Transport Phenomena in Biological System

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Lecture - 48
Unsteady State Heat Conduction

Welcome back in this class, let us start looking at unsteady state heat conduction. We already seen an example of steady state heat conduction. So, the next aspect is unsteady state heat conduction. What does unsteady mean we all know that the temperature that the radiation with time at a point in the system does not happen, variation of relevant properties at a point does not happen. The values remain the same with time.

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Again let us pose a problem solve it. All these will serve as problem solution strategies also. And we pick up the relevant principles relevant tools as we go along in the context of the problem itself. This problem reads us in a micro analysis system for the determination of an analyte the sample is first sprayed as 10 micro liter that has to be micro liter let me correct write away for you.

In a micro analysis system for the determination of an analyte the sample is first sprayed is 10 microliters spherical droplets into a heating zone. The droplet needs to be heated to 60℃ to complete a reaction that is a necessary step for the analysis. This is what we have here, the very standard situation, especially in micro analyses assuming that the properties of the sample drop are the same as that of water, since the sample is predominantly aqueous, so, that is not a bad assumption at all.

Estimate the time needed to reach the steady state temperature in the droplet, the droplet needs to reach a temperature of 60°C steady state for the reaction to have been completed at this step and we are asked to find the time that it takes to reach very relevant because heating rates must be adjusted such that there is not much time needed for the entire droplet to reach steady state. That is the whole idea here.

So, this is probably a question that is raised the design stage as to how much of heating should be there so that the time is minimized or if we have this much heating, is it good enough? How long does it take? Maybe there is a heater that is already available. But the company wants to use it, the designer wants to use it and wants to check whether the that will serve the purpose, very many different situations, for all those situations, this analysis becomes helpful. That is the beauty of this fundamental application is disrupt to us, rather, whichever it can be used in very many different ways.

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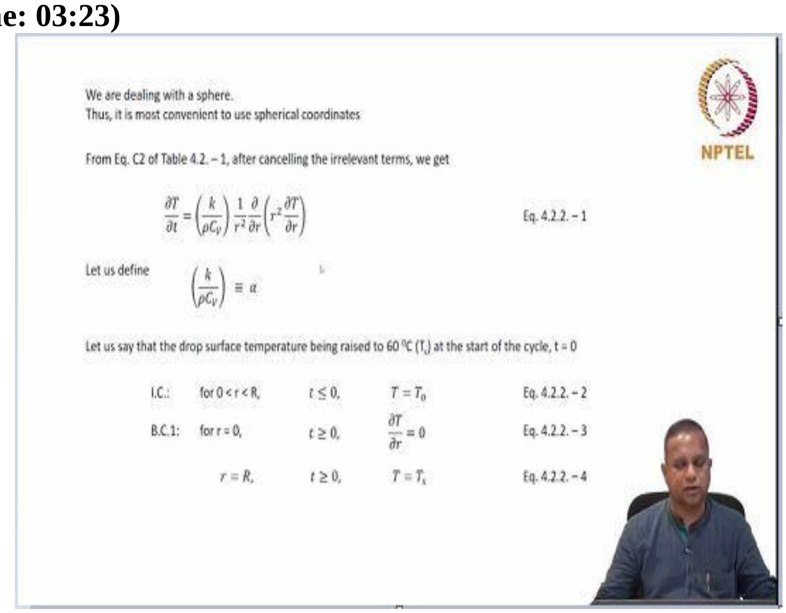

Let us go about solving this. We are of course dealing with a sphere. So obviously spherical coordinates are the ones that are relevant. We got use the equation of thermal energy here. So this go to that table, pick up the relevant equation. So in this case, these spherical coordinate equation and of course for laminar Newtonian fluid in this is water Newtonian fluid is perfectly fine, predominantly water Newtonian fluid is perfectly fine.

Constant ρ and k can be assumed in this case. So that is not a problem. So, equation C2, we could pick up from table 4.2- 1. And if we cancel the irrelevant terms, what I am going to ask you to do is, stop or pause the video here. Go back to that equation or pick up this equation C2 from 4.2 - 1. Cancel the irrelevant terms, see what you get, then we will continue. I think you need to pick this up. So why did you do this and then when you come back, we will see whether you got the same thing.

Go ahead. Take as much time as you want. Time is not a big, you know, time is not a big constraint here. I hope you got this as the remaining terms$\frac{\partial T}{\partial t}$. This is unsteady state the temperature inside the droplet continuously varies the temperature inside the system continuously varies. Therefore, this term is highly relevant equals

$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho C_V}\right)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \qquad (4.2.2\text{-}1)$$

So this, these would be the only terms that remain. And as you know, we have a time derivative here a partial differential equation along with the specific space derivatives. So, the mathematical effort that is needed is going to be quite significant to solve this. Let us solve this, I am not going to show you all the steps you need to now you have enough I have shown you individual steps each and every individual steps so far. So now I am going to let you figure it out yourself. But that is only way you learn.

