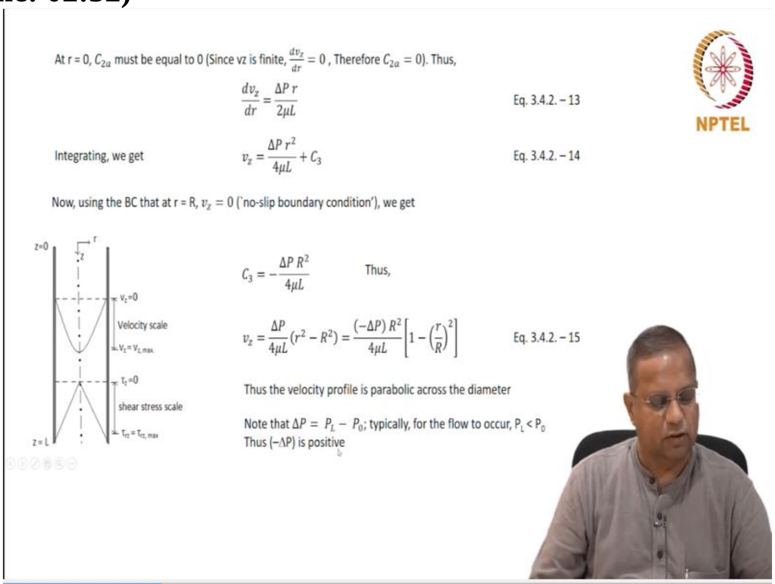

We did we applied the equations of motion to the situation where it is applicable then we got some insights that the pressure does not vary with the radius or the pressure does not vary with theta and therefore, the pressure across a cross section is same across the length will vary. Then we got the velocity profile through quite a bit of derivation please go through this interesting to solve if at all you are math oriented.

Otherwise you should know at least how to solve these things. When you have functions of ordinary differential equations of 2 different functions on either side they can only be equal if they are both equal to the same constant, and using that result from mathematics, then we could get to the velocity profile in fluid flow, the velocity profile is this v_z, this is vertical flow.

At $r = 0$, C_2 must be equal to 0.

Therefore

$$\frac{dv_z}{dr} = \frac{\Delta P\, r}{2\mu\, L} \qquad (3.4.2\text{-}13)$$

Integrating this, we get

$$v_z = \frac{\Delta P\, r^2}{4\mu\, L} + C_3 \qquad (3.4.2\text{-}14)$$

Now, using the BC that at $r = R$, $v_z = 0$ ('no slip boundary condition')

$$C_3 = -\frac{\Delta P\, R^2}{4\mu\, L}$$

Thus

$$v_z = \frac{\Delta P}{4\mu\, L}(r^2 - R^2) = \frac{(-\Delta P)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (3.4.2\text{-}15)$$

So this was the velocity profile and we also looked at the shear stress profile, shear stress profile we had looked at first, which turned out to be like this, this is the shear stress scale, this is the 0 value here. At the center it is 0, the velocity gradient there has to be 0. That is one of the boundary conditions by the way, and symmetry boundary conditions called then 0, it reaches a maximum linearly in either direction or in any which direction of the radius that you take.

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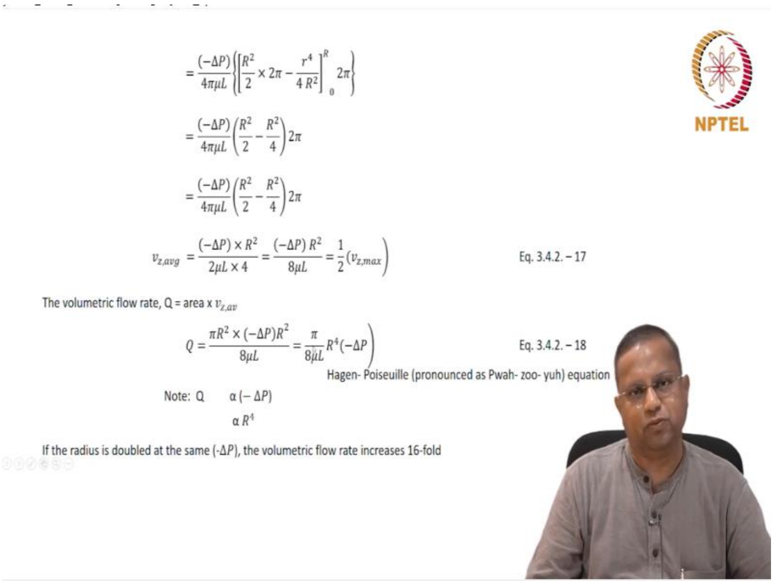

So, we also derived the expression for maximum velocity that is just by putting r = 0 the velocity maximum velocity is the center.

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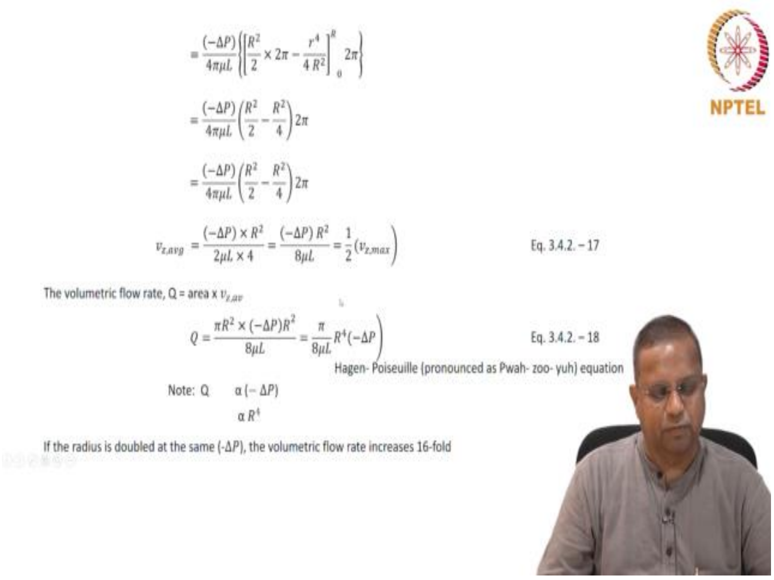

