Transport Phenomena in Biological Systems
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Lecture - 47
Temperature Profile in a Tissue

Welcome, in this class, let us start looking at temperature profile in a tissue as we go through this; see whether you are reminded of something else that we did earlier in the course. At some point in time it will be quite apparent till then at the back of your mind, keep seeing whether it is something this is somewhat similar to something else that we did in the earlier part of the course.

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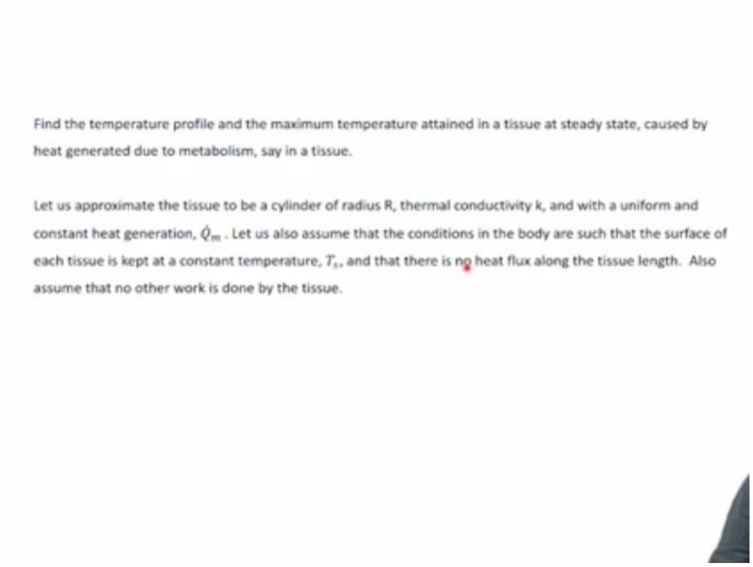

To do this, as we have been doing in this course, problem based learning and so on, I am going to present a problem then solve it give you the principles for solving it. As are in the context of solving this problem, and that way the learning is expected to be much better. So, the problem reads is something like this find the temperature profile and the maximum temperature attained in a tissue at steady state, highly relevant tissue or muscle may be caused by heat generation due to metabolism, say, attainment of a tissue caused by the heat generation generated due to metabolism.

This is what we are going to look at in this class. For that, let us approximate the tissue to be a cylinder of radius R, and thermal conductivity K and with a uniform constant heat generation $\dot{Q}_m$ is a muscle there is a tissue, let us say muscle tissue lot of blood supply to it and so on so forth rate and a lot of metabolic processes going on biochemical processes going on, there is a lot of heat generated because of those reactions. And that is what we are going to call as $\dot{Q}_m$ metabolic heat generation rate, and we are going to take it as a constant in this case.

Let us also assume that the conditions in the body are such that the surface of each tissue is kept at a constant temperature T_s is quite common. The body has temperature control mechanisms by which it effectively maintains the surface of each tissue at a constant temperature. In this case, we are going to call it T_s. Let us say that is 7.3℃, 37℃, whatever the normal body temperature that is, and 38 we will see whether it is and that there is no heat flux along the tissue length.

There is no axial heat flux; there is only a radial heat flux. Also assume that no other work is done by the tissue, the tissue is at rest there is no other work that is just the other aspects that are associated with the normal functioning of the tissue. This case of muscular tissue, the way we can approach this now that we have the equation of thermal energy, this we go to the appropriate coordinate system, pick up the equation cancel out the irrelevant terms, whatever remains is the governing equation for our case. That is the beauty of this.

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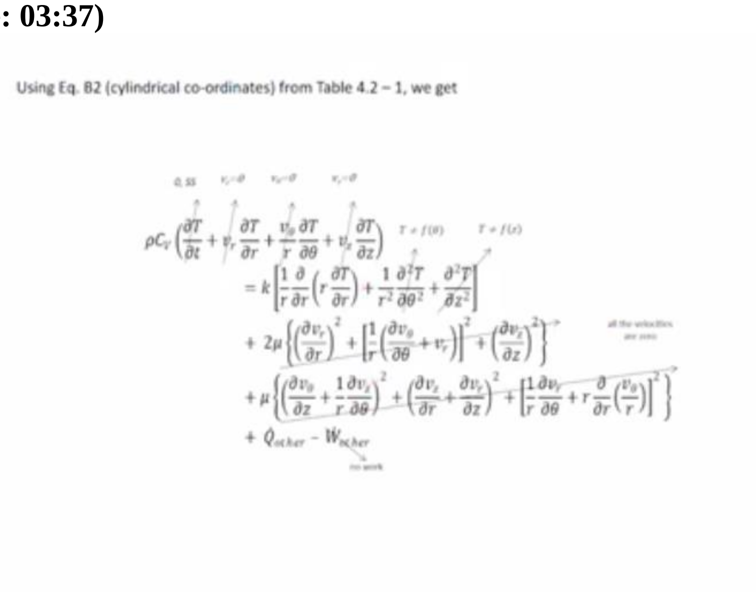