So let me show you the solution. It is not highly trivial So, I will show you the solution, but I want you to fill in the intervening steps is to that the simply way to learn. So this is equation 4.2.2 - 1. Let us define the non dimensional variables $\alpha = k/(\rho C_V)$, thermal diffusivity. Let us say that the drop surface temperature being raised to 60°C, Ts at the start of the cycle so the surface is already at 60 °C when you put in it is how the temperature varies inside the drop. That is a concern here.

We can consider the drop surface temperature being raised to 60 °C (T_s) at the start of the cycle, $t = 0$.

Thus

IC: For $0 < r < R$, $\quad t \le 0, \quad T = T_0$ (4.2.2-2)

BC 1: For $r = 0$, $\quad t \ge 0, \quad \dfrac{\partial T}{\partial r} = 0$ (4.2.2-3)

BC 2: For $r = R$, $\quad t \ge 0, \quad T = T_s$ (4.2.2-4)

Now we need to boundary conditions because it is a second order variation in space. So, we have the initial conditions, we have the boundary conditions, we can solve this of course, we would like to solve it in terms of non-dimensional variables makes the solution a lot more general.

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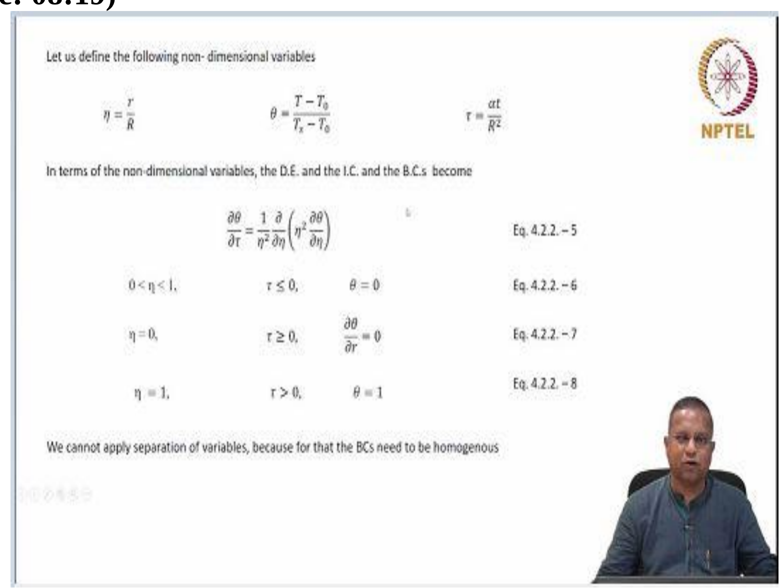

So, let us define the following non-dimensional variables η as r / R these distance variable a θ temperature variable as T - T0, the entire initial the temperature at which the entire droplet is initially divided by Ts, the surface temperature 60°C- T 0, that is what we are going to define our θ as, and then we are also going to define a τ, because there is variation of time. None of these show a variation of time τ as αt / R^2.

If we use non-dimensional variables defined as

$$\eta = \frac{r}{R}$$

$$\theta = \frac{T - T_0}{T_s - T_0}$$

and

$\tau = \alpha t / R^2$

Work out the dimensions, this is quite straightforward to C this quite straightforward to C, what are the dimensions of these to ensure that τ is indeed dimensionless. So, in terms of the non-dimensional variables, the differential equation and the initial boundary conditions will become, do work this out it is quite a few steps do work it out and check for

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial \theta}{\partial \eta}\right) \quad (4.2.2\text{-}5)$$

$$0 < \eta < 1, \qquad \tau \le 0, \qquad \theta = 0 \quad (4.2.2\text{-}6)$$

$$\eta = 0, \qquad \tau \ge 0, \qquad \frac{\partial \theta}{\partial r} = 0 \quad (4.2.2\text{-}7)$$

$$\eta = 1, \qquad \tau > 0, \qquad \theta = 1 \quad (4.2.2\text{-}8)$$

We cannot apply separation of variables to get the solution of the above differential equation because for that the BCs need to be homogenous. Thus, let us use the following transformation:

$$\theta'(\eta, \tau) = 1 - \theta(\eta, \tau) \quad (4.2.2\text{-}9)$$

So, we have the partial differential equation in terms of the non-dimensional variables, we non dimensionalize it and we have the initial and boundary conditions in terms of the non-dimensional variables, we can solve this and for solution, you will have to go back to your math course your differential equations course, see how to go about solving this form. Of course, here we cannot apply separation of variables? Check out why boundary conditions need to be homogeneous. Try to understand that better and then if you follow a method that will be suitable here, that is going to be something like this.