Then we got the average velocity, which turned out to be half the maximum velocity and then we got an expression for the flow rate in laminar flow, and this is the famous Hagen – Poiseuille equation

The volumetric flow rate, Q = Area × $v_{z,avg}$. Thus

$$Q = \frac{\pi R^2 \times (-\Delta P) R^2}{8\mu L} = \frac{\pi}{8\mu L} R^4 (-\Delta P) \qquad (3.4.2\text{-}18)$$

Thus

$$Q \quad \alpha \ (-\ \Delta P)$$
$$\alpha \ R^4$$

If the radius is doubled at the same $(-\ \Delta P)$, the volumetric flow rate increases 16-fold.

So, the flow rate is directly proportional to the pressure drop.

So, these are good insights that we saw of course, it is inversely proportional to μL and so on so forth. Typically, we would like to compare for a given situation or depends on your need for comparison, this is the relationship you can use this relationship whichever way you want. Then we looked at the shear stress and this is the shear stress profile that came later is it? That came later I suppose, we got the shear stress profile by looking at the other equation and that was a linear shear stress profile.

We had already discussed this and then we looked at the application to capillary flow, the capillary flow the only difference was the pressure drop is due to the capillary action the adhesive and the cohesive forces, the adhesive forces being much larger between the liquid and the solid surface compared to the cohesive forces between the liquid molecules and therefore, it gets the liquid arises on its own in a capillary situation.

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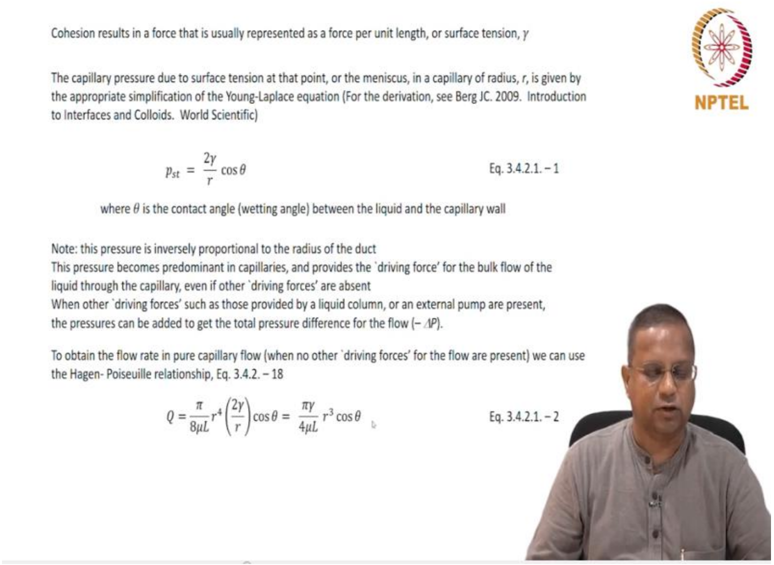

$$Q = \frac{\pi}{8\mu L} r^4 \left(\frac{2\gamma}{r} \right) \cos\theta = \frac{\pi\gamma}{4\mu L} r^3 \cos\theta \qquad (3.4.2.1\text{-}2)$$

Since the flow rate is a product of the cross-sectional area and the penetration velocity, the penetration velocity (v_p) can be obtained by dividing the above equation by the cross-sectional area πr^2

$$v_p = \frac{dL}{dt} = \frac{\gamma}{4\mu L} r \cos\theta \qquad (3.4.2.1\text{-}3)$$

In microfluidic situations, the above equation can be integrated to get the position of the liquid front along the capillary as a function of time.

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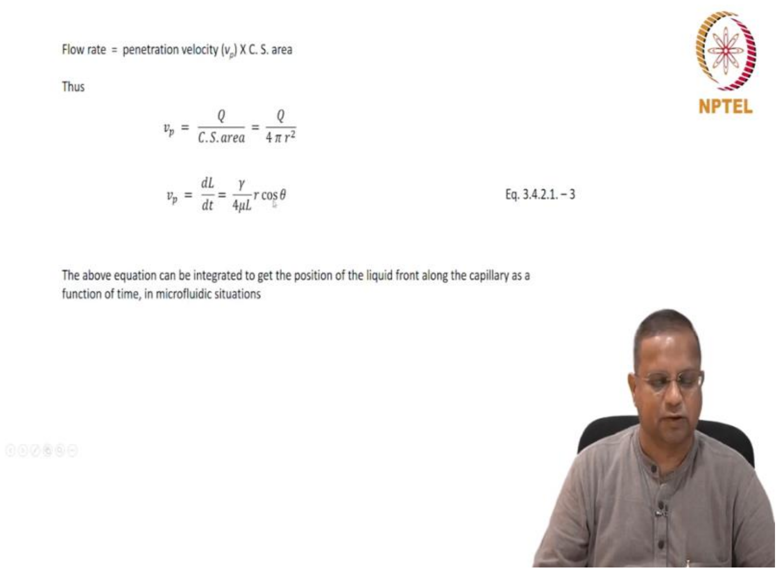

And then we also talked about the penetration velocity which is of interest in these studies, which can be found by flow rate divided by $4\pi r^2$ and we have an expression for that. And if you integrate this penetration velocity is nothing but, dL/dt of the length that has been travelled, the penetration length is of importance and studies and that can be obtained by integrating this expression.