So, will we can go to equation B2 cylindrical coordinates. I have already shown you the table I have requested you to make a copy of it. Please go to that table. Please see what B2 is can you pause the video here? Go back, go to your table pick out equation B2 and then restart the video and pause the video. What is that equation B2 in table 4.2 - 1 for cylindrical coordinates would have been this

Using Eq. B2 (cylindrical coordinates) from Table 4.2-1, in which we can cancel the irrelevant terms

$$\rho C_V\left(\underbrace{\frac{\partial T}{\partial t}}_{0,\ \text{SS}}+\underbrace{v_r}_{v_r=0}\frac{\partial T}{\partial r}+\underbrace{\frac{v_\theta}{r}}_{v_\theta=0}\frac{\partial T}{\partial \theta}+\underbrace{v_z}_{v_z=0}\frac{\partial T}{\partial z}\right)$$

$$=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}}_{T\neq f(\theta)}+\underbrace{\frac{\partial^2 T}{\partial z^2}}_{T\neq f(z)}\right]+\underbrace{2\mu\left\{\left(\frac{\partial v_r}{\partial r}\right)^2+\left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right)\right]^2+\left(\frac{\partial v_z}{\partial z}\right)^2\right\}}_{\text{All the velocities are zero}}$$

$$+\underbrace{\mu\left\{\left(\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)^2+\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)^2+\left[\frac{1}{r}\frac{\partial v_r}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2\right\}}_{\text{All the velocities are zero}}+\dot{Q}_{\text{other}}-\underbrace{\dot{W}_{\text{other}}}_{\text{No work}} \qquad (4.2.1\text{-}1)$$

So, let us apply let us see which of these terms are relevant for our case, this balance is certainly relevant. So this represents balance and we have taken this. So, let us start we are going to do a steady state analysis therefore, no time derivatives this first time goes to 0. There is no velocity at all there is no flow that is happening here. There is angular symmetry temperature does not vary with z as mentioned in the problem or we can take the temperature not to depend on z the axial distance none of these terms would remain because each one of those has velocity of the fluid, v_r, v_θ and v_z the velocity components of the fluid in this case. There is no fluid flowing, the blood is not a part of the tissue. Blood is not a part of the system that we have taken the, our focus here, and therefore, we do not have to worry about the blood flow at all.

Then these 2 terms would completely drop out there is of course $\dot{Q}_m$ other because there is metabolic heat that is being generated as mentioned that heat comes in even though blood is not a part of the system, those heat aspects coming into the tissue that is fine or they are being generated in the tissue that is fine that is being considered as a whole black box kind of a thing – $\dot{W}$ other there is no shaft work that is done in this case. So, this is what the equation reduces to and are we going to call it something no.

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Let us take $\dot{Q}_{other} = \dot{Q}_m$ = metabolic heat rate. Then, we can write

$$-\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = \dot{Q}_m \qquad \text{Eq. 4.2.1.} - 2$$

B.C. 1: at r = 0, T = finite or $\frac{dT}{dr} = 0$ Eq. 4.2.1. – 3

B.C. 2: at r = R, T = T_s Eq. 4.2.1. – 4

Integrating Eq. 4.2.1. – 2 once with B.C. 1, we get

$$\frac{dT}{dr} = -\frac{\dot{Q}_m}{2k}r$$

Integrating again with B.C. 2, we get

$$T = T_s + \frac{\dot{Q}_m R^2}{4k}\left[1-\left(\frac{r}{R}\right)^2\right] \qquad \text{Eq. 4.2.1.} - 5$$

That is the balance equation, we have brought it down to the terms that are relevant to us in terms of the terms that are relevant to us. $\dot{Q}_m$ metabolic heat rate that I already mentioned. Therefore, the only terms that are remaining here you know all these are gone to 0, the term remains of course, all these are going to 0 here you have $\dot{Q}_m$ other both on the same side, therefore, they did be negative if you take it to the other side, that is what has been done

If we take $\dot{Q}_{\text{other}} - \dot{Q}_m =$ metabolic heat rate, then

$$-\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = \dot{Q}_m \qquad (4.2.1\text{-}2)$$

So, the partial derivatives have been converted to total derivatives equals or ordinary derivatives whatever you want to call them equals $\dot{Q}_m$ So, this is the equation which describes whatever is happening. Equation 4.2.1 - 2. Now we need boundary conditions and, this is a steady state boundary conditions to solve this second order. Therefore, we need at least 2 boundary conditions to solve this, about the condition 1 at r = 0 the center point the same trick. So the temperature variation we are looking at.