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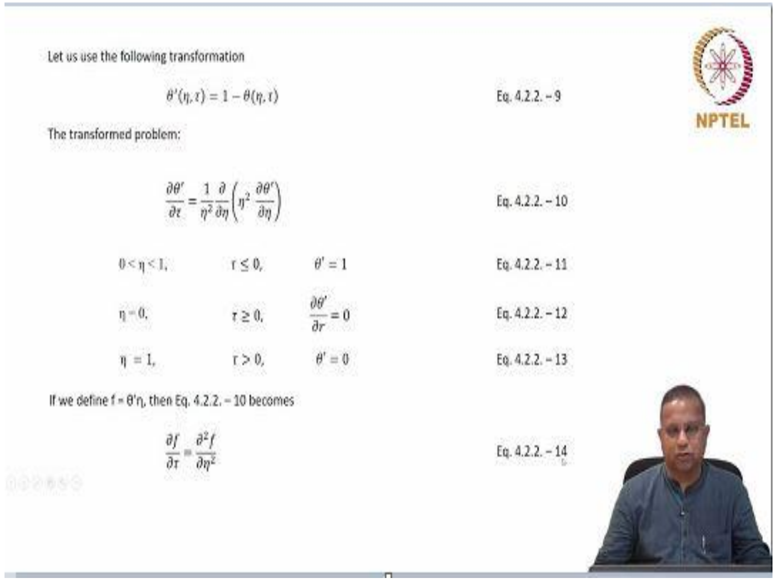

The transformed problem is

$$\frac{\partial \theta'}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial \theta'}{\partial \eta}\right) \tag{4.2.2-10}$$

$$0 < \eta < 1, \qquad \tau \le 0, \qquad \theta' = 1 \tag{4.2.2-11}$$

$$\eta = 0, \qquad \tau \ge 0, \qquad \frac{\partial \theta'}{\partial r} = 0 \tag{4.2.2-12}$$

$$\eta = 1, \qquad \tau > 0, \qquad \theta' = 0 \tag{4.2.2-13}$$

If we define $f = \theta'\eta$, then Eq. 4.2.2-10 becomes

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \eta^2} \tag{4.2.2-14}$$

And then this equation where we get into this form, a lot of us write it then differentiate it and see that it indeed reduces to this form. So, this is closer to the way things are given into the textbook because I have spent enough time to prep you to see how condensed things are in the textbook and other textbooks. My textbook is has a lot of details given maybe not here, maybe initially. And here the idea is to get you into that framework.

Because the information is available only in this form, you need to fill in all the gaps. Many people do not realize that you need to fill in so much gap that they are usually confused or wonder how can you go from this step to this step. This is equation 4.2.2 - 14.

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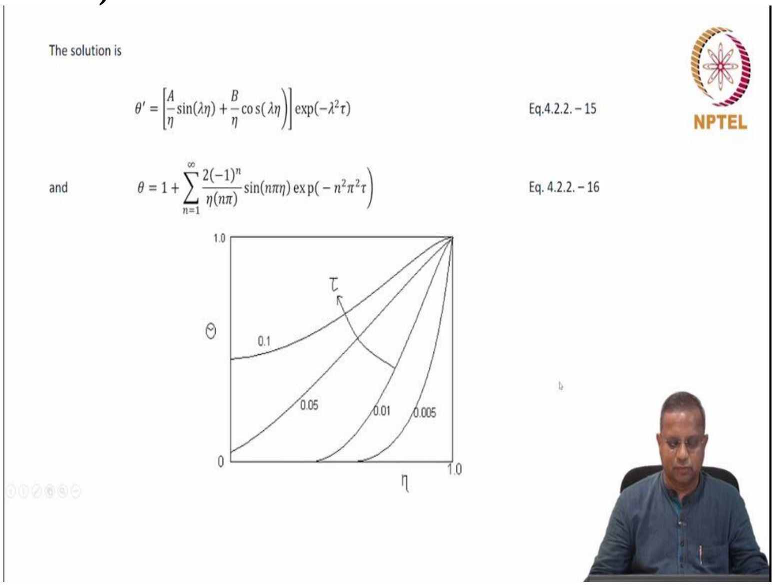

The solution is

$$\theta' = \left[\frac{A}{\eta}\sin(\lambda\eta) + \frac{B}{\eta}\cos(\lambda\eta)\right]\exp(-\lambda^2\tau) \qquad (4.2.2\text{-}15)$$

and

$$\theta = 1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\eta(n\pi)}\sin(n\pi\eta)\exp(-n^2\pi^2\tau) \qquad (4.2.2\text{-}16)$$

The variation of the non-dimensional temperature with non-dimensional distance at various values of non-dimensional time is given in Fig. 4.2.2-1.