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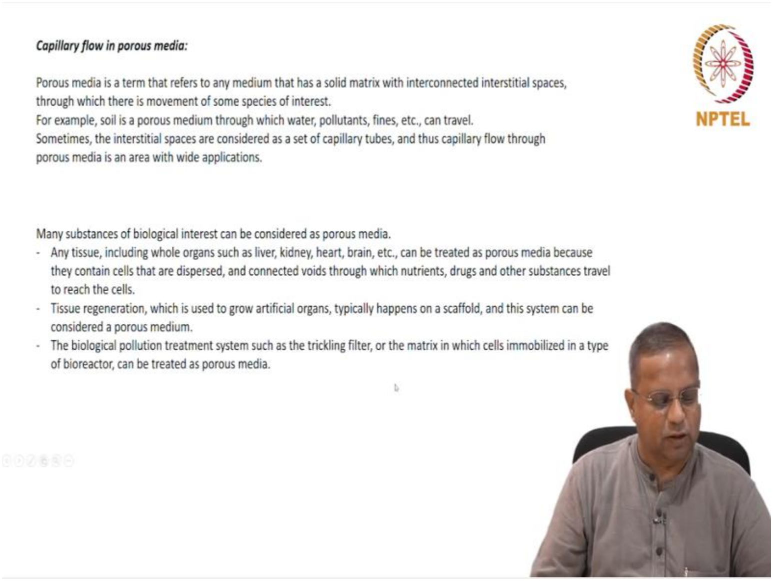

So, that is what we could do. So, the capillary flow in the porous media which is a set of interconnected voids of various shapes in a bed of solids that can be modelled as the bed with a lot of capillaries in them and you could get the same expression. And this model capillary flow in porous media can be used to describe flow across any tissue whole organs and in the soil of course, in tissue regeneration and so on so forth. There are various applications the application is only limited by our imagination. So, we have some fundamental concept which can be applied to various different aspects, that is the power of this course.

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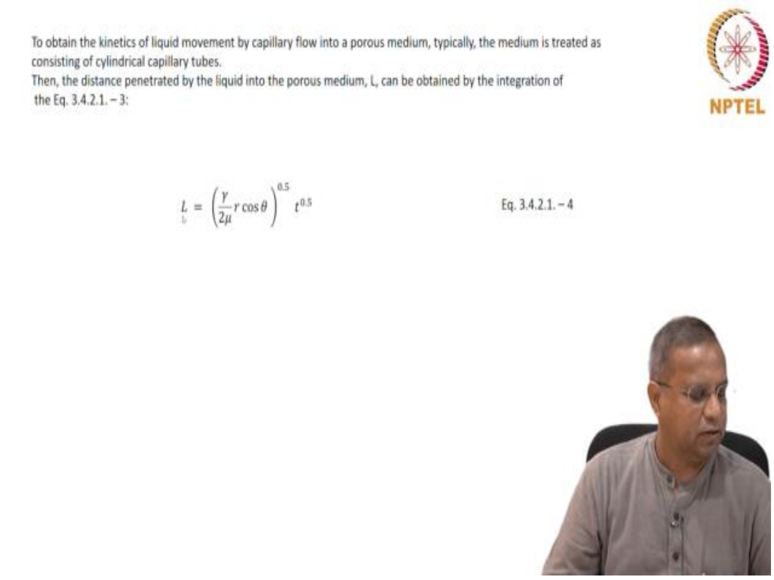

Then this was the expression that was given as the penetration length which could be helpful. Then, we looked at cuvette flow; cuvette flow is the flow between cylinders, concentric cylinders the flow happening in the space between the concentric cylinders in the annular space between the concentrate cylinders. And the flow is typically a tangential flow and this geometry has the advantage of providing very defined shear stresses on cells and so on so forth.

You could use it for that we have used it for that, others have also used it for that we have used it to cultivate cells under constant shear stress or a narrow range of shear stress define shear stress you could actually calculate the shear stress that the cells are experiencing. And that has given us a lot of insights into the behavior of cells. We derived the expression instead of going through that let me give you the final expression for the velocity.

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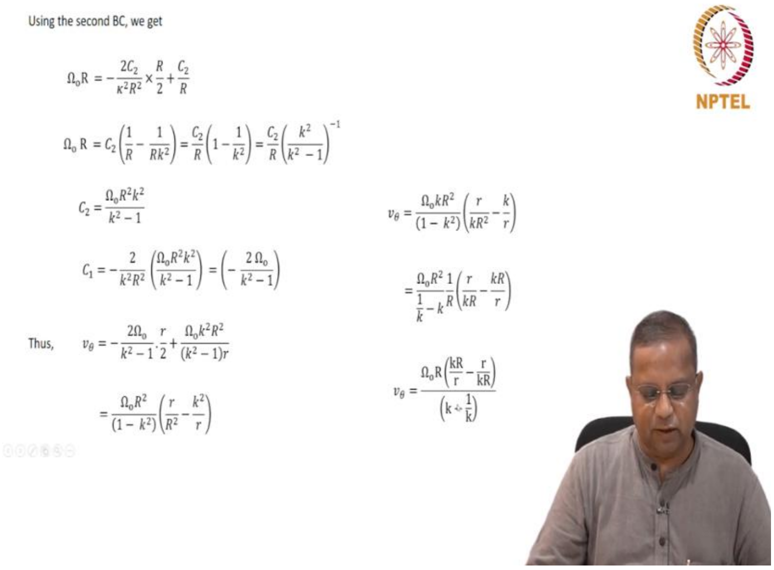

$$v_\theta = \frac{\Omega_0 R\left(\dfrac{kR}{r} - \dfrac{r}{kR}\right)}{\left(k - \dfrac{1}{k}\right)} \qquad (3.4.3\text{-}4)$$