The temperature at the center point must be the same irrespective of the direction we take to reach the center, the direction is what whatever we are taking whatever temperature profile is there is reality. So, that reality needs should not be affected by the direction that we take or the approach that we take, that is the bottom line for expecting a certain behavior at the center. Mathematically speaking, this reality can be reflected only if there is a maximum or a minimum temperature the center the physical system. And if there is a maximum or minimum, the derivative equals the derivative of the temperature with respect to the radius has to be equal to 0 you may know that

The boundary conditions are

$$\text{BC 1: At } r = 0,\ T = \text{finite or } \frac{dT}{dr} = 0 \qquad (4.2.1\text{-}3)$$

$$\text{BC 2: At } r = R,\ T = T_s \qquad (4.2.1\text{-}4)$$

Integrating once with BC 1, we get

$$\frac{dT}{dr} = -\frac{\dot{Q}_m}{2k} r$$

Integrating again with BC 2, we get

$$T = T_s + \frac{\dot{Q}_m R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (4.2.1\text{-}5)$$

We are going to call this equation 4.2.1 - 5. This is remind you of something. When we started out this lecture, I did mention watch out for some similarity with something that we have already done especially this term, somewhat this term and so on and so forth. Do you see some similarity? It is with laminar flow.

In a cylindrical pipe, the velocity distribution is a function of $1 - (r / R)^2$ a parabolic velocity distribution. So in this case of pure conduction in a must the equivalent of this laminar flow in a pipe for the moment of flux case we have a parabolic temperature profile similar to the parabolic velocity profile and this is the entire temperature profile of course you have a T_s essence also.

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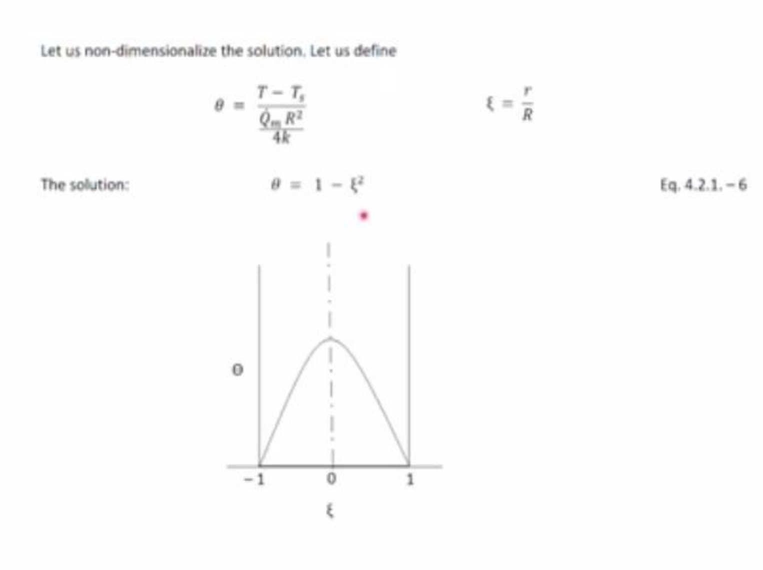

Now, let us attempt to express the results in a more general fashion. If we non-dimensionalise the variables

$$\theta = \frac{T - T_s}{\frac{\dot{Q}_m R^2}{4k}}$$

Then we also called ξ distance variable as r by the radius of the muscle R it is the tissue, the muscle the solution we already know it is just the same form the same solution as the case of laminar flow through a pipe.

$$\xi = \frac{r}{R}$$

the solution becomes

$$\theta = 1 - \xi^2 \qquad (4.2.1\text{-}6)$$

and therefore, if you plot theta versus ξ as the value 0 for ξ is going to - 1 to + 1 r / R, you know the radius at the center goes from - r we are converted to r / R into a distance variable and therefore, it goes from - 1 to 1 and you have this nice parabolic profile here. So, this is the temperature profile in out of pure conduction situation in a muscle which was assumed to be a cylinder.

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The rate of heat dissipation as a cylindrical surface for the tissue length L equals area times flux

The rate of heat dissipation at the cylindrical surface, for the tissue length, L is

$$\text{Area} \times \text{Flux} = 2\pi RL \times q_r|_{r=R} \tag{4.2.1-7}$$

$$= 2\pi RL\left(-k\frac{dT}{dr}\right)\bigg|_{r=R} = \pi R^2 L\dot{Q}_m \tag{4.2.1-8}$$

From Eq. 4.2.1-5, we can say that $T_{\max}$ occurs when $r = 0$.

Thus

$$T_{\max} = T_s + \frac{\dot{Q}_m R^2}{4k}$$

For $R = 1$ cm, $\dot{Q}_m = 5$ cal cm^{-3} h^{-1}, $k = 10^{-3}$ cal (cm.s.°C)$^{-1}$ and $T_s = 37$ °C, we get

$$T_{\max} = 37 + \frac{5 \times 1^2}{(4 \times 10^{-3})3600} = 37.3\ °\text{C}$$

And the temperature of the surface we are going to take is 37℃ So the units or cancel out or it will convert them into all constituents that have units and then cancel out, it does not make a difference in this case, that will turn out to be 37.3℃. So the center point of the muscle is 0.3℃ higher than the surface, the maximum temperature is 0.3℃ higher than the surface temperature. That is the insight that we get from this. It is a nice number to have, as long as your model appropriately describes your system, in this case, it does. I think we can stop here and continue in the next class see them.