So from if you have a variation of θ then you can pick out the variation of T just by a quick transformation here. So, but having it in form of θ makes a general θ versus η, θ versus ξ, θ versus ξ for various values of τ. You get this is θ.

If we use non-dimensional variables defined as

$$\eta = \frac{r}{R}$$

$$\theta = \frac{T - T_0}{T_s - T_0}$$

and this is for various values of the non-dimensional time at time = 0.005, this would be the profile of temperature in the droplet at τ = 0.01 this would be the profile of temperature point of θ that is which is nothing but temperature variable at 0.05 it will be like this at 0.1 it will be like this and so on and so forth. So as τ increases at some point in time, everything will reach 60℃, which is this well.

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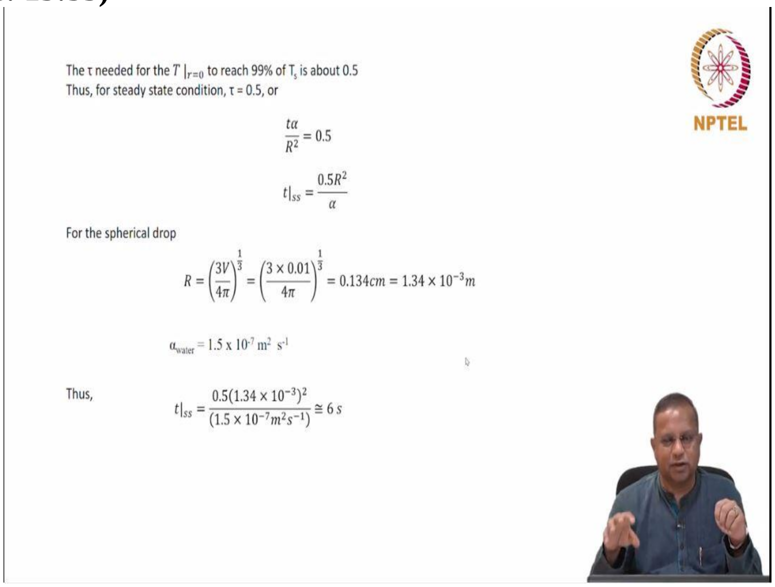

Now, we need a certain time we are working at a problem? So to have some feed of times involved the τ needed for the temperature of the center to say reach 99% if you are wondering 100 percent it is in final thing the way the problem is set up. So, for it reach 99% of the surface temperatures turns out to be about 0.5. So, that is what this one shows here it will go on you work out the various profiles for it to reach 99% here you need a τ of 0.5.

$\alpha t / R^2 = 0.5$ and $t|_{ss} = 0.5\ R^2 / \alpha$

So, for steady state conditions, we have 0.5 approximately steady state condition. So, $\alpha\, t / R^2$ which is τ is 0.5. So, the time that is needed for steady state is nothing but $0.5\ R^2 / \alpha$ in this transposing this equation and for a spherical drop, like the one that we consider here 10 microliters the radius

Now for the spherical drop

$$R = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = \left(\frac{3\times 0.01}{4\pi}\right)^{\frac{1}{3}} = 0.134\text{ cm} = 1.34 \times 10^{-3}\text{ m}$$

$\alpha_{water} \sim 1.5 \times 10^{-7}\ m^2\ s^{-1}$. Thus

$$t|_{ss} = \frac{0.5(1.34\times 10^{-3})^2}{(1.5\times 10^{-7}\, m^2 s^{-1})} \cong 6\text{ s}$$

Remember this is thermal diffusivity not the units here, meter square per second same as mass diffusivity or kinematic viscosity thermal diffusivity here it is nice remember that value for water $1.5*10^{-7}$ meter square per second and therefore, the time that is needed to reach steady state 0.5 R^2 divided by α terms are to be 6 seconds, quite decent the entire droplet is going to reach 99% or 60℃ in about 6 seconds, which is a good design, you can very quickly within 6 seconds it reaches that temperature.

So there is a temperature that is needed for the reaction to go to completion for the next step to work, so that is perfectly fine. I think we will stop here for this class, we have completed the chapter on thermal flux in third chapter. And we looked only at the heat flux as a function of its primary driving force a temperature gradient, conduction alone, all the velocities were 0 as you saw, but we needed that foundation for it to be complete, so that which we can use later.

So the primary driving force for heat flux is the temperature gradient and we saw on the temperature gradient cases in this chapter. When we meet next I will briefly review this chapter and then move on to charge flux see you.