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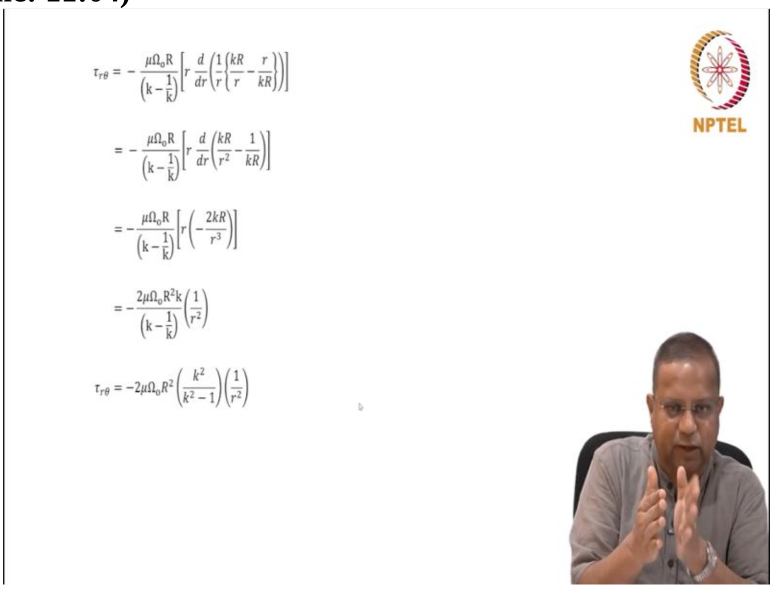

And we could also get the shear stress profile. Therefore, we have a clear idea as to what the cells are experiencing that

$$\tau_{r\theta} = -2\mu\Omega_0 R^2 \left(\frac{k^2}{k^2-1}\right)\left(\frac{1}{r^2}\right) \qquad (3.4.3\text{-}5)$$

.And as long as that R variation is 1, the shear stress is going to vary over a small range and that is very nice for us. Then we also could calculate the torque, we had an expression for that.

Then our next application was I think, we discussed the dimensional analysis using non dimensional numbers. We said that, even if you do not have insights into the process you could analyze on the basis of dimensions using the Buckingham pi theorem and get beautiful insights out of it.

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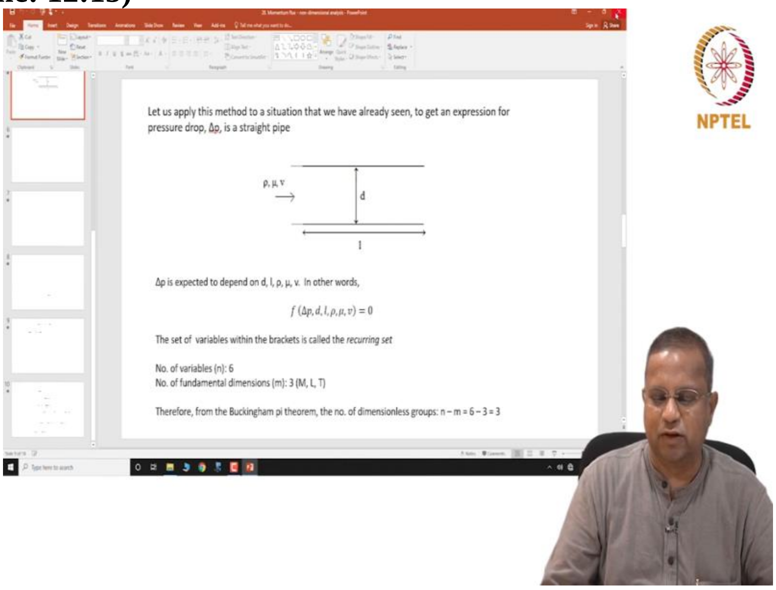

Then we had applied the dimensional analysis to a situation that we already know that is a flow through a pipe, the relationship between the pressure drop and the other relevant parameters of flow and the geometry and so on and so forth. The we got the relationship just by using we got the form of the relationship just by using or just by analyzing the dimensions of the various things using the Buckingham pi theorem.

And of course, methodology that was shown and as said that the methodology has steps which have proven by practice there is no mathematical proof for them, it has been proven by practice, it works most of the time or at least has worked in all cases that I have seen so far. So, that was what is illustrated to you, I think I will not take too much time on that you can go and use this. So, I was taking you from a very rigorous view to a completely.

You know, completely an approach that was based on experience macroscopic aspects and so on so forth. That is the dimensional analysis part of it, which also can give us this and then we switch back to that approach of getting insights, then we realized the mathematics can be quite daunting in certain cases, you could use numerical methods to solve such equations. However, those could also those are also not very easy to do in certain situations.

There are a different set of challenges that you need to overcome in a numerical solution. And when it is going to be used for design and operation alone, there are methods that are based on macroscopic aspects that work well. So, this was the dimensional analysis then we looked at unsteady state case and steady state flow, where the fluid is addressed initially and then you start the flow and you are interested in the time from the start to the time when steady state is achieved.

So, that is unsteady state and then we also looked at Poiseuille flow; then, before we get into the macroscopic aspects; we looked at turbulent flow at least a way to approach analysis of turbulent flow in a reasonably based on a reasonably fundamental approach and then we started making assumptions there when things got a little too tight.

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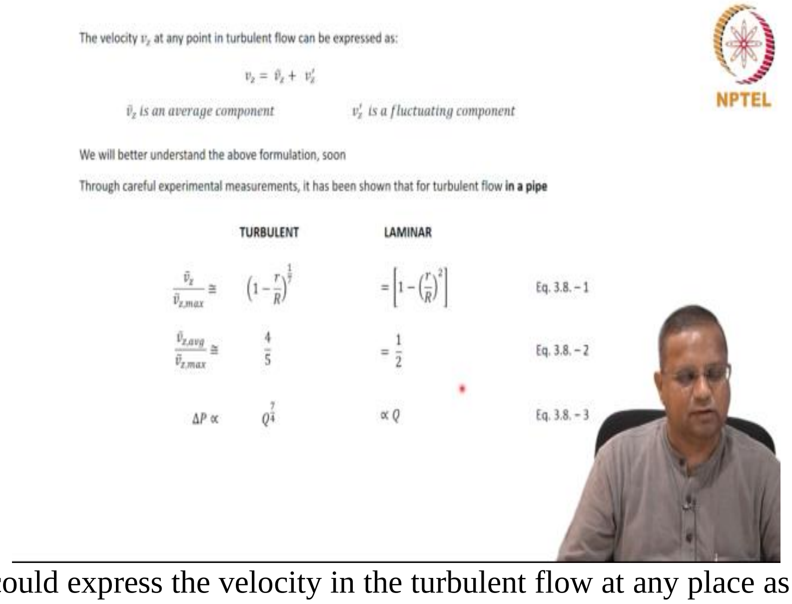

So we said we could express the velocity in the turbulent flow at any place as a sum of a time smooth component and fluctuating component or an average component and a fluctuating component. If you do that, then all we need to do is, of course there are some videos that were given and so on and so forth. All we needed to do was to replace the velocity I have shown it to you in this case, I think, briefly replace we talked about Reynolds stresses.

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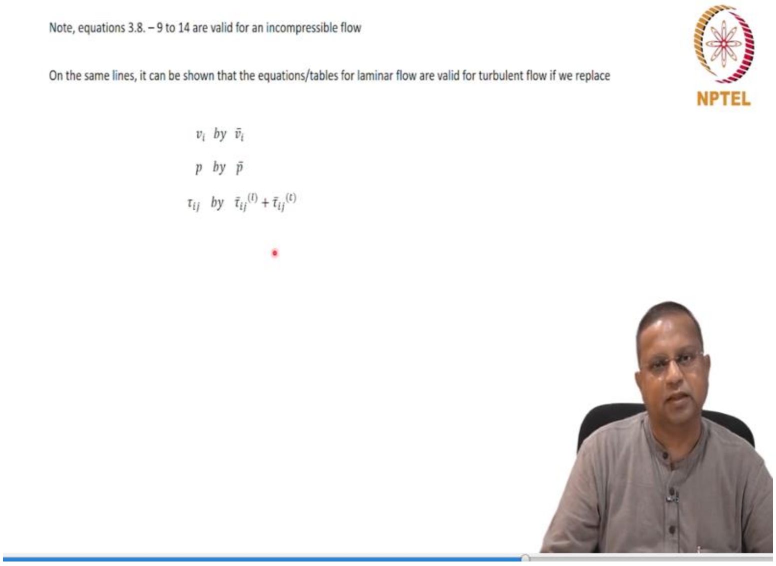

The velocity by an average velocity the equations of continuity equations of motion I mean the velocity replaced by the average velocity the pressure replaced by an average pressure and the shear stress you replace by a laminar component or the stress you replaced by a laminar component and a turbulent component sum, then the fundamental equations can be used to the extent possible.

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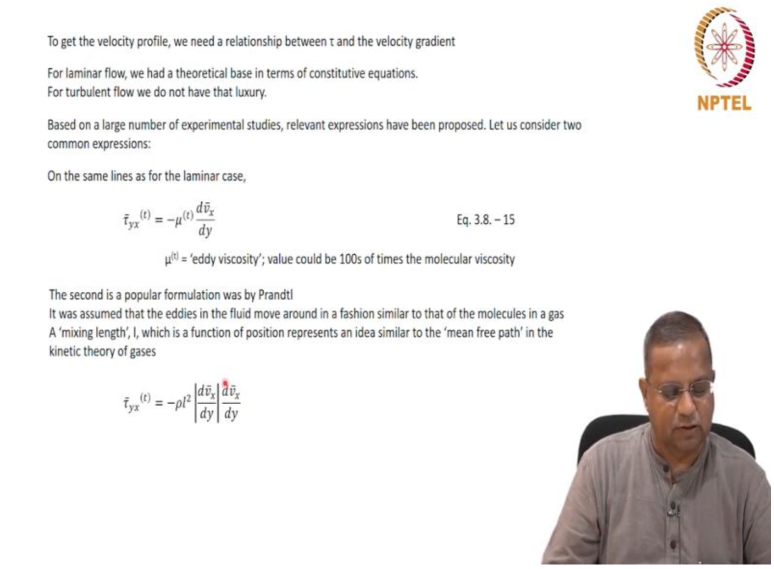

Then we said even that could get a little difficult so, people have used other means of getting some handle on the turbulent flow. So, one of the formulations is similar to Newton's law of viscosity, although this is not the Newton's law of viscosity because this is not molecular viscosity this is eddy viscosity which could be 100s of times the molecular viscosity, but this form seems to help, another form that is given by Prandtl was also shown as this

For laminar flow, we had a theoretical base in terms of constitutive equations. For turbulent flow, we do not have that luxury. Nevertheless, many expressions based on experimental observations have been proposed. Two are given below.

The first is on the same lines as for the laminar case.

$$\overline{\tau}_{yx}^{(t)} = -\mu^{(t)} \frac{d\overline{v}_x}{dy} \quad (3.8\text{-}15)$$

where $\mu^{(t)}$ is termed as 'eddy viscosity' and its value could be hundreds of times the molecular viscosity.

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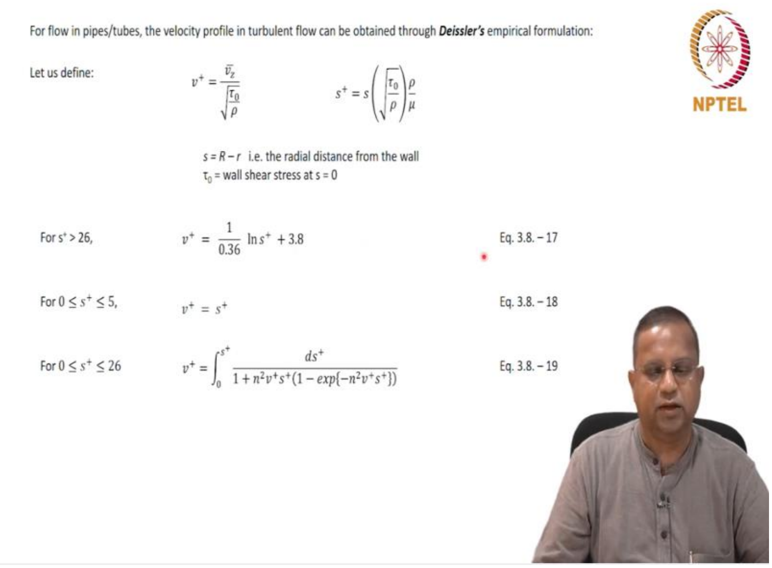

So, this was shown and of course, the Deissler's empirical formulations were also given in terms of your v^+ and s^+ which are non dimensional quantities. So, all this you have to get an idea as to the quantitative velocities shear stresses velocities in this case in turbulent flow in this case pipe flow. So, this is 1 and then we introduced the engineering Bernoulli equation and showed its applications to various practical situations.

So, engineering Bernoulli equation, although engineering Bernoulli equation is can be derived from basic momentum balance The applications over cross sections and so on so forth help us to get a lot of design aspects in a rather straightforward fashion easy fashion compared to and an in depth understanding and nothing beats an in depth understanding because there if you understand it in depth, the generality of application is so wide. So, always the aim of engineers is to go deeper and deeper and understand things better and better thereby improving the generality of applications for you know immediate applications.

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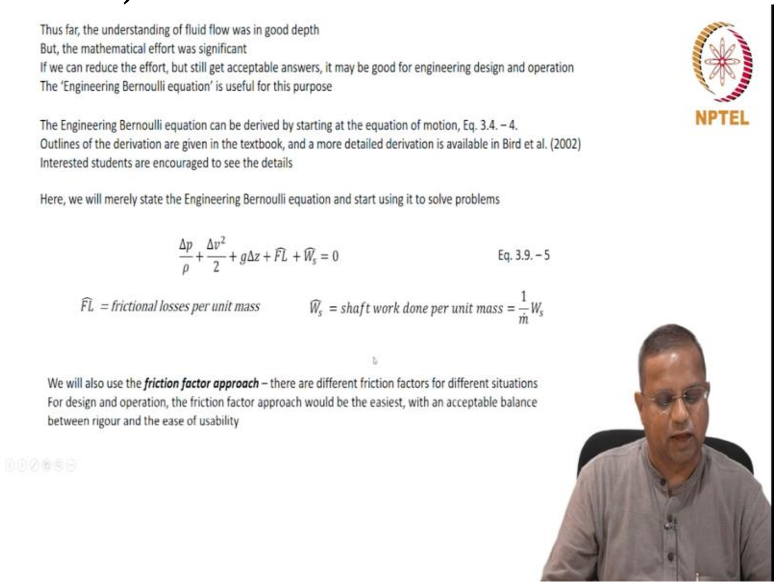

Some immediate applications such as this piping network before that the engineering Bernoulli equation was

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} + g\,\Delta x + \widehat{FL} + \widehat{W}_s = 0 \qquad (3.9\text{-}5)$$

where

$$\widehat{FL} = -\frac{1}{\dot{m}} \int (\tilde{\tau} : \vec{\nabla}\vec{v})\, dV$$

$$\widehat{W}_s = \frac{1}{\dot{m}} W_s$$

Equation 3.9-5 is a useful form of the engineering Bernoulli equation.

We add the friction losses per unit mass and this shaft work by unit mass, you get the engineering but not the equation. And we looked at the applications of this; we also said that we will look for a friction factor as it is called a friction factor.

That we defined in a certain way this is the shear stress versus the other stress, or the first by force kinetic force by unit area divided by the kinetic energy per unit volume. That is the way we define our friction factor.

$$\widehat{FL} = \frac{4\left(\frac{1}{2}\rho v_{\text{avg}}^2 f\right)L}{\rho D} = 4f\left(\frac{L}{D}\right)\left(\frac{v_{\text{avg}}^2}{2}\right) \qquad (3.9.1\text{-}8)$$

This $\widehat{FL}$ accounts for frictional losses at the pipe wall (skin friction). Equation 3.9.1-8 can be written as

$$\widehat{FL} = f\left(\frac{L}{\frac{D}{4}}\right)\left(\frac{v_{\text{avg}}^2}{2}\right)$$

This was the frictional loss per unit mass in a straight pipe this we actually derived and then this hydraulic radius which is cross sectional area divided by the wetted perimeter allowed us this formulation allowed us to extend the results to non-circular geometries. That way it is a simple formulation that is a powerful formulation.

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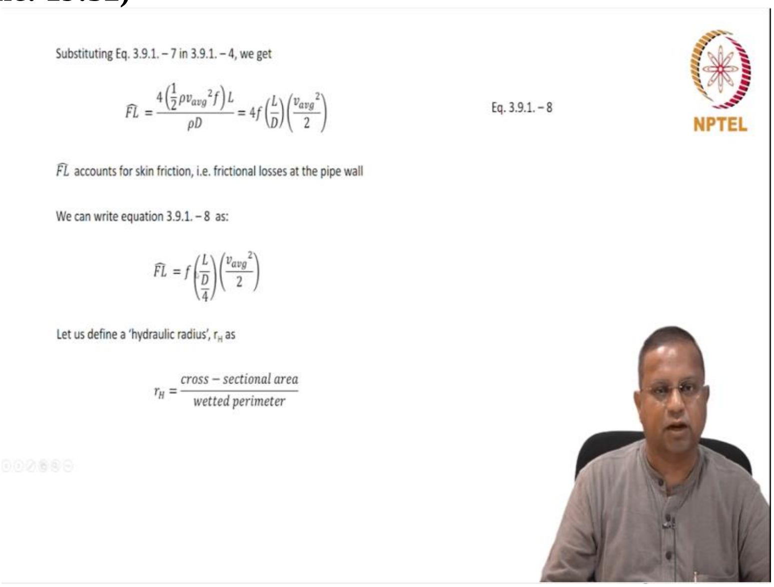

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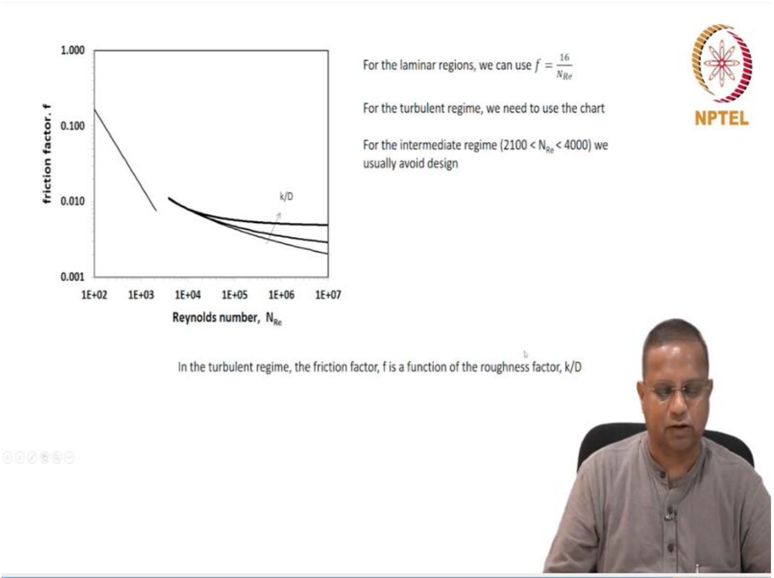

Then, we saw how we could get the value of friction factor through this fanning friction chart as it is called there is also a Moody's friction factor chart, Moody's chart as it is called. There is a slight difference the value of f a slightly different between the fanning friction factor and Moody's friction factor chart. However, let us not get into that, this is the fanning friction factor chart.

You need the Reynolds number and the roughness factor if at all is in turbulent flow to get the friction factor, if it is in laminar flow, all you need is 16 / N_{Re}, you will get the friction factor. So you have this and we saw applications this is engineering Bernoulli equation itself and we saw the applications to a piping network.

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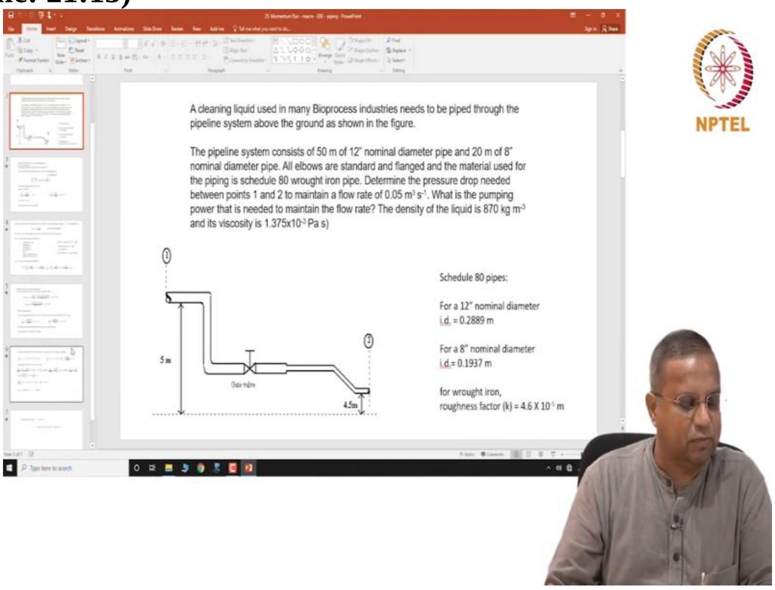

Here this piping network was analyzed to get the pumping power as well as the pressure drop both these design aspects were obtained by using or applying the engineering Bernoulli equation to this piping network. And since we are running short of time, let me quickly show you the various things I am not going to work out the problem I mean I had used I post this as a problem so that you could understand this better anyway we had worked out the problem please go and look at it. If you need to understand it further.

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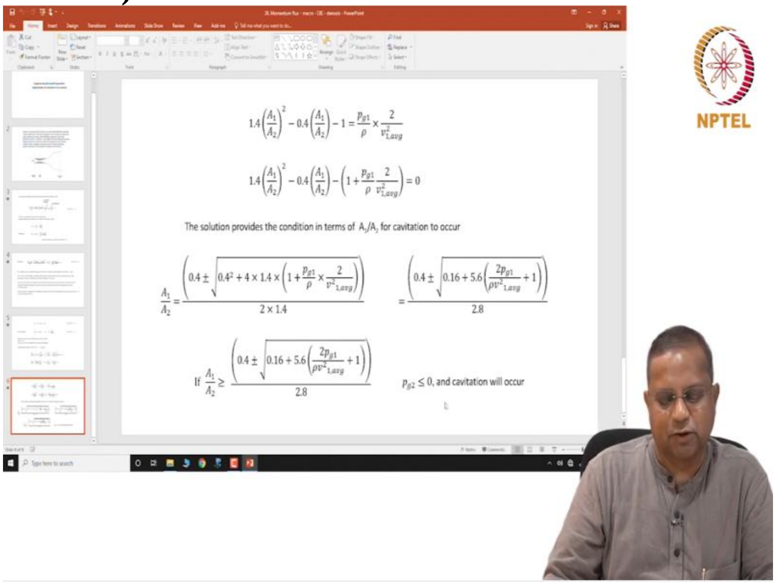

Then we looked at the application to a stenosis situations artery stenosis situation where interestingly, we could find the pressure at which cavitation would occur or the condition, the pressure gauge pressure at which the cavitation could occur we could use engineering Bernoulli equation to get in get that value. Then, we looked at the application of friction factor in the case where there is a relative motion between a solid and a fluid solid and a liquid at the same, let us say a body dropping through a liquid.

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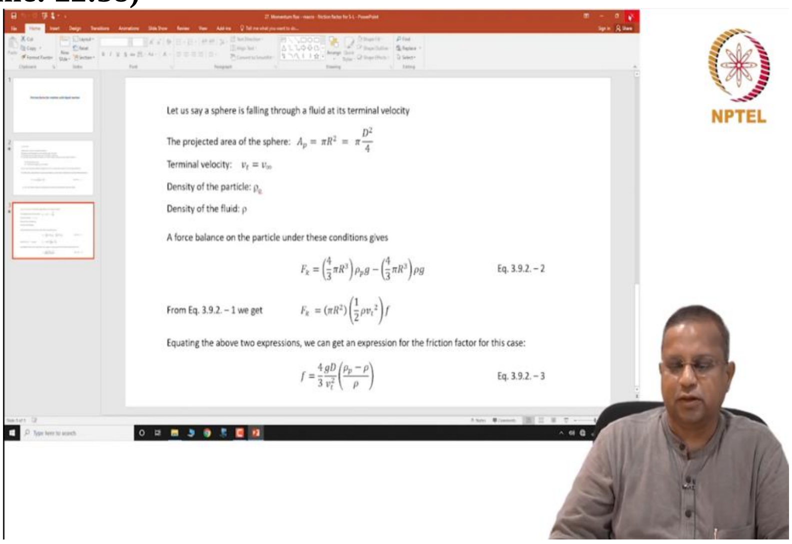

$$f = \frac{4}{3}\frac{gD}{v_t^2}\left(\frac{\rho_p - \rho}{\rho}\right) \qquad (3.9.2\text{-}3)$$

And then finally, we looked at the application of an engineering Bernoulli equation to find the friction factor for packed beds, once we found the friction factor for packed beds, we had done it over a couple of classes ultimately.

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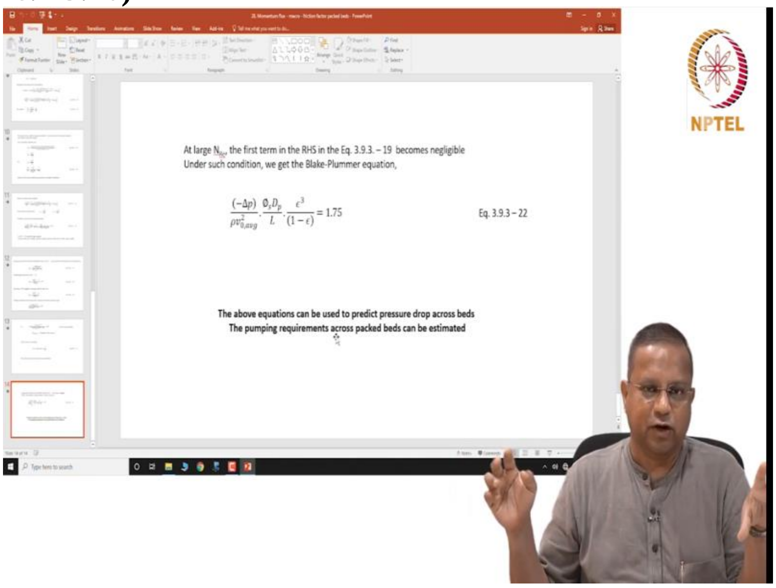

And then we could using the various geometric arguments, we could come up with the Ergun equation and the Kozeny-Carman equation under different conditions of flow. And also this equation can be used to predict pressure drop across the beds as well as the pumping requirements across the beds. So, this is what we did, we spanned a wide spectrum here, starting from fundamental aspects to very useful aspects for design and operation.

I think we will stop here in this class and when we start up again. In the next class, we will start looking at the next conserved quantity the flux of the next conserved quantity, which happens to be heat energy, see you.